An Introduction to SPHERICAL TRIGONOMETRY

The Solution of Triangles

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AN INTRODUCTION TO

SPHERICAL TRIGONOMETRY

THE SOLUTION OF TRIANGLES

By Frank Loxley Griffin

This pamphlet has been written to meet the existing emergency need for a treatment of spherical trigonometry, extremely brief but covering all points essential for the efficient solving of spherical triangles.

The sine law and the two cosine laws are derived at the outset and are used in deriving the customary formulas for the logarithmic solution of triangles, right and oblique. This procedure makes it unnecessary to devote much time or effort to the derivation of the formulas. Indeed, it renders immediately possible, though inconvenient, the solution of any spherical triangle, even before the preferable formulas have been obtained.

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SECTION ONE

The General Problem

§ 1. Importance of Spherical Triangles. On a sphere S innumerable curves of many kinds can be drawn connecting any two given points, A and B, of S. The shortest of all these curves is a great circle arc from A to B.* We may go from A to B in either direction along the circle, but unless A and B are 180° apart, one arc AB will be shorter than the other; and this more direct arc will be the shortest possible route from A to B on the surface.

Since the earth is approximately spherical, great circles on a sphere are important for navigation, both marine and aerial. Methods of determining the course to be followed in going from one point to another on the earth depend largely on the properties of "spherical triangles," whose sides are arcs of great circles. We need consider only angles and arcs less than 180°.

By the angle A between two sides, AB and AC, of a spherical triangle is meant the angle between the straight lines tangent to the arcs AB and AC at A. Those lines lie in the planes OAB and OAC and are perpendicular to the radius OA.

As in plane trigonometry, we shall here denote the angles of a spherical triangle by capital letters, A, B, C, and the opposite sides (arcs) by the corresponding small letters a, b, c. We shall find that if any three of these six "parts" of a triangle are given, even the three angles, the other three parts can be calculated, with sometimes more than one possible solution. The formulas needed will be analogous to but somewhat different from those used in plane trigonometry.

* A great circle of a sphere is one whose plane passes through the center O of the sphere. Unless A and B are ends of a diameter, they together with O determine a plane and there is one great circle through A and B. What if A and B are ends of a diameter?

§ 2. Some Essential Theorems from Solid Geometry. Let us list at the outset some geometric theorems that will be useful in understanding or proving trigonometric relations. We shall later refer to some of these explicitly by number, thus, *Th. 4* for theorem 4 in this list.

In any spherical triangle:

1. The sum of the three sides $< 360°$. (Each side may exceed $90°$.)
2. The sum of the three angles $> 180°$, but $< 540°$. (Each angle may exceed $90°$.)

Definition. The difference between $A + B + C$ and $180°$ is called the "spherical excess."

3. The area of the triangle is to the area of the sphere as the excess is to $720°$.
4. Any side contains the same number of angular units (degrees, etc.) as the angle which it subtends at the center of the sphere.

Definition. A point P is a pole of a great-circle arc p if P is an end of the diameter perpendicular to the plane of p. It follows that P is at a quadrant distance, $90°$, from every point of p.

5. If the vertices (A, B, C) of one spherical triangle are poles of the sides (a', b', c') of another triangle, then the vertices (A', B', C') of the second triangle are poles of the sides (a, b, c) of the first. If A and A' are on the same side of arc $B'C'$, and similarly for the other vertices, either triangle is called the polar of the other (Fig. 1).
6. In any spherical triangle and its polar, any side of one triangle is the supplement of the opposite angle of the other:

$$a = 180° - A', \quad a' = 180° - A; \quad \text{etc.} \tag{1}$$

7. If from the foot of a perpendicular p to a plane, a straight line m is drawn at right angles to any line l in the plane, and a line n is drawn joining the intersection of m and l to any point of the perpendicular p, then n is perpendicular to l.

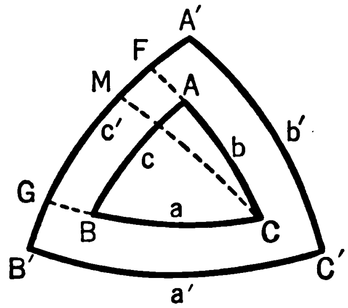

FIG. 1.

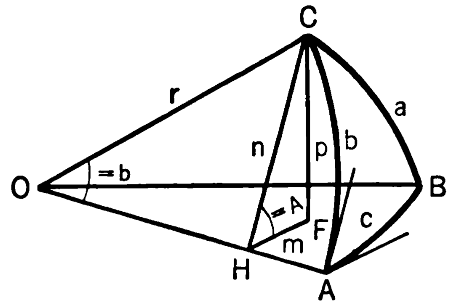

FIG. 2.

§ 3. The Sine Law. In any plane triangle the three sides are proportional to the sines of the opposite angles:

Plane $\triangle$: $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

In a spherical triangle the corresponding sine law is

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{2}$$

That is, *the sines of the three sides are proportional to the sines of the opposite angles.*

Proof: From any vertex C drop a perpendicular CF to the plane OAB of the opposite side c. From F draw a perpendicular FH to OA, and draw HC. Let p, m, n denote the lengths of these lines, as in Figure 2. In the right triangle FCH, HF and HC are perpendicular to OA (HF by construction and HC by *Th.* 7). Hence these lines are parallel to the tangents at A, whose included angle measures A. Thus, angle $FHC = A$. Also, the central angle $AOC =$ arc b (in degrees), and angle $BOC = a$. In the two right triangles HOC and FHC we have:

$$n = r \sin b, \qquad p = n \sin A, \qquad \therefore\ p = r \sin b \sin A.$$

Similarly, by dropping a perpendicular FK from F to OB and drawing KC, we could solve again for p on the other side, getting

$$p = r \sin a \sin B.$$

Equating values for p and dividing both sides by $r \sin A \sin B$:

$$r \sin a \sin B = r \sin b \sin A, \qquad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$

Since a and b are any two sides, we have more generally (2) above.

Remark. In the foregoing proof, based on Figure 2, the sides and angles are treated as each less than 90°. But the law (2) can be shown to be general, as can the following law also.

§ 4. The Cosine Law, for Sides. In any plane triangle the square of any side equals the sum of the squares of the other two sides minus twice the product of those sides by the cosine of their included angle. For instance, $a^2 = b^2 + c^2 - 2\,bc \cos A$.

The corresponding law in a spherical triangle is

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \tag{3}$$

That is, *the cosine of any side equals the product of the cosines of the other two sides, plus the product of the sines of those sides by the cosine of their included angle.*

Proof: The angle at C (Fig. 3) between the tangent lines CL ($= u$) and CM ($= v$) measures angle C of the spherical triangle ABC. In the right triangles COL and COM the acute angles at O are equal to sides b and a (in degrees); and we have at once:

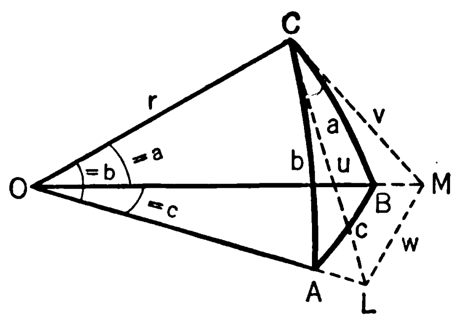

FIG. 3.

$$u = r \tan b, \qquad v = r \tan a,$$
$$\overline{OL} = r \sec b, \qquad \overline{OM} = r \sec a.$$

Now express LM by the cosine law in each of the plane triangles LMC and LMO:

(I) $w^2 = u^2 + v^2 - 2\,uv \cos C = r^2 (\tan^2 b + \tan^2 a - 2 \tan a \tan b \cos C)$,

(II) $w^2 = \overline{OL}^2 + \overline{OM}^2 - 2\,\overline{OL}\,\overline{OM} \cos c =$
$r^2 (\sec^2 b + \sec^2 a - 2 \sec a \sec b \cos c)$.

