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MATHEMATICS

A LECTURE DELIVERED AT COLUMBIA UNIVERSITY
IN THE SERIES ON SCIENCE, PHILOSOPHY AND ART
OCTOBER 16, 1907

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MATHEMATICS

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New York
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1907

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MATHEMATICS

In the early part of the last century a philosophic French mathematician, addressing himself to the question of the perfectibility of scientific doctrines, expressed the opinion that one may not imagine the last word has been said of a given theory so long as it can not by a brief explanation be made clear to the man of the street. Doubtless that conception of doctrinal perfectibility, taken literally, can never be realized. For doubtless, just as there exist now, so in the future there will abound, even in greater and greater variety and on a vaster and vaster scale, deep-laid and high-towering scientific doctrines that, in respect to their infinitude of detail and in their remoter parts and more recondite structure, shall not be intelligible to any but such as concentrate their life upon them. And so the noble dream of Gergonne can never literally come true. Nevertheless, as an ideal, as a goal of aspiration, it is of the highest value, and, though in no case can it be quite attained, it yet admits in many, as I believe, of a surprisingly high degree of approximation. I do not mind frankly owning that I do not share in the feeling of those, if there be any such, who regard their special subjects as so intricate, mysterious and high, that in all their sublimer parts they are absolutely inaccessible to the profane man of merely general culture even when he is led by the hand of an expert and condescending guide. For scientific theories are, each and all of them, and they will continue to be, built upon and about

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notions which, however sublimated, are nevertheless derived from common sense. These etherealized central concepts, together with their manifold bearings on the higher interests of life and general thought, can be measurably assimilated to the language of the common level from which they arose. And, in passing, I should like to express the hope that here at Columbia there may one day be established a magazine that shall have for its aim to mediate, by the help, if it may be found, of such pens as those of Huxley and Clifford, between the focal concepts and the larger aspects of the technical doctrines of the specialist, on the one hand, and the teeming curiosity, the great listening, waiting, eager, hungering consciousness of the educated thinking public on the other. Such a service, however, is not to be lightly undertaken. An hour, at all events, is hardly time enough in which to conduct an excursion even of scientific folk through the mazes of more than twenty hundred years of mathematical thought or even to express intelligibly, if one were competent, the significance of the whole in a critical estimate.

Indeed, such is the character of mathematics in its profounder depths and in its higher and remoter zones that it is well nigh impossible to convey to one who has not devoted years to its exploration a just impression of the scope and magnitude of the existing body of the science. An imagination formed by other disciplines and accustomed to the interests of another field may scarcely receive suddenly an apocalyptic vision of that infinite interior world. But how amazing and how edifying were such a revelation, if only it could be made. To tell the story of mathematics from Pythagoras and Plato to Hilbert and Lie and Poincaré; to recount and appraise the achievements of such as Euclid and Archimedes, Apollonius and Diophantus; to display and estimate the creations of Descartes and Leibniz and Newton; to dispose in genetic or-

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der, to analyze, to synthesize and evaluate, the discoveries of the Bernoullis and Euler, of Desargues and Pascal and Monge and Poncelet, of Steiner and Möbius and Plücker and Staudt, of Lobatschewsky and Bolyai, of W. R. Hamilton and Grassmann, of Laplace, Lagrange and Gauss, of Boole and Cayley and Hermite and Gordan, of Bolzano and Cauchy, of Riemann and Weierstrass, of Georg Cantor and Boltzmann and Klein, of the Peirces and Schröder and Peano, of Helmholtz and Maxwell and Gibbs; to explore, and then to map for perspective beholding and contemplation, the continent of doctrine built up by these immortals, to say nothing of the countless refinements, extensions and elaborations meanwhile wrought by the genius and industry of a thousand other agents of the mathetic spirit;—to do that would indeed be to render an exceeding service to the higher intelligence of the world, but a service that would require the conjoint labors of a council of scholars for the space of many years. Even the three immense volumes of Moritz Cantor's *Geschichte der Mathematik,* though they do not aspire to the higher forms of elaborate exposition and though they are far from exhausting the material of the period traversed by them, yet conduct the narrative down only to 1758. That date, however, but marks the time when mathematics, then schooled for over a hundred eventful years in the unfolding wonders of Analytic Geometry and the Calculus and rejoicing in the possession of these the two most powerful among the instruments of human thought, had but fairly entered upon her modern career. And so fruitful have been the intervening years, so swift the march along the myriad tracks of modern analysis and geometry, so abounding and bold and fertile withal has been the creative genius of the time, that to record even briefly the discoveries and the creations since the closing date of Cantor's work would require an addition to his great volumes of a score of volumes more.

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Indeed the modern developments of mathematics constitute not only one of the most impressive, but one of the most characteristic, phenomena of our age. It is a phenomenon, however, of which the boasted intelligence of a "universalized" daily press seems strangely unaware; and there is no other great human interest, whether of science or of art, regarding which the mind of the educated public is permitted to hold so many fallacious opinions and inferior estimates. The golden age of mathematics—that was not the age of Euclid, it is ours. Ours is the age in which no less than six international congresses of mathematics have been held in the course of nine years. It is in our day that more than a dozen mathematical societies contain a growing membership of over two thousand men representing the centres of scientific light throughout the great culture nations of the world. It is in our time that over five hundred scientific journals are each devoted in part, while more than two score others are devoted exclusively, to the publication of mathematics. It is in our time that the *Jahrbuch über die Fortschritte der Mathematik,* though admitting only condensed abstracts with titles, and not reporting on all the journals, has, nevertheless, grown to nearly forty huge volumes in as many years. It is in our time that as many as two thousand books and memoirs drop from the mathematical press of the world in a single year, the estimated number mounting up to fifty thousand in the last generation. Finally, to adduce yet another evidence of similar kind, it requires no less than the seven ponderous tomes of the forthcoming *Encyklopädie der Mathematischen Wissenschaften* to contain, not expositions, not demonstrations, but merely compact reports and bibliographic notices sketching developments that have taken place since the beginning of the nineteenth century. The Elements of Euclid is as small a part of mathematics as the Iliad is of literature; or as the sculpture of Phidias is of the

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world's total art. Indeed if Euclid or even Descartes were to return to the abode of living men and repair to a university to resume pursuit of his favorite study, it is evident that, making due allowance for his genius and his fame, and presupposing familiarity with the modern scientific languages, he would yet be required to devote at least a year to preparation before being qualified even to begin a single strictly graduate course.

It is not, however, by such comparisons nor by statistical methods nor by any external sign whatever, but only by continued dwelling within the subtle radiance of the discipline itself, that one at length may catch the spirit and learn to estimate the abounding life of modern mathesis: oldest of the sciences, yet flourishing to-day as never before, not merely as a giant tree throwing out and aloft myriad branching arms in the upper regions of clearer light and plunging deeper and deeper root in the darker soil beneath, but rather as an immense mighty forest of such oaks, which, however, literally grow into each other so that by the junction and intercrescence of limb with limb and root with root and trunk with trunk the manifold wood becomes a single living organic growing whole.

What is this thing so marvelously vital? What does it undertake? What is its motive? What its significance? How is it related to other modes and forms and interests of the human spirit?

What is mathematics? I inquire, not about the word, but about the thing. Many have been the answers of former years, but none has approved itself as final. All of them, by nature belonging to the "literature of knowledge," have fallen under its law and "perished by supersession." Naturally conception of the science has had to grow with the growth of the science itself.

