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Professor Renwick,
Columbia College,
with the respectful regards of
The Author –

NEW ELEMENTS

OF

GEOMETRY.

BY SEBA SMITH.

"GOD SAID, LET THERE BE LIGHT; AND THERE WAS LIGHT."

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THREE PARTS.

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PREFACE.

A FEW remarks of a personal and prefatory character, it may be proper in this place to address to the reader. Some thirty years ago, while in college, I had paid some little attention to Geometry, having gone with my class through three or four of the fifteen books of Euclid's Elements. But the knowledge obtained, even of the few books read, was somewhat superficial; and pursuits in after-life not requiring exercise in the science, thirty years disuse had suffered every demonstration and almost every principle derived from Euclid to fade from the mind. About two years and a half ago, JOHN A. PARKER, Esq., of New York, a gentleman whose life had mainly been passed in mercantile and commercial pursuits, applied to me, as an old acquaintance and friend, to examine some original papers, in which he claimed to have solved the most celebrated problem in mathematics, *the quadrature of the circle.* He had several years before discovered what he believed to be the true ratio of the circumference of a circle to its diameter, and had, during the interval, made repeated endeavors to have his papers examined, and his positions acknowledged by mathematicians. But he had found very few to give them even a slight examination, and none to concede the truth of his conclusions.

I took up his papers and read them with great care. I was at once much impressed with the boldness, strength, and originality of his reasoning, and finally convinced of the truth of his solution of that remarkable problem, which had long since been pronounced by mathematicians and learned societies as an impossibility. I became strongly interested in the whole subject of Geometry. I took down my old Euclid, and brushed off the dust of thirty years; I went to the bookstalls and bookstores and searched for different works on Geometry, till I had picked up fifteen or twenty, which I examined, some partially, some thoroughly, but all with a zest. Mr. PARKER's reasonings and demonstrations led to the conclusion that the circumference of a circle was not a line coinciding with the perimeter of the circle, as geometers had hitherto considered it, but a line wholly and perfectly outside of the circle, and consequently that it must be a magnitude entirely distinct from the circle, and must have breadth. In addition to these reasonings of Mr. PARKER, which were entirely original with him, I found upon research that the learned and acute mathematician, Dr. BARROW, had come to the decided conclusion that mathematical number always expressed *magnitude*. And I also found some remarks of Aristotle, which seemed to lead to the same conclusion.

Here a great question intensely pressed upon my mind,—if mathematical number always represents magnitude, mathematical *lines* represented by numbers are *magnitudes*, and must have breadth; and if they have breadth, is it not possible by some geometrical demonstration to prove *what that breadth is?* The thought pursued me day and night, for it would not leave me even in my sleeping hours. I set myself down steadily to the task for a year and a half, and the present volume is the result. The breadth of mathematical lines is not only perfectly established, but the whole subject of Geometry is simplified, cleared from obscurities and difficulties, and placed, as it were, on a new foundation. But let it not be supposed that the new laws and principles of Geometry, developed and demonstrated in this work, have been derived from hypothesis and theoretical reasoning. They rest not upon so unsafe a basis. They were reached by the pure methods of the inductive philosophy of Bacon. I went to work upon original diagrams with the Greek rule and compasses in my hand, and spent a long and laborious year in digging out my *facts*. I examined such varieties of geometrical forms as the imagination could suggest, and as patient thought and labor were able to investigate. I measured, computed and compared diameters, areas, and circumferences of plane figures, and diameters, solidities, and surfaces of solid figures, at the same time examining and comparing the roots of all these various quantities; and from the *facts* thus gradually collected and arranged, the general laws and principles of the science presented themselves clearly to view, and demanded the acknowledgment of their high prerogatives.

It is proper here also to remark, that a work made up almost entirely of new geometrical principles, and embracing a great variety of original arithmetical calculations, all prepared by one individual, without being revised by others, cannot reasonably be expected to be entirely free from errors. Some slip of the pen, some oversight of the eye, some figure missed, or some typographical error unperceived or uncorrected, may very probably be found to mar, in some degree, the work. But if nothing shall be discovered to invalidate the principles laid down, as they are intended to be explained, the Author trusts that minor errors, should such appear, will be charitably and cheerfully overlooked by the reader.

The work of Mr. Parker on the Quadrature of the Circle is in preparation for the press, and is expected soon to be published. It is therefore unnecessary, and would be hardly appropriate here for me to enter into any elaborate consideration of it. I have already expressed my conviction of the truth of his ratio of the circumference of a circle to its diameter. That ratio is 20612 for circumference, and 6561 for diameter, which is the smallest expression of the perfect ratio that can be given in whole numbers. This ratio, in a circle whose diameter is 1, gives for circumference 3.141594+. The approximate ratio obtained by geometers, and generally received as correct, is 3.141592+. Mr. Parker, it is seen, differs from this in the sixth decimal figure. And he shows conclusively that the method of geometers in obtaining this approximate ratio, which is by means of inscribed and circumscribed polygons, necessarily leads to an error in the sixth decimal place. To test the truth of his perfect ratio, Mr. PARKER has, with a bold conception and singular originality, applied it to some of the astronomical circles, and obtained remarkable and startling results, indicating that in the motions and periods of the heavenly bodies there are perfect mathematical relations much more wonderful and extensive than have yet been understood.

Hippocrates squared a portion of a circle more than two thousand years ago, in the figure called "the lune of Hippocrates;" and I have myself squared other portions of a circle by similar methods. And I think when the reader has seen, in the demonstrations and principles exhibited in the following pages, what perfect harmony prevails between the circle and all rectilineal figures, and how the circle controls all rectineal figures by one simple and uniform law, he will have no doubt that the whole circle may be perfectly squared.

SEBA SMITH.

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PART FIRST.