Equating, replacing $\sec^2 b$ by $1 + \tan^2 b$, etc., we get on cancelling:

$$-\tan a \tan b \cos C = 1 - \sec a \sec b \cos c.$$

Transposing the two negative terms and multiplying through by $\cos a \cos b$, we have equation (3) above.

Since this law holds for any side it gives us three formulas,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{4}$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \tag{5}$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{6}$$

Observe that, as in the case of the cosine law for plane triangles, this law (4), (5), (6) can be used to solve a spherical triangle if we have given the three sides a, b, c, or any two sides and their included angle. (After finding the third side from one of the formulas, the other angles can be found from the other two formulas.) The question of less laborious methods of solving will be discussed presently (§ 6).

§ 5. The Cosine Law, for Angles. In any spherical triangle there is also a cosine law for angles, in which the formulas read thus:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a, \tag{7}$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b, \tag{8}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \tag{9}$$

Probably the best plan for remembering the law is to state it in words as was done in § 4 for the cosine law for sides.*

This new cosine law is readily proved by considering the polar triangle $A'B'C'$ of the given triangle ABC (Fig. 1, p. 3).

The cosine law for sides, applied to the polar triangle (whose sides are a', b', c') gives

$$\cos a' = \cos b' \cos c' + \sin a' \sin b' \cos C'. \tag{10}$$

But by *Th.* 6, § 2, $a' = 180° - A$, etc.; also $C' = 180° - c$. Since the sine of $(180° - A) = \sin A$, while $\cos (180° - A) = -\cos A$, etc., we have at once

$$-\cos A = (-\cos B)(-\cos C) + \sin A \sin B (-\cos c).$$

Simplifying and multiplying through by -1 gives (7). Similarly for (8) and (9).

This cosine law for angles can be used to solve any spherical triangle if we have given the three angles A, B, C or any two angles and their included side. More convenient methods are available, however.

* Observe also that the first equation gives the second if we advance the letters cyclically, *i.e.*, change every a (or A) to b (or B, respectively), every b to a c, and every c to an a. Repeating this operation, the second equation gives the third, and the third gives the first again.

§ 6. Concerning the Solution of Spherical Triangles. We have seen that a spherical triangle can be solved in certain cases by the foregoing laws:

Given Parts	*Use*
3 sides; or 2 sides and included angle:	Cosine Law for Sides
3 angles; or 2 angles and included side:	Cosine Law for Angles

Suppose, however, that we have given some other combination of three parts, such as two angles and the side opposite one of them, say

$$A = 70°, \qquad B = 80°, \qquad a = 55°.$$

Using these values in the sine law we have:

$$\frac{\sin 55°}{\sin 70°} = \frac{\sin b}{\sin 80°} = \frac{\sin c}{\sin C}.$$

From this we could find side b (possibly two solutions). But since the angle-sum in a spherical triangle is not 180° as in a plane triangle, we do not know angle C and cannot complete the solution by the sine law alone. Substituting the given parts in (7), we have

$$\cos 70° = -\cos 80° \cos C + \sin 80° \sin C \cos 55°. \qquad \textbf{(11)}$$

Looking up the given functions and combining, we have

$$.56486 \sin C - .17365 \cos C = .34202. \qquad \textbf{(12)}$$

This equation is similar to one discussed in § 306 of the author's *Introduction to Mathematical Analysis* and can be solved by the method described there, viz. by determining two constants m and ϕ such that $m \sin (C - \phi)$ shall equal the left member of equation (12). Then, solving $m \sin (C - \phi) = .34202$ for $C - \phi$, we can find C. But this method of solving for C requires considerable planning and is rather inconvenient.

The problem of solving a spherical triangle when given two sides and the angle opposite one of them can be handled by a similar method, likewise inconvenient.

To summarize: We have found three basic laws for spherical triangles, the sine law, the cosine law for sides, and the cosine law for angles. By means of these three laws it is possible to solve any spherical triangle, if any three of its six parts are given. We could stop at this point and rely upon the rather complicated methods of solution now available with these laws. By studying further, however, we can derive from these three basic laws other formulas better adapted to the rapid solution of triangles through logarithmic methods. In Section II we shall do this for right spherical triangles and in Section III we shall deal with oblique spherical triangles.

EXERCISES

Solve the following spherical triangles for the parts specified.

1. Given $a = 77°$, $b = 49°$, $C = 32°$. Find c.

2. Find A, B, given $a = 48°$, $b = 88°$, $c = 70°$.

3. Find c, given $A = 70°$, $B = 75°$, $C = 65°$.

4. Given $B = 68°$, $C = 95°$, $a = 40°$. Find A.

SECTION TWO

Right Spherical Triangles

§ 7. Special Formulas Derived. If a spherical triangle has at least one right angle, it is called a right spherical triangle. (It is possible to have a bi-rectangular triangle with two right angles, or even a tri-rectangular triangle with all three angles right angles.)

In a right spherical triangle let $C = 90°$. Then $\sin C = 1$ and the sine law (§ 3) gives

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \sin c,$$

$$\mathbf{\sin a = \sin c \sin A,} \tag{13}$$

$$\mathbf{\sin b = \sin c \sin B.} \tag{14}$$

Further, $\cos C = \cos 90° = 0$, and equation (6) of the cosine law for sides gives

$$\mathbf{\cos c = \cos a \cos b.} \tag{15}$$

Similarly equation (9) of the cosine law for angles gives

$$0 = -\cos A \cos B + \sin A \sin B \cos c.$$

Transposing and dividing by $-\sin A \sin B$

$$\mathbf{\cos c = \operatorname{ctn} A \operatorname{ctn} B.} \tag{16}$$

The other two forms, (7) and (8), of the cosine law for angles give immediately

$$\mathbf{\cos A = \cos a \sin B,} \tag{17}$$

$$\mathbf{\cos B = \cos b \sin A.} \tag{18}$$

Four other important formulas, obtainable from the foregoing as indicated below, are:

$$\mathbf{\cos A = \tan b \operatorname{ctn} c,} \tag{19}$$

$$\mathbf{\cos B = \tan a \operatorname{ctn} c,} \tag{20}$$

and

$$\mathbf{\sin a = \tan b \operatorname{ctn} B,} \tag{21}$$

$$\mathbf{\sin b = \tan a \operatorname{ctn} A.} \tag{22}$$

Derivation of (19). Replacing $\cos a$ in (4) by $\cos c/\cos b$ from (15) and solving for $\cos A$, we find

$$\sin b \sin c \cos A = \frac{\cos c}{\cos b} - \cos b \cos c = \frac{\cos c\,(1 - \cos^2 b)}{\cos b},$$

$$\cos A = \frac{\cos c \sin b}{\sin c \cos b} = \tan b \operatorname{ctn} c.$$

The derivation of (20) is similar.

Derivation of (22). From the sine law, $\sin b = \sin a \sin B/\sin A$. In this expression replace $\sin B$ by $\cos A/\cos a$ from (17), and we have:

$$\sin b = \frac{\sin a \cos A}{\cos a \sin A} = \tan a \operatorname{ctn} A.$$

The derivation of (21) is similar.

§ 8. Solving Right Triangles by the Special Formulas. The foregoing ten formulas (13)–(22), printed in black-face type, each of which is suited to logarithmic work, can be used to solve quickly any right spherical triangle. Let us call c (opposite the right angle C) the hypotenuse, and a and b the legs, even though a or b may exceed c. The possible cases as to given parts (omitting $C = 90°$), and the best formulas to use in solving for the remaining parts and in checking, are shown in the following table.