A traditional conception, still current everywhere except in critical circles, has held mathematics to be the science of

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quantity or magnitude, where magnitude including multitude (with its correlate of number) as a special kind, signified whatever was "capable of increase and decrease and measurement." Measurability was the essential thing. That definition of the science was a very natural one, for magnitude did appear to be a singularly fundamental notion, not only inviting but demanding consideration at every stage and turn of life. The necessity of finding out how many and how much was the mother of counting and measurement, and mathematics, first from necessity and then from pure curiosity and joy, so occupied itself with these things that they came to seem its whole employment.

Nevertheless, numerous great events of a hundred years have been absolutely decisive against that view. For one thing, the notion of *continuum*—the "Grand Continuum" as Sylvester called it—that great central supporting pillar of modern Analysis, has been constructed by Weierstrass, Dedekind, Georg Cantor and others, without any reference whatever to quantity, so that number and magnitude are not only independent, they are essentially disparate. When we attempt to correlate the two, the ordinary concept of measurement as the repeated application of a constant finite unit, undergoes such refinement and generalization through the notion of Limit or its equivalent that counting no longer avails and measurement retains scarcely a vestige of its original meaning. And when we add the further consideration that non-Euclidian geometry employs a scale in which the unit of angle and distance, though it is a constant unit, nevertheless appears from the Euclidian point of view to suffer lawful change from step to step of its application, it is seen that to retain the old words and call mathematics the science of quantity or magnitude, and measurement, is quite inept as no longer telling either what the science has actually become or what its spirit is bent upon.

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Moreover, the most striking measurements, as of the volume of a planet, the growth of cells, the valency of atoms, rates of chemical change, the swiftness of thought, the penetrative power of radium emanations, are none of them done by *direct* repeated application of a unit or by any direct method whatever. They are all of them accomplished by one form or another of indirection. It was perception of this fact that led the famous philosopher and respectable mathematician, Auguste Comte, to define mathematics as "the science of indirect measurement." Here doubtless we are in presence of a finer insight and a larger view, but the thought is not yet either wide enough or deep enough. For it is obvious that there is an immense deal of admittedly mathematical activity that is not in the least concerned with measurement whether direct or indirect. Consider, for example, that splendid creation of the nineteenth century known as Projective Geometry: a boundless domain of countless fields where reals and imaginaries, finites and infinites, enter on equal terms, where the spirit delights in the artistic balance and symmetric interplay of a kind of conceptual and logical counterpoint,—an enchanted realm where thought is double and flows throughout in parallel streams. Here there is no essential concern with number or quantity or magnitude, and metric considerations are entirely absent or completely subordinate. The fact, to take a simplest example, that two points determine a line uniquely, or that the intersection of a sphere and a plane is a circle, or that any configuration whatever—the reference is here to ordinary space—presents two reciprocal aspects according as it is viewed as an ensemble of points or as a manifold of planes, is not a *metric* fact at all: it is not a fact about size or quantity or magnitude of any kind. In this domain it was *position* rather than size that seemed to some the central matter, and so it was proposed to call mathematics the science of measurement and *position.*

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Even as thus expanded, the conception yet excludes many a mathematical realm of vast extent. Consider that immense class of things known as Operations. These are limitless alike in number and in kind. Now it so happens that there are many systems of operations such that any two operations of a given system, if thought as following one another, together thus produce the same effect as some other single operation of the system. Such systems are infinitely numerous and present themselves on every hand. For a simple illustration, think of the totality of possible straight motions in space. The operation of going from point A to point B, followed by the operation of going from B to point C, is equivalent to the single operation of going straight from A to C. Thus the system of such operations is a closed system: combination, *i.e.*, of any two of the operations yields a third one, not without, but within, the system. The great notion of Group, thus simply exemplified, though it had barely emerged into consciousness a hundred years ago, has meanwhile become a concept of fundamental importance and prodigious fertility, not only affording the basis of an imposing doctrine—the Theory of Groups—but therewith serving also as a bond of union, a kind of connective tissue, or rather as an immense cerebro-spinal system, uniting together a large number of widely dissimilar doctrines as organs of a single body. But—and this is the point to be noted here—the abstract operations of a group, though they are very real things, are neither magnitudes nor positions.

This way of trying to come at an adequate conception of mathematics, namely, by attempting to characterize in succession its distinct domains, or its varieties of content, or its modes of activity, in the hope of finding a common definitive mark, is not likely to prove successful. For it demands an exhaustive enumeration, not only of the fields now occupied by the science, but also of those destined to

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be conquered by it in the future, and such an achievement would require a prevision that none may claim.

Fortunately there are other paths of approach that seem more promising. Everyone has observed that mathematics, whatever it may be, possesses a certain mark, namely, a degree of certainty not found elsewhere. So it is, proverbially, the exact science par excellence. Exact, no doubt, but in what sense? An excellent answer is found in a definition given about one generation ago by a distinguished American mathematician, Professor Benjamin Peirce: "Mathematics is the science which draws necessary conclusions,"—a formulation of like significance with the following fine *mot* by Professor William Benjamin Smith: "Mathematics is the universal art apodictic." These statements, though neither of them is adequate, are both of them telling approximations, at once foreshadowing and neatly summarising for popular use, the epoch-making thesis established by the creators of modern logic, namely, that mathematics is included in, and, in a profound sense, may be said to be identical with, Symbolic Logic. Observe that the emphasis falls on the quality of being "necessary," *i.e.,* correct logically, or valid formally.

But why are mathematical conclusions correct? Is it that the mathematician has a reasoning faculty essentially different in kind from that of other men? By no means. What, then, is the secret? Reflect that conclusion implies premises, that premises involve terms, that terms stand for ideas or concepts or notions, and that these latter are the ultimate material with which the spiritual architect, called the Reason, designs and builds. Here, then, one may expect to find light. The apodictic quality of mathematical thought, the correctness of its conclusions as conclusions, are due, not to any special mode of ratiocination, but to the character of the concepts with which it deals. What is that distinctive characteristic? The answer is: precision

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and completeness of determination. But how comes the mathematician by such completeness? There is no mysterious trick involved: some concepts admit of such precision and completeness, others do not; the mathematician is one who deals with those that do.

The matter, however, is not quite so simple as it sounds, and I bespeak your attention to a word of caution and of further explanation. The ancient maxim, *ex nihilo nihil fit,* may well be doubted where it seems most obviously valid, namely, in the realm of matter, for it may be that matter has evolved from something else; but the maxim cannot be ultimately denied where its application is least obvious, namely, in the realm of mind, for without principia in the strictest sense, doctrine is, in the strictest sense, impossible. And when the mathematician speaks of complete determination of concepts and of rigor of demonstration, he does not mean that the undefined and the undemonstrated have been or can be entirely eliminated from the foundations of his science. He knows that such elimination is impossible; he knows, too, that it is unnecessary, for some undefinable ideas are perfectly clear and some undemonstrable propositions are perfectly precise and certain. It is in terms of such concepts that a definable notion, if it is to be mathematically available, must admit of complete determination, and in terms of such propositions that mathematical discourse secures its rigor. It is, then, of such indefinables among ideas and such indemonstrables among propositions—paradoxical as the statement may appear—that the foundations of mathematics in its ideal conception are composed; and whatever doctrine is logically constructible on such a basis is mathematics either actually or potentially. I am not asserting that the substructure herewith characterized has been brought to completion. It is on the conception of it that the accent is here designed to fall, for it is the conception as such that at once affords

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to fundamental investigation a goal and a guide and furnishes the means of giving the science an adequate definition.