N.B. It is suggested that students pause at this point to find out for themselves which of the formulas (13)–(22) *could conveniently be used* to calculate the remaining parts of each following triangle, and then see whether their selection of formulas agrees with the table below:

I. Given $a = 78°$, $c = 82°$.
II. Given $c = 100°$, $A = 81°$.
III. Given $A = 75°$, $B = 115°$.

Case	*Given Parts*	*Formulas for Solving*	*Checking*
I	Two legs (a; b)	(15), (21), (22)	(16)
II	Hypotenuse; leg (say c; a)	(13), (15), (20)	(18)
III	Leg; opposite angle (say a; A)	(13), (17), (22)	(14)
IV	Leg; adjacent angle (say a; B)	(17), (20), (21)	(19)
V	Hypotenuse; angle (say c; A)	(13), (16), (19)	(21)
VI	Two angles (A; B)	(16), (17), (18)	(15)

There would be modifications of this list in Cases II–V for other selections of suitable letters under each type. The list is given here simply to bring out the possibility of solving in each general case by means of some of the ten formulas.

It is not advisable to use this list mechanically, as a much more convenient plan is available which makes it possible to pick out and write down the appropriate formula in any case without even memorizing the ten special formulas, (13)–(22). This very desirable objective is attained by using two easily remembered rules which cover all ten special formulas, as was discovered by Napier, the inventor of logarithms.

EXERCISES

1. Given c and b, to find a, A, B, which of the formulas (13) – (23) would be needed?

2. (*a*) Like Ex. 1 if given b and B, to find c, a, and A.

(*b*) Similarly, if given b and A, to find c, a, and B.

3. Given $b = 74° 15'$, $A = 82° 30'$, find the other parts.

§ **9. Napier's Rules.** Omit the right angle C, and replace the hypotenuse (c) and the other angles (A and B) by their complements, abbreviated: co.c, co.A, co.B. Then write these modified parts and the other two parts (a and b) in a circular arrangement in the relative order in which the parts occur around the triangle:

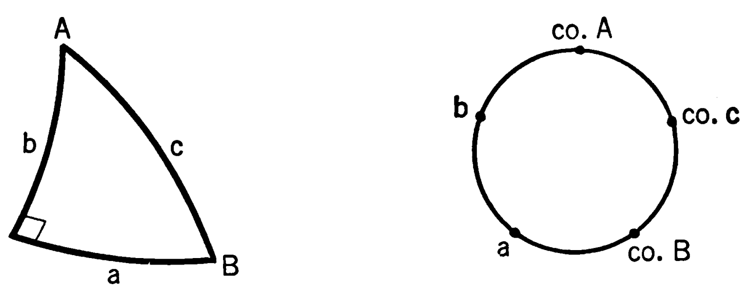

FIG. 4.

Any one of these five "parts" in the circular arrangement may now be considered as the "middle part," the two next to it being then called the "adjacent parts," and the remaining two the "opposite parts." With this understanding, Napier's two rules which cover all ten of the special formulas (§ 8) are:

I *The sine of the middle part = the product of the tangents of the two adjacent parts*;

II *The sine of the middle part = the product of the cosines of the two opposite parts.*

To illustrate: Take a as the middle part. Then the adjacent parts are b and co.B. (See Fig. 4.) Rule I gives

$$\sin a = \tan b \tan (\text{co.}B) = \tan b \operatorname{ctn} B,$$

since the tangent of the complement of B is the cotangent of B. This equation repeats the special formula (21).

Also, with a as the middle part, the opposite parts are co.c and co.A, whose cosines are the sines of c and A. Thus Rule II gives

$$\sin a = \cos (\text{co.}c) \cos (\text{co.}A) = \sin c \sin A,$$

which repeats special formula (13).

EXERCISE

1. Taking b as the middle part, verify as in the foregoing illustration that Rules I and II furnish two more of the special formulas. Likewise take co.A, co.c, and co.B, in turn, as the middle part, and thus verify that Rules I and II cover the entire list of special formulas.

§ 10. Use of Napier's Rules in Solving Triangles. Given two parts of a right spherical triangle, to find some specified third part, we first choose carefully some one of these three parts (or the complement where needed) as the "middle part" in such a way that the other two parts shall both be "adjacent parts" or both "opposite parts." The corresponding rule gives the equation needed in solving for the required part.

In studying the following illustrations, re-draw Figure 4 and notice carefully where the various parts come in the circular arrangement.

Example I. Given $a = 60°$, $A = 70°$. Apply Napier's rules to get the formulas needed for finding: (1) c; (2) b; (3) B.

(1) Looking at a, co.A, and co.c in Fig. 4, we choose a as the middle part, which makes co.A and co.c opposite parts. Rule II then gives: $\sin a = \cos (\text{co.}A) \cos (\text{co.}c)$. But co.$A$ = co.70° = 20°; also, the cosine of the complement of c is equal to $\sin c$. Hence the equation simplifies to

$$\sin 60° = \cos 20° \sin c.$$

(2) Looking at a, co.A, and b in Figure 4, we choose b as the

middle part, making a and co.A adjacent parts. By Rule I: $\sin b = \tan a \tan(\text{co.}A)$ which reduces to

$$\sin b = \tan 60° \tan 20°.$$

(3) With a, co.A, and co.B, we take co.A as the middle part, with a and co.B as opposite parts. Then, by Rule II: $\sin(\text{co.}A) = \cos a \cos(\text{co.}B)$, which gives

$$\sin 20° = \cos 60° \sin B.$$

The equations thus obtained in (1), (2), (3), will be solved shortly (§ 11).

For any other pair of given parts and any required third part, we would proceed in similar fashion. It is important always to consider the circular order of "parts" in Figure 4, in order to choose the "middle part" effectively.

If angle A were obtuse, say $A = 140°$ with $a = 170°$, then in (1) above $\sin a$ would be $\sin 170°$, and co.A would be $-50°$. But $\cos(-50°) = \cos 50°$. In (2) above, however, where we have $\tan(\text{co.}A)$, this would be $\tan(-50°)$, which equals $-\tan 50°$, a negative value. Here $\tan a$ would be $\tan 170°$ ($= -\tan 10°$), and the two negatives would combine to give a positive value for $\sin b$. Similarly in (3) above, we should have $\sin(\text{co.}A) = \sin(-50°) = -\sin 50°$, and $\cos a = \cos 160° = -\cos 20°$, giving a positive value for $\sin B$. Sometimes, however, the $-$ signs do not disappear. (Example II, § 11, will cover such a case.)

EXERCISES

In each following case choose a suitable middle part, write the corresponding formula by one of Napier's Rules, insert the given numerical values, and write in simplified form an expression for a function of the required part.

1. Given $a = 70°$, $b = 40°$, to find c.

2. To find B when given $a = 100°$ and $c = 85°$.

3. Given $b = 120°$, $B = 105°$, to find c.

4. To find A if given $B = 75°$ and $c = 110°$.

5. Given $A = 72°$, $B = 108°$, to find:
(i) a; (ii) b; (iii) c.

§ 11. Practical Solution of Right Spherical Triangles. After getting a numerical formula involving each unknown part alone, as in § 10, (1), (2), (3), we carry out the calculations, preferably by logarithms. We must carefully keep track of any negative signs that may enter with a cosine, tangent, or cotangent of an obtuse angle or side greater than 90°.

When a part (angle or side) is found from its cosine, tangent, or cotangent, the sign of the function shows in what quadrant the part should be taken. When, however, a part is found from its sine, there is an ambiguity: either the first or second quadrant may be used unless excluded by some further consideration. Thus there may be more than one triangle which contains the given parts.

In deciding how to pair values found for different parts in such ambiguous cases, and also in excluding a second (supposed) solution in some cases, the following facts, stated here without proof, are sometimes helpful.

1. A leg and its opposite angle are in the same quadrant.
2. The two legs, a and b, are in the same quadrant if $c < 90°$; but are in different quadrants if $c > 90°$.
3. The sum of any two sides exceeds the third side.
4. The sum of the oblique angles, A and B, exceeds 90° and is less than 270°; but the absolute difference of these angles is less than 90°.
5. Two unequal angles are opposite unequal sides; the greater angle is opposite the greater side.

Example I. Solve completely the right spherical triangle in which $a = 60°$, $A = 70°$.

By Napier's Rules we found in § 10:

$$\sin c = \frac{\sin 60°}{\cos 20°}, \quad \sin b = \tan 60° \tan 20°, \quad \sin B = \frac{\sin 20°}{\cos 60°}.$$

Performing these operations by logarithms we get finally:

$$\log \sin c = 9.96454 - 10, \qquad \log \sin b = 9.79963 - 10,$$
$$\log \sin B = 9.83508 - 10.$$

The tables then give the following values of c, b, B, less than 90°:

$$c_1 = 67° \, 9'.6, \qquad b_1 = 39° \, 4'.9, \qquad B_1 = 43° \, 9'.6.$$

Another possibility is the supplement of each, denoted by a subscript 2:

$$c_2 = 112° \, 50'.4, \quad b_2 = 140° \, 55'.1, \quad B_2 = 136° \, 50'.4.$$

The question arises finally whether, in a valid triangle for the given a and A, the set c_1, b_1, and B_1 should go together, likewise c_2, b_2, B_2, or whether some mixture of the two sets would be needed. By Fact 1 above: b_1 and B_1 are present or else b_2 and B_2; but not b_1 and B_2, etc. By Fact 2 or 3: c_1 requires b_1 since $a = 60°$; and c_2 requires b_2. Thus, in this problem, the possible triangles are:

(1) $a = 60°$, $A = 70°$, $c = 67° \, 9'.6$, $b = 39° \, 4'.9$, $B = 43° \, 9'.6$.
(2) $a = 60°$, $A = 70°$, $c = 112° \, 50'.4$, $b = 140° \, 55'.1$, $B = 136° \, 50'.4$.

Example II. Given $a = 160°$, $B = 80°$, find A, b, c.

1. To find A from a, B: Rule II, with co.A as middle part:
   $\cos A = \cos a \sin B$, $\therefore \cos A = -\cos 20° \sin 80°$.
2. To find b from a, B: Rule I with a as middle part:
   $\sin a = \tan b \operatorname{ctn} B$, $\tan b = \sin a \tan B = \sin 160° \tan 80°$.
3. To find c from a, B: Rule I with co.B as middle part:
   $\cos B = \tan a \operatorname{ctn} c$, $\operatorname{ctn} c = \cos B \operatorname{ctn} a = -\cos 80° \operatorname{ctn} 20°$.

In part (1) above, $\cos a$ is $\cos 160°$ which equals $-\cos 20°$. Thus $\cos A$ must be negative and A must be obtuse ($A > 90°$). Similarly for $\operatorname{ctn} c$ and c in part (3).

The logarithmic work may be arranged compactly as follows. We omit writing -10 where the characteristic has the form $9 \ldots\ldots - 10$, and simply keep it in mind. Where working with a negative function such as $\cos 160°$ ($= -\cos 20°$), we write a small letter n after the logarithm to avoid overlooking the $-$ sign for the final function. Two n's in the same product would neutralize each other.

(1)		(2)		(3)	
$\cos a$	9.97299 n	$\sin a$	9.53405	$\cos B$	9.23967
$\sin B$	9.99335	$\tan B$	0.75368	$\operatorname{ctn} a$	0.43893 n
$\cos A$	9.96634 n	$\tan b$	0.28773	$\operatorname{ctn} c$	9.67860 n
$A = 180° - 22° \, 16'$		$b = 62° \, 43'.6$		$c = 180° - 64° \, 29'.7$	
$A = 157° \, 44'$				$c = 115° \, 30'.3$	

To check A, b, c: try Rule I with co.A as middle part:	$\tan b$	0.28773
	$\operatorname{ctn} c$	9.67860 n
$\cos A = \tan b \operatorname{ctn} c.$	$\cos A$	9.96633 n

These logarithms check closely enough. A further check on the parts themselves can be made by using one given part with two calculated parts.

EXERCISES

In Exercises 1–10 find the remaining parts of the right triangle (or triangles) having each following pair of given parts (besides $C = 90°$). Check in each case.

1. $A = 75° 12'$, $B = 52° 27'$.

2. $A = 168° 24'$, $B = 101° 6'$.

3. $a = 98° 12'$, $b = 109° 40'$.

4. $a = 81° 36'.5$, $b = 130°$.

5. $a = 62° 53'.4$, $c = 85° 14'.3$.

6. $b = 117° 12'$, $c = 78° 45'$.

7. $b = 71° 2'.1$, $A = 95° 14'.8$.

8. $b = 21° 8'$, $B = 33° 40'$.

9. $c = 69° 48'.4$, $A = 83° 55'$.

10. $c = 145° 8'$, $B = 100° 12'$.

11.–14. In Exercises 1–4, p. 12, carry out the computations needed and find each specified part.

15. Similarly find a, b, c in Exercise 5, p. 12, and check.

16. From a point A on the equator near the Galapagos Islands, in longitude 90° *W.*, a ship heads for a point B near Hawaii in latitude 20° *N.* and longitude 155° *W.* If the ship follows the great circle arc AB, find the distance it must go and the angle A northward from the equator at which it must start. A $1'$ arc = 1 nautical mile. (Hint: Denote by C the point where the meridian 155° *W.* meets the equator, 65° west of A. Then we have a right spherical triangle with $C = 90°$, $b = \text{arc } AC = 65°$, and $a = \text{arc } CB =$ latitude of $B = 20°$. We are to find c and A.)

17. Like Ex. 16 for a flight from a point A on the equator near Singapore in longitude 103° *E.* to B near Tokio in latitude 36° *N.* and longitude 140° *E.*

18. Similar to Ex. 16 for a voyage from a point A on the equator in longitude 0° to a point B off Buenos Aires in latitude 35° *S.* and longitude 55° *W.*

19. Similar to Ex. 16 for a flight from A on the equator near the Amazon River in longitude 50° *W.* to B near Capetown in latitude 34° *S.* and longitude 20° *E.*

SECTION THREE

Oblique Spherical Triangles

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§ 12. **Formulas Needed.** As was pointed out in § 6, any spherical triangle can be solved, rather inconveniently, by means of the three basic laws: the sine law; the cosine law for sides, and the cosine law for angles. It is preferable to replace the cosine laws by other formulas adapted to logarithmic methods. The new formulas, deducible from the cosine laws by methods to be explained in §§ 18–19, resemble the formulas used in the logarithmic solution of a plane triangle when three sides or two sides and their included angle are given.* There are four sets of formulas needed here, of two main types.

(I) *The half-angle formulas, in terms of the sides.* Let s denote the semi-perimeter, $s = \frac{1}{2}(a + b + c)$, and define a quantity r such that

$$r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \tag{23}$$

Then it is found that

$$\tan\tfrac{1}{2}A = \frac{r}{\sin(s-a)}, \qquad \tan\tfrac{1}{2}B = \frac{r}{\sin(s-b)}, \tag{24}$$

with a similar equation for $\tan\frac{1}{2}C$.