On the other hand, actually to realize the conception requires that the foundation to be established shall both include every element that is essential and exclude every one that is not. For a foundation that subsequently demands or allows superfoetation of hypotheses is incomplete; and one that contains the non-essential is imperfect. Of the two problems thus presented, it is the latter, the problem of exclusion, of reducing principles to a minimum, of applying Occam's Razor to the pruning away of non-essentials,—it is that problem that taxes most severely both the analytic and the constructive powers of criticism. And it is to the solution of that problem that the same critical spirit of our time, which in other fields is reconstructing theology, burning out the dross from philosophy, and working relentless transformations of thought on every hand, has directed a chief movement of modern mathematics.

Apart from its technical importance, which can scarcely be overestimated, the power, depth and comprehensiveness of the modern critical movement in mathematics, make it one of the most significant scientific phenomena of the last century. Double in respect to origin, the movement itself has been composite. One component began at the very centre of mathematical activity, while the other took its rise in what was then erroneously regarded as an alien domain, the great domain of symbolic logic.

A word as to the former component. For more than a hundred years after the inventions of Analytical Geometry and the Calculus, mathematicians may be said to have fairly rioted in applications of these instruments to physical, mechanical and geometric problems, without concerning themselves about the nicer questions of fundamental

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principles, cogency, and precision. In the latter part of the eighteenth century the efforts of Euler, Lacroix and others to systematize results served to reveal in a startling way the necessity of improving foundations. Constructive work was not indeed arrested by that disclosure. On the contrary new doctrines continued to rise and old ones to expand and flourish. But a new spirit had begun to manifest itself. The science became increasingly critical as its towering edifices more and more challenged attention to their foundations. Manifest already in the work of Gauss and Lagrange, the new tendence, under the powerful impulse and leadership of Cauchy, rapidly develops into a momentous movement. The Calculus, while its instrumental efficacy is meanwhile marvelously improved, is itself advanced from the level of a tool to the rank and dignity of a science. The doctrines of the real and of the complex variable are grounded with infinite patience and care, so that, owing chiefly to the critical constructive genius of Weierstrass and his school, that stateliest of all the pure creations of the human intellect—the Modern Theory of Functions with its manifold branches—rests today on a basis not less certain and not less enduring than the very integers with which we count. The movement still sweeps on, not only extending to all the cardinal divisions of Analysis but, through the agencies of such as Lobatschewsky and Bolyai, Grassmann and Riemann, Cayley and Klein, Hilbert and Lie, recasting the foundations of Geometry also. And there can scarcely be a doubt that the great domains of Mechanics and Mathematical Physics are by their need destined to a like invasion.

In the light of all this criticism, mathematics came to appear as a great ensemble of theories, compendent no doubt, interpenetrating each other in a wondrous way, yet all of them distinct, each built up by logical processes on its own appropriate basis of pure hypotheses, or assumptions,

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or postulates. As all the theories were thus seen to rest equally on hypothetical foundations, all were seen to be equally legitimate; and doctrines like those of Quaternions, non-Euclidian geometry and Hyperspace, for a time suspected because based on postulates not all of them traditional, speedily overcame their heretical reputations and were admitted to the circle of the lawful and orthodox.

It is one thing, however, to deal with the principal divisions of mathematics severally, underpinning each with a foundation of its own. That, broadly speaking, has been the plan and the effect of the critical movement as thus far sketched. But it is a very different and a profounder thing to underlay all the divisions at once with a single foundation, with a foundation that shall serve as a support, not merely for all the *divisions* but for something else, distinct from each and from the sum of all, namely, for the *whole,* the science itself, which they constitute. It is nothing less than that achievement which, unconsciously at first, consciously at last, has been the aim and goal of the other component of the critical movement, that component which, as already said, found its origin and its initial interest in the field of symbolic logic. The advantage of employing symbols in the investigation and exposition of the formal laws of thought is not a recent discovery. As everyone knows, symbols were thus employed to a small extent by the Stagirite himself. The advantage, however, was not pursued; because for two thousand years the eyes of logicians were blinded by the blazing genius of the "master of those that know." With the single exception of the reign of Euclid, the annals of science afford no match for the tyranny that has been exercised by the logic of Aristotle. Even the important logical researches of Leibniz and Lambert and their daring use of symbolical methods were powerless to break the spell. It was not till 1854 when George Boole, having invented an algebra to trace and illuminate the

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subtle ways of reason, published his symbolical "Investigation of the Laws of Thought," that the revolution in logic really began. For, although for a time neglected by logicians and mathematicians alike, it was Boole's work that inspired and inaugurated the scientific movement now known and honored throughout the world under the name of Symbolic Logic.

It is true, the revolution has advanced in silence. The discoveries and creations of Boole's successors, of C. S. Peirce, of Schröder, of Peano and of their disciples and peers, have not been proclaimed by the daily press. Commerce and politics, gossip and sport, accident and crime, the shallow and transitory affairs of the exoteric world,—these have filled the columns and left no room to publish abroad the deep and abiding things achieved in the silence of cloistral thought. The demonstration by symbolical means of the fact that the three laws of Identity, Excluded Middle and Non-contradiction are absolutely independent, none of them being derivable from the other two; the discovery that the syllogism is not deducible from those laws but has to be postulated as an independent principle; the discovery of the astounding and significant fact that false propositions imply all propositions and that true ones, though not implying, are implied by, all; the discovery that most reasoning is not syllogistic, but is asyllogistic, in form, and that, therefore, contrary to the teaching of tradition, the class-logic of Aristotle is not adequate to all the concerns of rigorous thought; the discovery that Relations, no less than Classes, demand a logic of their own, and that a similar claim is valid in the case of Propositions: no intelligence of these events nor of the immense multitude of others which they but meagrely serve to hint and to exemplify, has been cabled round the world and spread broadcast by the flying bulletins of news. Even the scientific public, for the most part accustomed to viewing the

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mind as only the instrument and not as a subject of study, has been slow to recognize the achievements of modern research in the minute anatomy of thought. Indeed it has been not uncommon for students of natural science to sneer at logic as a stale and profitless pursuit, as the barren mistress of scholastic minds. These men have not been aware of what certainly is a most profound, if indeed it be not the most significant, scientific movement of our time. In America, in England, in Germany, in France, and especially in Italy—supreme histologist of the human understanding—the deeps of mind and logical reality have been explored in our generation as never before in the history of the world. Owing to the power of the symbolic method, not only the foundations of the Aristotelian logic -the Calculus of Classes—have been recast, but side by side with that everlasting monument of Greek genius, there rise today two other structures, fit companions of the ancient edifice, namely, the Logic of Relations and the Logic of Propositions.

And what are the entities that have been found to constitute the base of that triune organon? The answer is surprising: a score or so of primitive, indemonstrable, propositions together with less than a dozen undefinable notions, called logical constants. But what is more surprising—for here we touch the goal and are enabled to enunciate what has been justly called "one of the greatest discoveries of our age"—is the fact that the basis of logic is the basis of mathematics also. Thus the two great components of the critical movement, though distinct in origin and following separate paths, are found to converge at last in the thesis: Symbolic Logic is Mathematics, Mathematics is Symbolic Logic, the twain are one.