(II) *The half-side formulas, in terms of the angles.* Let S denote $\frac{1}{2}(A + B + C)$, and define R so that

$$R = \sqrt{\frac{\cos(S-A)\cos(S-B)\cos(S-C)}{-\cos S}}. \tag{25}$$

* See the author's *Introduction to Mathematical Analysis*, §§ 158–159, or any text on plane trigonometry.

(The fraction under this radical is positive, for $S > 90°$ and each difference, $S - A$, etc. is less than $90°$.) The half-sides are given by

$$\text{ctn}\ \tfrac{1}{2}\,a = \frac{R}{\cos\,(S - A)}, \qquad \text{ctn}\ \tfrac{1}{2}\,b = \frac{R}{\cos\,(S - B)}, \tag{26}$$

with a similar equation for ctn $\frac{1}{2}\,c$.

(III)–(IV). The remaining two sets are given in § 14.

§ 13. Solution of Oblique Triangles: Cases I, II. *Case I. Given the three sides: a, b, c.*

To find the angles A, B, C by logarithmic methods use the half-angle formulas (24) with (23).

Example I. Given $a = 50°$, $b = 80°$, $c = 110°$.

Here $s = \frac{1}{2}\,(50° + 80° + 110°) = 120°$, $s - a = 70°$, $s - b = 40°$, $s - c = 10°$.

$$r = \sqrt{\frac{\sin 70°\ \sin 40°\ \sin 10°}{\sin 120°}}; \quad \tan \tfrac{1}{2}\,A = \frac{r}{\sin 70°}, \text{ etc.} \tag{27}$$

The logarithmic work runs as follows. The sine law is used to check the angles obtained.

sin 70°	9.97299
sin 40°	9.80807
sin 10°	9.23967
	9.02073
sin 120°	9.93753
r^2	9.08320
r	9.54160

r	9.54160
sin 70°	9.97299
tan $\frac{1}{2}\,A$	9.56861

$\frac{1}{2}\,A = 20°\ 19'.3$
$A = 40°\ 38'.6$

r	9.54160
sin 40°	9.80807
tan $\frac{1}{2}\,B$	9.73353

$\frac{1}{2}\,B = 28°\ 25'.9$
$B = 56°\ 51'.8$

r	9.54160
sin 10°	9.23967
tan $\frac{1}{2}\,C$	0.30193

Check: $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$.

$\frac{1}{2}\,C = 63°\ 29'$,
$C = 126°\ 58'$

numerator	9.88425	9.99335	9.97299
denominator	9.81381	9.92291	9.90254
fraction	0.07044	0.07044	0.07045

Case II. Given the three angles: A, B, C.

To find the sides a, b, c use the half-side formulas (26) with

(25). The arrangement of the logarithmic work is substantially identical with that in Case I above.

If the angles are $A = 50°$, $B = 80°$, $C = 110°$, we have $S = 120°$, $S - A = 70°$, $S - B = 40°$, $S - C = 10°$. Since $\cos S$ is negative, the denominator, $-\cos S$, in (25) is positive and the solution is real.

EXERCISES

1. Find and check the angles of a spherical triangle whose sides are: $a = 107° \, 32'.2$, $b = 87° \, 40'.8$, $c = 96° \, 37'$.

2. Like Ex. 1 for the triangle: $a = 73° \, 15'$, $b = 118° \, 7'$, $c = 158° \, 0'$.

3. Find and check the sides of a spherical triangle whose angles are: $A = 112° \, 42'.5$, $B = 79° \, 23'.5$, $C = 92° \, 14'$.

4. Like Ex. 3 for the triangle: $A = 52° \, 11'$, $B = 106° \, 0'$, $C = 122° \, 39'$.

§ **14. Napier's Analogies.** Two other sets of new formulas needed in solving spherical triangles are the following, called Napier's Analogies, numbered below as *N1*, *N2*, etc.* They greatly resemble the law of tangents in plane trigonometry:

N 1:
$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \operatorname{ctn} \tfrac{1}{2} C, \tag{28}$$

N 2:
$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \operatorname{ctn} \tfrac{1}{2} C. \tag{29}$$

These formulas are useful when we know two sides and their included angle, a, b, C (§ 15). The other two are:

N 3:
$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c, \tag{30}$$

N 4:
$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c. \tag{31}$$

These are useful when we know two angles and their included side.

§ **15. Solution of Oblique Triangles: Cases III, IV.** *Case III. Given two sides and included angle, a, b, C.*

* As to the derivation of these formulas, see § 19.

By Napier's first two analogies, $N\,1$, $N\,2$, we can find $\frac{1}{2}(A+B)$, and $\frac{1}{2}(A-B)$, and hence A and B by adding and subtracting. The third side can then be found from another analogy, say $N\,4$, solved thus for $\tan\frac{1}{2}c$:

$$\tan\tfrac{1}{2}c = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}\tan\tfrac{1}{2}(a-b). \qquad (32)$$

The sine law can then be used as a check.

Example I. Given $a = 112°\,17'$, $b = 84°\,3'$, $C = 95°$. Here $a - b = 28°\,14'$, $a + b = 196°\,20'$. Thus, $N\,1$ and $N\,2$ give:

$$\tan\tfrac{1}{2}(A+B) = \frac{\cos 14°\;\;7'}{\cos 98°\,10'}\operatorname{ctn} 47°\,30',$$

$$\tan\tfrac{1}{2}(A-B) = \frac{\sin 14°\;\;7'}{\sin 98°\,10'}\operatorname{ctn} 47°\,30'.$$

The computation runs as follows:

cos 14° 7′	9.98668	sin 14° 7′	9.38721
cos 98° 10′	9.15245 n	sin 98° 10′	9.99557
	0.83423 n		9.39164
ctn 47° 30′	9.96205	ctn 47° 30′	9.96205
$\tan\frac{1}{2}(A+B)$	0.79628 n	$\tan\frac{1}{2}(A-B)$	9.35369

$\frac{1}{2}(A+B) = 180° - 80°\,55'.1$
$\quad = 99°\,4'.9$

$\frac{1}{2}(A+B) = 99°\,4'.9$
$\frac{1}{2}(A-B) = 12°\,43'.4$

$A = 111°\,48'.3$
$B = 86°\,21'.5$

sin 99° 4′.9	9.99452
sin 12° 43′.4	9.34290
	0.65162
tan 14° 7′	9.40052
$\tan\frac{1}{2}c$	0.05214

We next write (32):

$$\tan\tfrac{1}{2}c = \frac{\sin 99°\,4'.9}{\sin 12°\,43'.4}\tan 14°\,7'.$$

$\frac{1}{2}c = 48°\,25'.9$
$c = 96°\,51'.8$

Check

$$\frac{\sin 112°\,17'}{\sin 111°\,48'.3} = \frac{\sin 84°\,3'}{\sin 86°\,21'.5} = \frac{\sin 96°\,51'.8}{\sin 95°}(?)$$

numer.	9.96629	9.99765	9.99687
denom.	9.96776	9.99912	9.99834
fraction	9.99853	9.99853	9.99853

Case IV. Given two angles and included side, A, B, c.

The method is closely similar to that in Case III above. We use Napier's third and fourth analogies, *N 3*, *N 4*, to find $\frac{1}{2}(a+b)$, $\frac{1}{2}(a-b)$ and hence a and b. Then we find the third angle by using *N 2*, solved for ctn $\frac{1}{2} C$:

$$\operatorname{ctn} \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B). \qquad (33)$$

The sine law again affords a check.

EXERCISES

1. Find and check the remaining parts of the spherical triangles which have the following given parts:

(i) $a = 86° 31'$,	$b = 70° 15'$,	$C = 108° 54'$.
(ii) $a = 73° 46'$,	$c = 91° 8'$,	$C = 66° 14'$.
(iii) $c = 61° 12'.2$,	$A = 100° 22'.6$,	$B = 81° 6'.8$.
(iv) $b = 95° 0'$,	$A = 122° 14'$,	$C = 89° 2'$.