Is it really so? Does the identity exist in fact? Is it true that so simple a unifying foundation for what has hitherto been supposed two distinct and even mutually

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alien interests has been actually ascertained? The basal masonry is indeed not yet completed but the work has advanced so far that the thesis stated is beyond dispute or reasonable doubt. Primitive propositions appear to allow some freedom of choice, questions still exist regarding relative fundamentality, and statements of principles have not yet crystallized into settled and final form; but regarding the nature of the data to be assumed, the smallness of their number and their adequacy, agreement is substantial. In England, Russell and Whitehead are successfully engaged now in forging "chains of deduction" binding the cardinal matters of Analysis and Geometry to the premises of General Logic, while in Italy the *Formulaire de Mathématiques* of Peano and his school has been for some years growing into a veritable encyclopedia of mathematics wrought by the means and clad in the garb of symbolic logic.

But is it not incredible that the concept of number with all its distinctions of cardinal and ordinal, fractional and whole, rational and irrational, algebraic and transcendental, real and complex, finite and infinite, and the concept of geometric space, in all its varieties of form and dimensionality, is it not incredible that mathematical ideas, surpassing in multitude the sands of the sea, should be precisely definable, each and all of them, in terms of a few logical constants, in terms, *i.e.,* of such indefinable notions as *such that, implication, denoting, relation, class, propositional function,* and two or three others? And is it not incredible that by means of so few as a score of premises (composed of ten principles of deduction and ten other indemonstrable propositions of a general logical nature), the entire body of mathematical doctrine can be strictly and formally deduced?

It is wonderful, indeed, but not incredible. Not incredible in a world where the mustard seed becometh a tree, not

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incredible in a world where all the tints and hues of sea and land and sky are derived from three primary colors, where the harmonies and the melodies of music proceed from notes that are all of them but so many specifications of four generic marks, and where three concepts—energy, mass, motion, or mass, time, space—apparently suffice for grasping together in organic unity the mechanical phenomena of a universe.

But the thesis granted, does it not but serve to justify the cardinal contentions of the depreciators of mathematics? Does it not follow from it that the science is only a logical grind, suited only to narrow and straitened intellects content to tramp in treadmill fashion the weary rounds of deduction? Does it not follow that Schopenhauer was right in regarding mathematics as the lowest form of mental activity, and that he and our own genial and enlightened countryman, Oliver Wendell Holmes, were right in likening mathematical thought to the operations of a calculating machine? Does it not follow that Huxley's characterization of mathematics as "that study which knows nothing of observation, nothing of induction, nothing of experiment, nothing of causation," is surprisingly confirmed by fact? Does it not follow that Sir William Hamilton's famous and terrific diatribe against the science finds ample warrant in truth? Does it not follow, as the Scotch philosopher maintains, that mathematics regarded as a discipline, as a builder of mind, is inferior? That devotion to it is fatal to the development of the sensibilities and the imagination? That continued pursuit of the study leaves the mind narrow and dry, meagre and lean, disqualifying it both for practical affairs and for those large and liberal studies where moral questions intervene and judgment depends, not on nice calculation by rule, but on a wide survey and a balancing of probabilities?

The answer is, no. Those things not only do not follow

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but they are not true. Every count in the indictment, whether explicit or only implied, is false. Not only that, but the opposite in each case is true. On that point there can be no doubt; authority, reason and fact, history and theory, are here in perfect accord. Let me say once for all that I am conscious of no desire to exaggerate the virtues of mathematics. I am willing to admit that mathematicians do constitute an important part of the salt of the earth. But the science is no catholicon for mental disease. There is in it no power for transforming mediocrity into genius. It cannot enrich where nature has impoverished. It makes no pretense of creating faculty where none exists, of opening springs in desert minds. *"Du bist am Ende—was du bist."* The great mathematician, like the great poet or great naturalist or great administrator, is born. My contention shall be that where the mathetic endowment is found, there will usually be found associated with it, as essential implications in it, other endowments in generous measure, and that the appeal of the science is to the whole mind, direct no doubt to the central powers of thought, but indirectly through sympathy of all, rousing, enlarging, developing, emancipating all, so that the faculties of will, of intellect and feeling learn to respond, each in its appropriate order and degree, like the parts of an orchestra to the "urge and ardor" of its leader and lord.

As for Hamilton and Schopenhauer, those detractors need not detain us long. Indeed but for their fame and the great influence their opinions have exercised over "the ignorant mass of educated men," they ought not in this connection to be noticed at all. Of the subject on which they presumed to pronounce authoritative judgment of condemnation, they were both of them ignorant, the former well nigh proudly so, the latter unawares, but both of them, in view of their pretensions, disgracefully ignorant. Lack of knowledge, however, is but a venial sin, and English-

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speaking mathematicians have been disposed to hope that Hamilton might be saved in accordance with the good old catholic doctrine of invincible ignorance. But even that hope, as we shall see, must be relinquished. In 1835 William Whewell, then fellow and tutor of Trinity College, Cambridge, published an appreciative pamphlet entitled "Thoughts on the Study of Mathematics as a Part of a Liberal Education." The author was a brilliant scholar. "Science was his forte," but "omniscience his foible," and his reputation for universal knowledge was looming large. That reputation, however, Hamilton regarded as his own prerogative. None might dispute the claim, much less share the glory of having it acknowledged on his own behalf. Whewell must be crushed. In the following year Sir William replies in the Edinburg Review, and such a show of learning! The reader is apparently confronted with the assembled opinions of the learned world, and—what is more amazing—they all agree. Literati of every kind, of all nations and every tongue, orators, philosophers, educators, scientific men, ancient and modern, known and unknown, all are made to support Hamilton's claim, and even the most celebrated mathematicians seem eager to declare that the study of mathematics is unworthy of genius and injures the mind. Whewell was overwhelmed, reduced to silence. His promised rejoinder failed to appear. The Scotchman's victory was complete, his fame enhanced, and his alleged judgment regarding a great human interest of which he was ignorant has reigned over the minds of thousands of men who have been either willing or constrained to depend on borrowed estimates. But even all this may be condoned. Jealousy, vanity, parade of learning, may be pardoned even in a philosopher. Hamilton's deadly sin was none of these, it was sinning against the light. In October, 1877, A. T. Bledsoe, then editor of the Southern Review—unfortunately too little

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known—published an article in that journal in which he proved beyond a reasonable doubt—I have been at the pains to verify the proof—that Hamilton by studied selections and omissions deliberately and maliciously misrepresented the great authors from whom he quoted —d'Alembert, Blaise Pascal, Descartes and others— distorting their express and unmistakable meaning even to the extent of complete inversion. This same verdict regarding Hamilton's vandalism, in so far as it relates to the works of Descartes, was independently reached by Professor Pringsheim and in 1904 announced by him in his *Festrede* before the Munich Academy of Sciences. As for Schopenhauer, I regret to say that a similar charge and finding stand against him also. For not only did he endorse without examination and re-utter Hamilton's tirade in the strongest terms, thus reinforcing it and giving it currency on the continent, but, as Pringsheim has shown, the German philosopher, by careful excision from the writings of Lichtenberg, converts that distinguished physicist's just strictures on the then flourishing but wayward Combinatorial School of mathematics into a severe condemnation of mathematicians in general and of the science itself, which, nevertheless, in the opening but omitted line of the very passage from which Schopenhauer quotes, is characterized by Lichtenberg as *"eine gar herrliche Wissenschaft."* Regarding the question of the intrinsic merit of the estimate of mathematics which these two most famous and influential enemies of the science have made so largely current in the world that it fairly fills the atmosphere and people take it in unconsciously as by a kind of cerebral suction, I shall speak in another connection. What I desire to emphasize here is the fact that neither the vast, splendid, superficial learning of the pompous author of "The Philosophy of the Conditioned" nor the pungence and pith, brilliance and intrepidity of the author of "Die

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Welt als Wille" can avail to constitute either of them an authority in a subject in which neither was informed and in which both stand convicted falsifiers of the judgments and opinions of other men.