§ 16. Solution of Oblique Triangles: Cases V, VI. *Case V. Given two sides, angle opposite one: a, b, A.*

No further formulas are needed. We find angle B from the sine law, then use Napier's Analogies I and III to calculate C and c, and finally check by further use of the sine law.

In finding B from its sine we have the possibility of two angles, acute and obtuse. But we can determine whether both are admissible by applying the following theorem which can be deduced from the cosine law for sides (proof omitted here):

Theorem. *If* b *is closer to 90° than* a *is, there will be two solutions; but if* b *is farther from 90° than* a *is, there will be only one solution for* B, *viz. with* B *in the same quadrant as* b.

A corresponding theorem holds, with the letters interchanged, when we are given a, b, B, and are solving for A.

Example I. Given $a = 64° 52'$, $b = 124° 16'$, $A = 104° 27'$. Here b is farther from 90° than a is. Hence B can have only one value, — in the second quadrant.

$$\sin B = \frac{\sin b \sin A}{\sin a} = \frac{\sin 124° 16' \sin 104° 27'}{\sin 64° 52'}.$$

Using logarithms:

$$\log \sin B = 9.94644; \quad B = 180° - 62° \, 7'.6 = 117° \, 52'.4.$$

In the other formulas we need these values:

$$b + a = 189° \, 8', \qquad b - a = 59° \, 24',$$
$$B + A = 222° \, 19'.4, \qquad B - A = 13° \, 25'.4.$$

With $B > A$, Analogies I and III, give for ctn $\frac{1}{2} C$ and tan $\frac{1}{2} c$:

$$\operatorname{ctn} \tfrac{1}{2} C = \frac{\cos \frac{1}{2} (b + a)}{\cos \frac{1}{2} (b - a)} \tan \tfrac{1}{2} (B + A) = \frac{\cos 94° \, 34' \tan 111° \, 9'.7}{\cos 29° \, 42'},$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2} (B + A)}{\cos \frac{1}{2} (B - A)} \tan \tfrac{1}{2} (b + a) = \frac{\cos 111° \, 9'.7 \tan 94° \, 34'}{\cos 6° \, 42'.7}.$$

The further logarithmic work is shown below.

cos 94° 34′	8.90102 n	cos 111° 9′.7	9.55751 n
tan 111° 9′.7	0.41217 n	tan 94° 34′	1.09760 n
	9.31319		0.65511
cos 29° 42′	9.93884	cos 6° 42′.7	9.99701
ctn $\frac{1}{2} C$	9.37435	tan $\frac{1}{2} c$	0.65810

$\frac{1}{2} C = 76° \, 40'.7$ $\qquad$ $\frac{1}{2} c = 77° \, 36'.4$

$C = 153° \, 21'.4$ $\qquad$ $c = 155° \, 12'.8$

Check:

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

numerator	9.91720	9.62246
denominator	9.94644	9.65170
fraction	9.97076	9.97076

Case VI. Given two angles, side opposite one: A, B, a.

This case is solved in the same way as Case V. In fact, exactly the same formulas are used. The sine law is, however, solved for sin b instead of sin B:

$$\sin b = \sin B \sin a \div \sin A.$$

Again there may be two admissible triangles or only one. The question is settled by a theorem like that above, with capital letters and small letters interchanged.

Example II. Given $A = 62° \, 23'$, $B = 66° \, 31'$, $a = 116° \, 47'$. Here B is closer to 90° than A is: there are two possible triangles, one with $b < 90°$, one with $b > 90°$. For each of these values of b, say b_1 and b_2, we carry out a logarithmic calculation like that in Example I above.

From the sine law we obtain log sin $b = 9.96569 - 10$.

$$b_1 = 67° \, 31'.4 \qquad\qquad b_2 = 112° \, 28'.6.$$

We use these values with $a = 116° \, 47'$ to find $\frac{1}{2}(a + b)$ and $\frac{1}{2}(a - b)$ for each triangle:

In Triangle 1:	In Both:	In Triangle 2:
$\frac{1}{2}(a+b) = 92° \, 9'.2$	$\frac{1}{2}(A+B) = 64° \, 27'$	$\frac{1}{2}(a+b) = 114° \, 37'.8$
$\frac{1}{2}(a-b) = 24° \, 37'.8$	$\frac{1}{2}(A-B) = -2° \, 4'$	$\frac{1}{2}(a-b) = 2° \, 9'.2$

$$\operatorname{ctn} \tfrac{1}{2} C_1 = \frac{\cos 92° \, 9'.2}{\cos 24° \, 37'.8} \tan 64° \, 27';$$

$$\operatorname{ctn} \tfrac{1}{2} C_2 = \frac{\cos 114° \, 37'.8}{\cos 2° \, 9'.2} \tan 64° \, 27';$$

$$\tan \tfrac{1}{2} c_1 = \frac{\cos 64° \, 27'}{\cos (-2° \, 4')} \tan 92° \, 9'.2;$$

$$\tan \tfrac{1}{2} c_2 = \frac{\cos 64° \, 27'}{\cos (-2° \, 4')} \tan 114° \, 37'.8.$$

Using logarithms we find for the two triangles:

$\frac{1}{2} C_1 = 85° \, 3'.5$	$\frac{1}{2} c_1 = 85° \, 1'.25$	$\frac{1}{2} C_2 = 41° \, 6'.1$	$\frac{1}{2} c_2 = 43° \, 16'.2$
$C_1 = 170° \, 7'$	$c_1 = 170° \, 2'.5$	$C_2 = 82° \, 12'.2$	$c_2 = 86° \, 32'.4.$

The sine law for b, B, c, C affords a check for each triangle.

EXERCISES

In these triangles note at the outset whether one or two solutions are to be expected. Find and check the remaining parts in each admissible triangle.

1. $a = 78° \, 20'$, $b = 68° \, 12'$, $A = 84° \, 19'.6$.

2. $a = 76° \, 2'$, $b = 80° \, 7'$, $A = 100° \, 49'$.

3. $A = 102° \, 38'$, $B = 109° \, 6'$, $a = 53° \, 42'$.

4. $A = 70° \, 53'$, $B = 98° \, 13'$, $a = 122° \, 55'$.

5. $B = 95° \, 18'$, $C = 81° \, 44'$, $b = 148° \, 24'$.

§ 17. Concerning the Area of a Spherical Triangle. When the three angles are known, we may find the spherical excess,

$E = A + B + C - 180°$. Then, as stated in § 2, the area of the triangle can be found from the formula,

$$\frac{\text{area of triangle}}{\text{area of sphere}} = \frac{E}{720°}.$$

There are various other formulas for the area, but in such a brief course as this, we can get along with one formula.

§ 18. Derivation of the Half-angle and Half-side Formulas. Because of the brevity of this course we shall simply indicate the steps by which formulas (24) and (26) are derived from the cosine laws. For any angle A, whether in a triangle or not:

$$\sin^2 \tfrac{1}{2} A = \tfrac{1}{2}(1 - \cos A), \qquad \cos^2 \tfrac{1}{2} A = \tfrac{1}{2}(1 + \cos A). \quad \textbf{(34)}$$

For an angle of a spherical triangle the cosine law for sides gives

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \quad \textbf{(35)}$$

$$1 - \cos A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c} = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

The numerator of this last fraction equals $2 \sin \frac{1}{2}(b - c + a) \sin \frac{1}{2}(a + c - b)$. If we let $a + b + c = 2s$, then $a + c - b = 2s - 2b$, and $b - c + a = 2s - 2c$. Thus the numerator becomes $2 \sin (s - c) \sin (s - b)$; and

$$\sin^2 \tfrac{1}{2} A = \frac{\sin (s - c) \sin (s - b)}{\sin b \sin c}.$$

In like manner it is shown that

$$\cos^2 \tfrac{1}{2} A = \frac{\sin s \sin (s - a)}{\sin b \sin c}.$$

$$\tan^2 \tfrac{1}{2} A = \frac{\sin (s - c) \sin (s - b)}{\sin s \sin (s - a)}$$

$$= \frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s \sin^2 (s - a)}.$$

This last fraction, without the denominator factor $\sin^2(s-a)$, is r^2 as defined in (23). Thus we have the first of the formulas (24). Similarly for the others.