As to Huxley and Holmes, the case is different. Both of them were generous, genial and honest, and to their opinions on any subject we gladly pay respect qualified only as the former's judgment regarding mathematics was qualified by Sylvester himself:

> "Verständige Leute kannst du irren sehn
> In Sachen nämlich, die sie nicht verstehn."

In relation to Huxley's statement that mathematical study knows nothing of observation, induction, experiment, and causation, it ought to be borne in mind that there are two kinds of observation: outer and inner, objective and subjective, material and immaterial, sensuous and sense-transcending; observation, that is, of physical things by the bodily senses, and observation, by the inner eye, by the subtle touch of the intellect, of the entities that dwell in the domain of logic and constitute the objects of pure thought. For, phrase it as you will, there is a world that is peopled with ideas, ensembles, propositions, relations, and implications, in endless variety and multiplicity, in structure ranging from the very simple to the endlessly intricate and complicate. That world is not the product but the object, not the creature but the quarry of thought, the entities composing it—propositions, for example,—being no more identical with thinking them than wine is identical with the drinking of it. Mind or no mind, that world exists as an extra-personal affair,—Pragmatism to the contrary notwithstanding. It appears to me to be a radical error of pragmatism to blink the fact that the most fundamental of spiritual things, namely, curiosity, never poses as a maker of

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truth but is found always and only in the attitude of seeking it. Indeed truth might be defined to be the presupposition or the complement of curiosity—as that without which curiosity would cease to be what it is. The constitution of that extra-personal world, its intimate ontological make-up, is logic in its essential character and substance as an independent and extra-personal form of being, while the study of that constitution is logic pragmatically, in its character, *i.e.,* as an enterprise of mind. Now—and this is the point I wish to stress—just as the astronomer, the physicist, the geologist, or other student of objective science looks abroad in the world of sense, so, not metaphorically speaking but literally, the mind of the mathematician goes forth into the universe of logic in quest of the things that are there; exploring the heights and depths for facts—ideas, classes, relationships, implications, and the rest; observing the minute and elusive with the powerful microscope of his Infinitesimal Analysis; observing the elusive and vast with the limitless telescope of his Calculus of the Infinite; making guesses regarding the order and internal harmony of the data observed and collocated; testing the hypotheses, not merely by the complete induction peculiar to mathematics, but, like his colleague of the outer world, resorting also to experimental tests and incomplete induction; frequently finding it necessary, in view of unforeseen disclosures, to abandon a once hopeful hypothesis or to transform it by retrenchment or by enlargement:—thus, in his own domain, matching, point for point, the processes, methods and experience familiar to the devotee of natural science.

Is it replied that it was not observation of the objects of pure thought but the other kind, namely, sensuous observation, that Huxley had in mind, then I rejoin that, nevertheless, observation by the inner eye of the things of thought *is* observation, not less genuine, not less difficult,

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not less rich in its objects and disciplinary value, than is sensuous observation of the things of sense. But this is not all, nor nearly all. Indeed for direct beholding, for immediate discerning, of the things of mathematics there is none other light but one, namely, psychic illumination, but mediately and indirectly they are often revealed or at all events hinted by their sensuous counterparts, by indications within the radiance of day, and it is a great mistake to suppose that the mathetic spirit elects as its agents those who, having eyes, yet see not the things that disclose themselves in solar light. To facilitate eyeless observation of his sense-transcending world, the mathematician invokes the aid of physical diagrams and physical symbols in endless variety and combination; the logos is thus drawn into a kind of diagrammatic and symbolical incarnation, gets itself externalized, made flesh, so to speak; and it is by attentive physical observation of this embodiment, by scrutinizing the physical frame and make-up of his diagrams, equations and formulae, by experimental substitutions in, and transformations of, them, by noting what emerges as essential and what as accidental, the things that vanish and those that do not, the things that vary and the things that abide unchanged, as the transformations proceed and trains of algebraic evolution unfold themselves to view,—it is thus, by the laboratory method, by trial and by watching, that often the mathematician gains his best insight into the constitution of the invisible world thus depicted by visible symbols. Indeed the importance to the mathematician of such sensuous observation cannot be overrated. It is not merely that the craving to see has led to the construction of the manifold models, ingenious and noble, of Schilling and others, illustrating important parts of Higher Geometry, Analysis Situs, Function Theory and other doctrines, but the annals of the science are illustrious with achievements made possible by facts first noted by

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the physical eye. To take a simple example from ancient days, it was by observation of the fact that the squares of certain numbers are each the sum of two other squares, the detection and collection of these numbers by the method of trial, observation of the fact that apparently all and only the numbers of such triplets are measures of the sides of right triangles,—it was thus, by observation and experiment, by the method of incomplete induction, common to the experimental sciences, that the Pythagorean theorem, now familiar throughout the world, was discovered. It was by Leibniz's observation of the definitely lawful manner in which the coefficients of a system of equations enter their solution that the suggestion came of a notion on the basis of which there has grown up in our time an imposing theory, an algebra built up on algebra—the colossal doctrine of Determinants. It was the observation, the detection by the eye of Lagrange and Boole and Eisenstein, of the fact that linear transformation of certain algebraic expressions leaves certain functions of their coefficients absolutely undisturbed in form, unaltered in frame of constitution, that gave rise to the concept, and therewith to the morphological doctrine, of Invariants, a theory filling the heavens like a light-bearing ether, penetrating all the branches of geometry and analysis, revealing everywhere abiding configurations in the midst of change, everywhere disclosing the eternal reign of the law of Form. It was in order to render evident to sensuous observation and to keep constantly before the physical eye the pervasive symmetry of mathematical thought that Hesse in the employment of homogeneous coördinates set the example, since then generally followed, of replacing a variety of different letters by repetitions of a single one distinguished by indices or subscripts,—a practice yet further justified on grounds both of physical and of intellectual economy. It was by sensuous observation that Clerk Maxwell, in the beginning

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of his wondrous career, detected a lack of symmetry in the then recognized equations of electro-dynamics and by that observed fact together with a discriminating sense of the scientific significance of esthetic intimations, that he was led to remove the seeming blemish by the addition of a term, antedating experimental justification of his daring deed by twenty years: an example of prescience not surpassed by that of Adams and Leverrier who, while engaged in the study of planetary disturbance, each of them about the same time and independently of the other, felt the then unknown Neptune "trembling on the delicate thread of their analysis" and correctly informed the astronomer where to point his telescope in order to behold the planet. One might go on to cite the theorem of Sturm in Equation Theory, the "Diophantine theorems of Fermat" in the Theory of Numbers, the Jacobian "doctrine of double periodicity" in Function Theory, Legendre's law of reciprocity, Sylvester's reduction of Euler's problem of the Virgins to the form of a question in Simple Partitions, and so on and on, thus continuing indefinitely the story of the great rôle of observation, experiment and incomplete induction, in mathematical discovery. Indeed it is no wonder that even Gauss, "facile princeps matematicorum," even though he dwelt aloft in the privacy of a genius above the needs and ways of other minds, yet pronounced mathematics "a science of the eye."

Indeed the time is at hand when at least the academic mind should discharge its traditional fallacies regarding the nature of mathematics and thus in a measure promote the emancipation of criticism from inherited delusions respecting the kind of activity in which the life of the science consists. Mathematics is no more the art of reckoning and computation than architecture is the art of making bricks or hewing wood, no more than painting is the art of mixing colors on a palette, no more than the science of geol-

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ogy is the art of breaking rocks, or the science of anatomy the art of butchering.

Did not Babbage or somebody invent an adding machine? And does it not follow, say Holmes and Schopenhauer, that mathematical thought is a merely mechanical process? Strange how such trash is occasionally found in the critical offering of thoughtful men and thus acquires circulation as golden coin of wisdom. It would not be sillier to argue that, because Stanley Jevons constructed a machine for producing certain forms of logical inference, therefore all thought, even that of a philosopher like Schopenhauer or that of a poet like Holmes, is merely a thing of pulleys and levers and screws, or that the pianola serves to prove that a symphony by Beethoven or a drama by Wagner is reducible to a trick of mechanics.

But far more pernicious, because more deeply imbedded and persistent, is the fallacy that the mathematician's mind is but a syllogistic mill and that his life resolves itself into a weary repetition of *A* is *B*, *B* is *C*, therefore *A* is *C*; and *Q.E.D.* That fallacy is the *Carthago delenda* of regnant methodology. Reasoning, indeed, in the sense of compounding propositions into formal arguments, is of great importance at every stage and turn, as in the deduction of consequences, in the testing of hypotheses, in the detection of error, in purging out the dross from crude material, in chastening the deliverances of intuition, and especially in the final stages of a growing doctrine, in welding together and concatenating the various parts into a compact and coherent whole. But, indispensable in all such ways as syllogistic undoubtedly is, it is of minor importance and minor difficulty compared with the supreme matters of Invention and Construction. *Begriffbildung,* the resolution of the nebula of consciousness into star-forms of definite ideas; discriminating sensibility to the logical significances, affinities and bearings of these; susceptibility to the delicate

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intimations of the subtle or the remote; sensitiveness to dim and fading tremors sent below by breezes striking the higher sails; the ability to grasp together and to hold in steady view at once a multitude of ideas, to transcend the individuals and, compounding their forces, to seize the resultant meaning of them all; the ability to summon not only concepts but doctrines, marshalling them and bringing them to bear upon a single point, like great armies converging to a critical centre on a battle field. These and such as these are the powers that mathematical activity in its higher rôles demands. The power of ratiocination, as already said, is of exceeding great importance but it is neither the base nor the crown of the faculties essential to "Mathematicised Man." When the greatest of American logicians, speaking of the powers that constitute the born geometrician, had named Conception, Imagination, and Generalization, he paused. Thereupon from one in the audience there came the challenge, "What of Reason?" The instant response, not less just than brilliant, was: "Ratiocination—that is but the smooth pavement on which the chariot rolls." When the late Sophus Lie, great comparative anatomist of geometric theories, creator of the doctrines of Contact Transformations, and Infinite Continuous Groups, and revolutionizer of the Theory of Differential Equations, was asked to name the characteristic endowment of the mathematician, his answer was the following quaternion: *Phantasie, Energie, Selbstvertrauen, Selbstkritik.* Not a word, you observe, about ratiocination. *Phantasie,* not merely the fine frenzied fancy that gives to airy nothings a local habitation and a name, but the creative imagination that conceives ordered realms and lawful worlds in which our own universe is as but a point of light in a shining sky; *Energie,* not merely endurance and doggedness, not persistence merely, but mental *vis viva,* the kinetic, plunging, penetrating power of intellect; *Selbstvertrauen* and

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Selbstkritik, self-confidence aware of its ground, deepened by achievement and reinforced until in men like Richard Dedekind, Bernhard Bolzano and especially Georg Cantor it attains to a spiritual boldness that even dares leap from the island shore of the Finite over into the all-surrounding boundless ocean of Infinitude itself, and thence brings back the gladdening news that the shoreless vast of Transfinite Being differs in its logical structure from that of our island home only in owning the reign of more *generic* law.

Indeed it is not surprising, in view of the polydynamic constitution of the genuinely mathematical mind, that many of the major heroes of the science, men like Desargues and Pascal, Descartes and Leibniz, Newton, Gauss and Bolzano, Helmholtz and Clifford, Riemann and Salmon and Plücker and Poincaré, have attained to high distinction in other fields not only of science but of philosophy and letters too. And when we reflect that the very greatest mathematical achievements have been due, not alone to the peering, microscopic, histologic vision of men like Weierstrass, illuminating the hidden recesses, the minute and intimate structure of logical reality, but to the larger vision also of men like Klein who survey the kingdoms of geometry and analysis for the endless variety of things that flourish there, as the eye of Darwin ranged over the flora and fauna of the world, or as a commercial monarch contemplates its industry, or as a statesman beholds an empire; when we reflect not only that the Calculus of Probability is a creation of mathematics but that the master mathematician is constantly required to exercise judgment—judgment, that is, in matters not admitting of certainty—balancing probabilities not yet reduced nor even reducible perhaps to calculation; when we reflect that he is called upon to exercise a function analogous to that of the comparative anatomist like Cuvier, comparing theories

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and doctrines of every degree of similarity and dissimilarity of structure; when, finally, we reflect that he seldom deals with a single idea at a time, but is for the most part engaged in wielding organized hosts of them, as a general wields at once the divisions of an army or as a great civil administrator directs from his central office diverse and scattered but related groups of interests and operations; then, I say, the current opinion that devotion to mathematics unfits the devotee for practical affairs should be known for false on *a priori* grounds. And one should be thus prepared to find that as a fact Gaspard Monge, creator of descriptive geometry, author of the classic "Applications de l'analyse à la géométrie"; Lazare Carnot, author of the celebrated works, "Géométrie de position," and "Réflexions sur la Métaphysique du Calcul infinitesimal"; Fourier, immortal creator of the "Théorie analytique de la chaleur"; Arago, rightful inheritor of Monge's chair of geometry; and Poncelet, creator of pure projective geometry; one should not be surprised, I say, to find that these and other mathematicians in a land sagacious enough to invoke their aid, rendered, alike in peace and in war, eminent public service.

To speak at length, if that were necessary, of Huxley's deliverance that the study of mathematics "knows nothing of causation," the "law of my song and the hastening hour forbid." Suffice it to say in passing that when the mathematician seeks the consequences of given suppositions, saying 'when these precede, those will follow,' and when, having plied a circle, a sphere or other form chosen from among infinitudes of configurations, with some transformation among infinite hosts at his disposal, he speaks of its 'effect,' then, I submit, he is employing the language of causation with as nice propriety as it admits of in a world where, as everyone knows, except such as still enjoy the blessings of a juvenile philosophy, the best we can say is

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that the ceaseless shuttles fly back and forth, and streams of events without original source flow on without ultimate termination. Indeed it is a certain and signal lesson of science in all its forms everywhere that the language of cause and effect, except in the sense of facts being lawfully implied in other facts, has no indispensable use.