The half-side formulas (26) are derived in like manner, starting from the value of $\cos a$ as given by the cosine law for angles:

$$\cos a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}.$$

§ 19. Derivation of Napier's Analogies. Formulas (28)–(31) can be derived in the following manner. Starting with the half-angle formulas (24), we have, after minor simplifications:

$$\tan \tfrac{1}{2} A \tan \tfrac{1}{2} B = \frac{r^2}{\sin (s-a) \sin (s-b)} = \frac{\sin (s-c)}{\sin s}.$$

Adding this product $\tan \frac{1}{2} A \tan \frac{1}{2} B$ to 1, also subtracting the same product from 1, and then dividing the sum by the difference, and reducing:

$$\frac{1 + \tan \frac{1}{2} A \tan \frac{1}{2} B}{1 - \tan \frac{1}{2} A \tan \frac{1}{2} B} = \frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)}. \tag{36}$$

Multiplying numerator and denominator of the first fraction by $\cos \frac{1}{2} A \cos \frac{1}{2} B$ and using the "Addition Formulas" for the cosine, we simplify the first fraction in (36) to $\cos \frac{1}{2}(A-B) \div \cos \frac{1}{2}(A+B)$. Also the fraction on the right side of (36) simplifies to

$$\frac{2 \sin \frac{1}{2}(2s-c) \cos \frac{1}{2} c}{2 \cos \frac{1}{2}(2s-c) \sin \frac{1}{2} c} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2} c}.$$

Equating the simplified values gives:

$$\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2} c},$$

which is equivalent to formula (30).

The other Napier Analogies are derived in a like manner.

SECTION FOUR

Application to Navigation

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§ 20. Great Circle Track Between Ports. In planning a voyage or flight from a point A to a very distant point B, a navigator needs to lay out the great circle route, technically called the great circle "track" from A to B. He needs to know: (1) The distance AB to be traveled; (2) the direction in which to start from A; (3) the location of several points along the track; (4) the direction in which he should be traveling when passing through each of those points. This information can be found approximately from maps, or more accurately by spherical trigonometry.

To illustrate, suppose that A and B in Fig. 5 (page 26) represent certain points near San Francisco and Tokio, respectively:

Point	*Latitude*	*Longitude*
A	$37° 40'$ *N.*	$123° \ 0'$ *W.*
B	$34° 50'$ *N.*	$139° 50'$ *E.*

Then, if C represents the North Pole, $EFGH$ the equator, FBC the meridian through B, and GAC the meridian through A, we know:

$$\text{arc } GA = 37° 40', \qquad \text{arc } AC = 90° - 37° 40' = 52° 20';$$
$$\text{arc } FB = 34° 50', \qquad \text{arc } BC = 90° - 34° 50' = 55° 10'.$$

Also, since A and G are $123°$ *west* of the Greenwich meridian (not shown), while B and F are $139° 50'$ *east* of Greenwich, the equatorial arc $GHEF$ from G to F (by the long way around, crossing the Greenwich meridian) is $123° + 139° 50' = 262° 50'$. Hence the direct arc $GF = 360° - 262° 50' = 97° 10'$. This arc GF equals the angle C, at the pole, between the meridians of A and B.

Hence in the spherical triangle ABC we know two sides and their included angle:

$$a = 55° 10', \qquad b = 52° 20', \qquad C = 97° 10'.$$

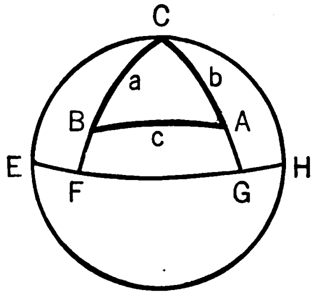

FIG. 5.

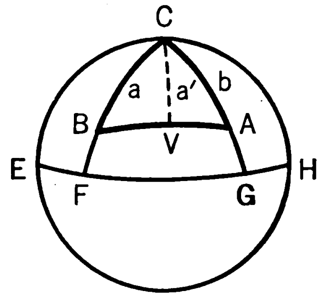

FIG. 6.

The required great circle distance AB is simply the third side of this triangle. The initial direction in which the navigator should leave A is determined by the angle A which the track AB makes with the meridian AC, *i.e.*, with the north direction at A.

The "course" of a ship at any point P is defined as the angle from the northward meridian at P (measured clockwise, *i.e.*, with east as 90°) to the forward track of the ship. Thus, in Fig. 5 if angle A were 58° the "course" would be $360° - A$, or 302°.

§ 21. Great Circle Distance and Initial Course. Let us find the distance AB and the initial course at A in the San Francisco–Tokio problem illustrated by Fig. 5.

We have given two sides and their included angle, as listed in (37) above. If we use the logarithmic method of § 15, Case III, we shall need the values, $\frac{1}{2}(a - b) = 1° 25'$, $\frac{1}{2}(a + b) = 53° 45'$, $\frac{1}{2} C = 48° 35'$. Substituting these values in (28), (29), and (31) gives:

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos 1° 25'}{\cos 53° 45'} \operatorname{ctn} 48° 35',$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin 1° 25'}{\sin 53° 45'} \operatorname{ctn} 48° 35',$$

$$\tan \tfrac{1}{2} c = \frac{\sin \tfrac{1}{2}(A + B)}{\sin \tfrac{1}{2}(A - B)} \tan 1° 25'.$$

Working out the values by logarithms gives (page 28, Ex. 1):

$$A = 57° 42'.4, \quad B = 54° 36'.5, \quad c = 74° 27'.3. \qquad \textbf{(38)}$$

Hence the initial course is $360° - A = 302° 17'.4$; and the great circle distance is $74° 27'.3$ or $4467'.3$. Since a $1'$ arc of a great circle is one nautical mile, the distance is 4467 miles, approximately.

Navigators more commonly find the distance by using a modification of the cosine law known as the "haversine formula" and then find angle A by using the sine law or the haversine formula again. (§ 23.)

§ 22. Positions Along the Track. To calculate several points along the track a navigator first finds the point V where the track is nearest the pole C. The meridian CV is perpendicular to the track (Fig. 6). Thus in triangle ACV for the preceding problem we have angle $AVC = 90°$, $b = 52° 20'$ (known originally), and angle $A = 57° 42'.4$ (found as explained in § 21). This right triangle can therefore be solved for a' (= arc CV) and for angle ACV. The latitude of V is then known, being $90° - a'$; and the longitude of V (west) is angle ACV plus 123°, the given longitude of A.