I have not spoken of "Applied Mathematics," and that for the best of reasons: there is, strictly speaking, no such thing. The term indeed exists, and, in a conservative practical world that cares but little for "The nice sharp quillets of the law," it will doubtless persist as a convenient designation for something that never existed and never can. It is of the very essence of the practician type of mind not to know aught as it is in itself nor aught as self-justified but to mistake the secondary and accidental for the primary and essential, to blink and elude the presence of *immediate* worth, and being thus blind to instant and immanent ends, to revel in means and uses and applications, requiring all things to excuse their being by extraneous and emanant effects,—vindicating the stately elm by its promise of lumber, or the lily by its message of purity, or the flood of Niagara by its available energy, or even knowledge itself by the worldly advantage and the power which it gives. I am told that even the deep and exquisite terminology of art has been to some extent invaded by such barbarous and shallow phrases as 'applied music,' 'applied architecture,' 'applied sculpture,' 'applied painting,' as if Beauty, virgin mother of art, could, without dissolution of her essential character, consciously become the willing drudge and paramour of Use. And I suppose we are fated yet to hear of applied glory, applied holiness, applied poetry—*i.e.,* poetry that is consciously pedagogic or that aims at a moral and thereby sinks or rises to the level of a sermon—of applied joy, applied ontology, yea, of applied inapplicability itself.

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It is in implications and not in applications that mathematics has its lair. Applied mathematics is mathematics simply or is not mathematics at all. To think aright is no characteristic striving of a class of men; it is a common aspiration; and Mechanics, Mathematical Physics, Mathematical Astronomy, and the other chief *Anwendungsgebiete* of mathematics, as Geodesy, Geophysics, and Engineering in its various branches, are all of them but so many witnesses to the truth of Riemann's saying that "Natural science is the attempt to comprehend nature by means of exact concepts." A gas molecule regarded as a minute sphere or other geometric form, however complicate; stars and planets conceived as ellipsoids or as points, and their orbits as loci; time and space, mass and motion and impenetrability; velocity, acceleration and energy; the concepts of norm and average;—what are these but mathematical notions? And the wondrous garment woven of them in the loom of logic—what is that but mathematics? Indeed every branch of so-called applied mathematics is a mixed doctrine, being thoroughly analyzable into two disparate parts: one of these consists of determinate concepts formally combined in accordance with the canons of logic, *i.e.,* it is mathematics and not natural science viewed as matter of observation and experiment; the other *is* such matter and is natural science in that conception of it and not mathematics. No fibre of either component is a filament of the other. It is a fundamental error to regard the term Mathematicisation of thought as the importation of a tool into a foreign workshop. It does not signify the transition of mathematics conceived as a thing accomplished over into some outlying domain like physics, for example. Its significance is different radically, far deeper and far wider. It means the growth of mathematics itself, its extension and development from within; it signifies the continuous revelation, the endlessly progressive coming into view, of

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the static universe of logic; or, to put it dynamically, it means the evolution of intellect, the upward striving and aspiration of thought everywhere, to the level of cogency, precision and exactitude. This self-propagation of the rational logos, the springing up of mathetic rigor even in void and formless places, in the very retreats of chaos, is to my mind the most impressive and significant phenomenon in the history of science, and never so strikingly manifest as in the last half hundred years. Seventy-two years ago, even Comte, the stout advocate of mathematics as constituting "the veritable point of departure for all rational scientific education, general or special," expressed the opinion that we should never "be in position by any means whatever to study the chemical composition of the stars." In less than twenty-five years thereafter that negative prophecy was falsified by the chemical genius of Bunsen fortified by the mathematics of Kirchoff. Not only has mathematics grown, in the domain of Physics, into the vast proportions of Rational Dynamics, but the derivative and integral of the Calculus, and Differential Equations, are more and more finding subsistence in Chemistry also, and by the work of Nernst and others even the foundations of the latter science are being laid in mathematico-physical considerations. Merely to sketch most briefly the mathematical literature that has grown up in the field of Political Economy requires twenty-five pages of the above mentioned *Encyklopädie* of mathematics. Similar sketches for Statistics and Life Insurance require no less than thirty and sixty-five pages respectively. Even in the baffling and elusive matter of Psychology, the work of Herbart, Fechner, Weber, Wundt and others confirms the hope that the soil of that great field will some day support a vigorous growth of mathematics. It seems indeed as if the entire surface of the world of human consciousness were predestined to be covered over, in varying degrees of luxuriance, by the flora of mathetic science.

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But while mathematics may spring up and flourish in any and all experimental and observational fields, it is by no means to be expected that 'experiment and observation' will ever thus be superseded. Such domains are rather destined to be occupied at the same time by two tenants, mathematical science and science that is not mathematical. But while the former will serve as an ideal standard for the latter, mathematics has neither the power nor the disposition to disseize experiment and observation of any holdings that are theirs by the rights of conquest and use. Between mathematics on the one hand and non-mathematical science on the other, there can never occur collision or quarrel, for the reason that the two interests are ultimately discriminated by the kind of curiosity whence they spring. The mathematician is curious about definite naked relationships, about logically possible modes of order, about varieties of implication, about completely determined or determinable functional relationships, considered solely in and of themselves, considered, that is, without the slightest concern about any question whether or no they have any external or sensuous validity or other sort of validity than that of being logically thinkable. It is the aggregate of things thinkable logically that constitutes the mathematician's universe, and it is inconceivably richer in mathetic content than can be any outer world of sense such as the physical universe according to which we chance to have our physical being.

This mere speck of a physical universe in which the chemist, the physicist, the astronomer, the biologist, the sociologist, and the rest of nature students, find their great fields and their deep and teeming interests, may be a realm of invariant uniformities, or laws; it may be a mechanically organic aggregate, connected into an ordered whole by a tissue of completely definable functional relationships; and it may not. It may be that the universe eternally has been and is a genuine cosmos; it may be that the external

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sea of things immersing us, although it is ever changing infinitely, changes only lawfully, in accordance with a system of immutable rules of order that constitute an invariant at once underived and indestructible and securing everlasting harmony through and through; and it may not be such. The student of nature assumes, he rightly assumes, that it is; and, moved and sustained by characteristic appropriate curiosity, he endeavors to find in the outer world what are the elements and what the relationships assumed by him to be valid there. The mathematician as such does not make that assumption and does not seek for elements and relationships in the outer world.

Is the assumption correct? Undoubtedly it is admissible, and as a working hypothesis it is undoubtedly exceedingly useful or even indispensable to the student of external nature; but is it true? The mathematician as man does not know although he cares. Man as mathematician neither knows nor cares. The mathematician does know, however, that, if the assumption be correct, every relationship that is valid in nature is, *in abstractu,* an element in his domain, a subject for his study. He knows, too, at least he strongly suspects, that, if the assumption be not correct, his domain remains the same absolutely, and the title of mathematics to human regard "would remain unimpeached and unimpaired" were the universe without a plan or, having a plan, if it "were unrolled like a map at our feet, and the mind of man qualified to take in the whole scheme of creation at a glance."

The two realms, of mathematics, of natural science, like the two curiosities and the two attitudes, the mathematician's and the nature student's, are fundamentally distinct and disparate. To think logically the logically thinkable —that is the mathematician's aim. To assume that nature is thus thinkable, an embodied rational logos, and to discover the thought supposed incarnate there—these

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are at once the principle and the hope of the student of nature.

Suppose the latter student is right and that the outer universe really is an embodied logos of reason, does it follow that all the logically thinkable is incorporated in it? It seems not. Indeed there appears to be many a rational logos. A cosmos, a harmoniously ordered universe, one that through and through is self-compatible, can hardly be the whole of reason materialized and objectified. At all events the mathematician has delight in the conceptual construction and in the contemplation of divers systems that are inconsistent with one another though each is thoroughly self-coherent. He constructs in thought a summitless hierarchy of hyperspaces, an endless series of ordered worlds, worlds that are possible and logically actual. And he is content not to know if any of them be otherwise actual or actualized. There is, for example, a Euclidian geometry and there are infinitely many kinds of non-Euclidian. These doctrines, regarded as *true* descriptions of some one actual space, are incompatible. In our universe, to be specific, if it be as Plato thought and natural science takes for granted, a geometrized or geometrizable affair, then one of these geometries may be, none of them may be, not all of them can be, objectively valid. But in the infinitely vaster world of pure thought, in the world of mathesis, all of them are valid; there they co-exist, there they interlace and blend among themselves and others as differing strains of a hypercosmic harmony.

It is from some such elevation, not the misty lowland of the sensuously and materially Actual, but from a mount of speculation lawfully rising into the azure of the logically Possible, that one may glimpse the dawn heralded by the avowal of Leibniz: *"Ma métaphysique est toute mathématique."* Time fails me to deal fittingly with the great theme herewith suggested, but I cannot quite forbear to

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express briefly my conviction that, apart from its service to kindred interests of thought as a standard of clarity, rigor and certitude, mathematics is and will be found to be an inexhaustible quarry of material—of ontologic types, of ideas and problems, of distinctions, discriminants and hints, evidences, analogies and intimations—all for the exploitation and use of Philosophy, Psychology, and Theology. The allusion is not to such celebrated alliances of philosophy and mathesis as flourished in the school of Pythagoras and in the gigantic personalities of Plato, Descartes, Spinoza, and Leibniz, nor to the more technical mathematico-philosophical researches and speculations of our own time by such as C. S. Peirce, Russell, Whitehead, Peano, G. Cantor, Couturat and Poincaré, glorious as were those alliances and important as these researches are. The reference is rather to the unappreciated fact that the measureless accumulated wealth of the realm of exact thought is at once a marvelous mine of subject matter and a rich and ready arsenal for those great human concerns of reflective and militant thought that is none the less important because it is not exact.

For the vindication of that claim, a hint or two must here suffice. The modern mathematical concepts of number, time, space, order, infinitude, finitude, group, manifold, functionality, and innumerable hosts of others, the varied processes of mathematics, and the principles and modes of its growth and evolution, all of these or nearly all still challenge and still await those kinds of analysis that are proper to the philosopher and the psychologist. The psychology of Euclidian, non-Euclidian, and hyper spaces, the question of the intuitability of the latter, the secret of their having become not only indispensable in various branches of mathematics but instrumentally useful in other fields also, as in the kinetic theory of gases; the question, for example, why it is that while *thought* maintains a straightforward

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course through four-dimensional space, *imagination* travels through it on a zig-zag path, of two logically identical configurations, being partially or completely blind to the one, yet perfectly beholding the other; the evaluation and adjustment of the contradictory claims of Poincaré and his school on the one hand and of Mach and his disciples on the other, the former contending that Modern Analysis is a "free creation of the human spirit" guided indeed but not constrained by experience of the external world, being merely kept by this from aimless wandering in wayward paths; while the latter maintain that mathematical concepts, however tenuous or remote or recondite, have been literally evolved continuously in accordance with the needs of the animal organism and with environmental conditions out of the veriest elements (feelings) of physical life, and accordingly that the purest offspring of mathematical thought may trace a legitimate lineage back and down to the lowliest rudiments of physical and physiological experience:—these problems and such as these are, I take it, problems for the student of mind as mind and for the student of psycho-physics.

Regarding the relations of mathesis to the former "queen of all the sciences," I have on this occasion but little to say. I do not believe that the declined estate of Theology is destined to be permanent. The present is but an interregnum in her reign and her fallen days will have an end. She has been deposed mainly because she has not seen fit to avail herself promptly and fully of the dispensations of advancing knowledge. The aims, however, of the ancient mistress are as high as ever, and when she shall have made good her present lack of modern education and learned to extend a generous and eager hospitality to modern light, she will reascend, and will occupy with dignity as of yore an exalted place in the ascending scale of human interests and the esteem of enlightened men. And

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mathematics, by the character of her inmost being, is especially qualified, I believe, to assist in the restoration. It was but little more than a generation ago that the mathematician, philosopher and theologian, Bernhard Bolzano, dispelled the clouds that throughout all the foregone centuries had enveloped the notion of Infinitude in darkness, completely sheared the great term of its vagueness without shearing it of its strength, and thus rendered it forever available for the purposes of logical discourse. Whereas, too, in former times the Infinite betrayed its presence not indeed to the faculties of Logic but only to the spiritual Imagination and Sensibility, mathematics has shown, even during the life of the elder men here present,—and the achievement marks an epoch in the history of man,—that the structure of Transfinite Being is open to exploration by the organon of Thought. Again, it is in the mathematical doctrine of Invariance, the realm wherein are sought and found configurations and types of being that, amid the swirl and stress of countless hosts of transformations, remain immutable, and the spirit dwells in contemplation of the serene and eternal reign of the subtile law of Form, it is there that Theology may find, if she will, the clearest conceptions, the noblest symbols, the most inspiring intimations, the most illuminating illustrations, and the surest guarantees of the object of her teaching and her quest, an Eternal Being, unchanging in the midst of the universal flux.

It is not, however, by any considerations or estimates of utility in any form however high it be or essential to the worldly weal of man; it is not by evaluating mastery of the processes of measurement and computation, though these are continuously vital everywhere to the conduct of practical life; nor is it by strengthening the arms of natural science and speeding her conquests in a thousand ways and a hundred fields; nor yet by extending the empire of the

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human intellect over the realms of number and space and establishing the dominion of thought throughout the universe of logic; it is not even by affording argument and fact and light to theology and so contributing to the advancement of her supreme concerns;—it is not by any of these considerations nor by all of them that Mathematics, were she called upon to do so, would rightly seek to vindicate her highest claims to human regard. It requires indeed but little penetration to see that no science, no art, no doctrine, no human activity whatever, however humble or high, can ultimately succeed in justifying itself in terms of measurable fruits and emanant effects, for these remain always to be themselves appraised, and the process of such attempted vindication is plainly fated to issue only in regression without an end. Such Baconian apologetic, when offered as final, quite mistakes the finest mood of the scientific spirit and is beneath the level of academic faith. Science does not seek emancipation in order to become a drudge, she consents to serve indeed but her service aims at freedom as an end.

Man has been so long a slave of circumstance and need, he has been so long constrained to seek license for his summit faculties, in lower courts without appeal, that a sudden transitory moment of release sets him trembling with distrust and fear, an occasional imperfect vision of the instant dignity of his spiritual enterprises is at once obscured by doubt, and he straightway descends into the market places of the world to excuse or to justify his illumination, pleading some mere utility against the ignoring or the condemnation of an insight or an inspiration whose worth is nevertheless immediate and no more needs and no more admits of utilitarian justification than the breaking of morning light on mountain peaks or the bounding of lambs in a meadow.

The solemn cant of Science in our day and her sombre

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visage are but the lingering tone and shade of the prison-house, and they will pass away. Science is destined to appear as the child and the parent of freedom blessing the earth without design. Not in the ground of need, not in bent and painful toil, but in the deep-centred play-instinct of the world, in the joyous mood of the eternal Being, which is always young, Science has her origin and root; and her spirit, which is the spirit of genius in moments of elevation, is but a sublimated form of play, the austere and lofty analogue of the kitten playing with the entangled skein or of the eaglet sporting with the mountain winds.

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