To solve the right triangle ACV for a' we use Napier's Rule II, choosing a' as the middle part, with co.b and co.A as opposite parts.*

$$\sin a' = \cos \text{co.}b \cos \text{co.}A; \qquad \sin a' = \sin 52° 20' \sin 57° 42'.4. \qquad \textbf{(39)}$$

This gives a'. To find angle ACV, denoted here by C', we choose co.b as the middle part, with co.A and co.C' as adjacent parts. By Rule I:

$$\sin \text{co.}b = \tan \text{co.}A \tan \text{co.}C'; \text{ctn } C' = \cos 52° 20' \tan 57° 42'.4. \qquad \textbf{(40)}$$

Using logarithms we find from (39) and (40):

$$a' = 42° 0', \qquad C' = 45° 57'.9. \qquad \textbf{(41)}$$

Thus the latitude of V is $90° - a' = 48° 0'$ $N.$, and the longitude is $123° + C' = 168° 57'.9$ $W.$

To find positions along the track, say at intervals of 10° of longitude, we could now consider other right triangles with CV as one leg and with the other leg along VA or VB. Each such right triangle would contain the known leg a' and a known angle of 10° or 20°, etc., as chosen at C. Hence each such triangle could be solved for its hypotenuse. Subtracting the latter from 90° would give the latitude of the point on the track whose

* The lettering is different here from that in Fig. 4. Here b is the hypotenuse; hence we have co.b.

longitude was virtually chosen in taking the 10° or 20° angle. Each such solution would give two points, one on VA and one on VB.

EXERCISES

1. Do the necessary logarithmic work and verify the values of A, B, and c listed in equation (38).

2. Similarly verify the values of a' and C' listed in (41).

3. Find the distance and the initial course for the great circle track from A to B in each following case:

(*a*) A (Lat., 40° *N.*; Long., 80° *W.*),
B (Lat., 50° *N.*; Long., 5° *E.*);
(*b*) A (Lat., 38° 20′ *N.*; Long., 65° *W.*),
B (Lat., 0° 0′; Long., 20° *W.*);
(*c*) A (Lat., 6° 0′ *S.*; Long., 35° 5′ *W.*),
B (Lat., 15° 0′ *N.*; Long., 17° 0′ *W.*)

[Part (*c*) gives the track from Natal to Dakar.]

4. (*a*) In Ex. 3 (*a*) find the point V farthest north on the track.
(*b*) Find a point D on this track, whose longitude is 10° west of that of V.

5. (*a*) In Ex. 3 (*c*) show that the point V farthest north lies far beyond B on the track extended. (*b*) Find a point on the track in longitude 30° *W*.

§ 23. The Haversine Formula. The quantity $\frac{1}{2}(1 - \cos A)$ is called the *haversine* of A, written *hav* A:

$$\textbf{hav A} = \tfrac{1}{2}\,(\mathbf{1 - \cos A}). \qquad \textbf{(42)}$$

Thus, hav $60° = \frac{1}{2}\left(1 - \frac{1}{2}\right) = .25000$; hav $90° = \frac{1}{2}(1 - 0) = .50000$; hav $120° = \frac{1}{2}\left(1 + \frac{1}{2}\right) = .75000$; etc. As A runs from 0° to 180°, hav A runs from 0 to 1, continually increasing.

Five-place tables of haversines and their logarithms are available, for angles usually running at intervals of 10′ from 0° to 180°. A common arrangement is shown here:

	0′		10′		....	50′	
	Value	Log	Value	Log	....	Value	Log
60°	.25000	.39794	.25126	.40012	....	.25632	.40897
61°	.25760	.41094	.25887	.41308	....	.26398	.42157

Here the characteristics of the logarithms are omitted; but they are evident in each case from the tabulated "Value."

The "haversine formula," which is much used in solving spherical triangles, is derived from the cosine law for sides,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \tag{43}$$

From (42), $\cos A = 1 - 2 \operatorname{hav} A$. Substituting this in (43), with a like expression for $\cos a$, we get

$$1 - 2 \operatorname{hav} a = \cos b \cos c + \sin b \sin c\ (1 - 2 \operatorname{hav} A). \tag{44}$$

The right member of (44) equals $(\cos b \cos c + \sin b \sin c) - 2 \sin b \sin c \operatorname{hav} A$, or $\cos (b - c) - 2 \sin b \sin c \operatorname{hav} A$. Replacing $\cos (b - c)$ by $1 - 2 \operatorname{hav} (b - c)$, we cancel the 1's and divide through by -2, getting

$$\mathbf{hav}\ \boldsymbol{a} = \mathbf{hav}\ (\boldsymbol{b} - \boldsymbol{c}) + \mathbf{sin}\ \boldsymbol{b}\ \mathbf{sin}\ \boldsymbol{c}\ \mathbf{hav}\ \boldsymbol{A}. \tag{45}$$

In (45) we may, if we wish, write $(c - b)$ in place of $(b - c)$. This is desirable in case $c > b$.

It is well to state in words the general law represented by (45): *The haversine of any side of a spherical triangle equals the haversine of the difference of the other two sides, plus the product of the sines of those sides by the haversine of their included angle.*

To illustrate the use of this "haversine law" in a navigation problem let us calculate again the great-circle distance in the San Francisco–Tokio problem of §§ 20–21. As in (37):

$$a = 55^\circ\ 10', \qquad b = 52^\circ\ 20', \qquad C = 97^\circ\ 10',$$

and we want c, the third side of the triangle. The formula, like (45) but starting with c is:

$$\operatorname{hav} c = \operatorname{hav} (a - b) + \sin a \sin b \operatorname{hav} C. \tag{46}$$

i.e., $\operatorname{hav} c = \operatorname{hav} 2^\circ\ 50' + \sin 55^\circ\ 10' \sin 52^\circ\ 20' \operatorname{hav} 97^\circ\ 10'$.

We calculate the product term by logarithms and add the resulting *value* to the tabulated *value* of hav $2^\circ\ 50'$ to get the *value* of hav c. Then we look up c.

Value	Log
sin 55° 10′	9.91425
sin 52° 20′	9.89849
hav 97° 10′	9.75003
.36540 ⟵	9.56277

product	= .36540
hav 2° 50′	= .00061
hav c	= .36601
c	= 74° 27′.36
c	= 4467′ +.

The distance is about 4467 nautical miles, as found before.

The initial course, $360° - A$, can now be found by using this value of c in the haversine law, starting with a as in (45). Since $c - b = 22° 7'.36$,
$\text{hav } 55° 10' = \text{hav } 22° 7'.36 + \sin 52° 20' \sin 74° 27'.36 \text{ hav } A.$
The table gives hav $55° 10' = .21440$, hav $22° 7'.36 = .03681$. Transposing, and solving for hav A:

$$\text{hav } A = \frac{.17759}{\sin 52° 20' \sin 74° 27'.36}.$$

Looking up logarithms we get log. hav $A = 9.36711 - 10$, whence

$$A = 57° 42'.4 \qquad \text{(as in § 21).}$$

EXERCISES

1. (a–c) Use the haversine law to solve Ex. 3 (a–c), p. 28.

2. Similarly find the great circle distance and initial course from Portland, Ore., to Berlin, given these positions: P(45° 31′ 0″ N; 122° 40′ 39″ W), B(52° 31′ 31″ N; 13° 12′ 51″ E).

3. Likewise find the distance and initial course for each following flight or voyage:

(a) From Miami (25° 46′ N; 80° 11′ W)
to Lisbon (38° 42′ N; 9° 11′ W);
(b) From New York (40° 46′ N; 73° 52′ W)
to Los Angeles (33° 57′ N; 118° 22′ W);
(c) From Washington (38° 52′ N; 77° 3′ W)
to Mexico (19° 26′ N; 99° 7′ W);
(d) From Chicago (41° 50′ N; 87° 49′ W)
to Fairbanks (64° 51′ N; 147° 44′ W);
(e) From Rio de Janeiro (22° 54′ S; 43° 10′ W)
to Panama (8° 57′ N; 79° 32′ W);
(f) From St. Johns, N.F. (47° 34′ N; 52° 41′ W)
to Gibraltar (36° 6′ N; 5° 21′ W);
(g) From Honolulu (21° 18′ N; 157° 52′ W)
to Singapore (1° 17′ N; 103° 51′ E).

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