THE PHILOSOPHY OF GEOMETRY.

SECTION I.

IMPORTANCE OF THE SCIENCE, AND ITS DIFFICULTIES.

"The invention of forms," says Lord Bacon, the great founder of inductive philosophy, "is of all other parts of knowledge the worthiest to be sought, if it be possible to be found." And in the same connection he adds: "As for the possibility, they are ill-discoverers that think there is no land when they can see nothing but sea."

Plato also regarded *forms* as the true object of knowledge; but in the judgment of Bacon he "lost the real fruit of his opinion, by considering of forms as absolutely *abstracted from matter;*" by which means he was led into theological speculations, "wherewith all his natural philosophy is infected."

In the opinion of Pythagoras, the study of the mathematics, including geometry, was "the first step toward wisdom." The pupils in his school first became mathematicians; and after they had made sufficient progress in geometrical science, they were conducted to the study of nature, the investiga' primary principles, and the consideration of the attributes of Deity.

Plato arrived at such a reverence for geometry, that he had inscribed over the door of his academy where he taught philosophy, "Let no one who is ignorant of geometry enter here." And when his opinion was asked concerning the probable employment of Deity, he is said to have replied, "He geometrizes continually;" by which he undoubtedly meant that the great Author of nature established and governs the universe by geometrical laws.

Also the learned and pious Dr. Barrow held geometry in such estimation, that he considered the contemplation of it as not unworthy of the Deity; and in publishing an edition of the works of Apollonius, he inscribed it with the words, "God himself geometrizes. O Lord, how great a geometer art thou!"

And in testimony of the truth and immutability of the principles of geometry, Aristotle, the great master of ancient philosophies, declared that "the poles of the world will be sooner removed out of their places, and the fabric of nature destroyed, than the foundations of geometry fail, or its conclusions be convinced of falsity."

And yet this first and most important of the sciences—most important, because lying at the foundation of all other sciences—so clear in its principles, so certain in its conclusions, so venerable for its antiquity, hoary with the lapse of thousands of years, and honored in every age by the earnest investigations of the master-minds of men—this grand fabric of geometry, so beautiful in its proportions, and so magnificent in extent, rests in part on a false foundation. One of the corner-stones upon which it was first erected was given with false dimensions, and must be removed, and its place supplied by the true corner-stone, before the structure can be made perfect throughout, and present an unbroken harmony in all its relations.

To remove that false corner-stone and supply the true, is the object intended by the present treatise. Should I be met at the threshold by the incredulous world, and reproachfully or satirically asked, in the words of Paul, "Who is sufficient for these things?" I shall reply only in the humble spirit of the same apostle, when he declared that "God hath chosen the foolish things of the world to confound the wise; and God hath chosen the weak things of the world to confound the things which are mighty."

The error, which I allege to exist in the fundamental principles of geometry, is embodied in the first definitions of the science, as given by Euclid, and as adopted and followed in the many hundreds of works written on the subject for the last two thousand years. Euclid, and I believe all other geometers who have written hitherto, take their stand upon these definitions, viz.:

"A line is length without breadth;" and "a surface is length and breadth without thickness."

I meet these definitions at once, and declare that every mathematical line has a *breadth*, as definite, as measurable, and as clearly demonstrable, as its length; and that every mathematical surface has a *thickness* as definite, as measurable, and as clearly demonstrable, as its length or breadth.

Will it be answered here, that the demonstrations given by geometers are clearly and unquestionably true, and therefore, if there be an error in one or two of their definitions or assumed principles it affects not their conclusions, and must be a matter of but little consequence? Though such an answer could scarcely be expected to come from a mind imbued with sound principles of philosophy, it may still be worth while to dwell upon it for a moment. Astronomers got along very well before Galileo's time, upon the hypothesis that the earth was the center of the solar system. They calculated eclipses on that hypothesis, and the eclipses came out right, and verified their calculations. Was it, therefore, a matter of but little consequence, that their whole system was based on a false foundation? The conclusions which they could reach from that foundation were clearly and unquestionably true; but there are many vast and important truths in the science of astronomy, which they never could have reached till their fundamental error was discovered, and the earth allowed to revolve about the sun.

So there are many important truths in geometry demonstrated in this work, and many more doubtless yet to be demonstrated, which never could have been reached till the true nature of lines and surfaces was discovered and their proper quantities demonstrated.

It is certainly unphilosophical to admit that a truth, lying at the foundation of any science, can be unimportant. Professor Playfair has well and forcibly said: "The truths of geometry are all necessarily connected with one another, and the system of such truths can never be rightly explained, unless that connection be accurately traced wherever it exists. It is upon this, that the beauty and peculiar excellence of the mathematical sciences depend. It is this, which, by preventing any one truth from being single and insulated, connects the different parts so firmly, that they must all stand, or fall together."

If it is a truth, therefore, that mathematical lines have a definite and measurable breadth, and that mathematical surfaces have a definite and measurable thickness, that truth must unquestionably be of great importance to the science of geometry and to all mathematics. And the want of a knowledge of this truth, I think, has hitherto prevented the true relation between numbers, magnitudes, and forms, from being clearly and properly understood. It is not strange therefore, that while a fundamental principle in mathematics remained shrouded in darkness, the professors of that science should have been led into a thousand laborious and useless speculations, upon questions in which that unknown principle was necessarily involved. Indeed from these causes, the mathematical sciences, like a very luxuriant vine left without pruning, have run out into immense quantities of foliage, bearing comparatively but little fruit. This state of things has become a reproach to mathematics. The writer of an able article in the Edinburgh Encyclopedia remarks, "The luxuriance of modern analytical speculation is arrived at such a point as to startle the most industrious, and to render an equally perfect knowledge of all its parts, *no longer attainable by one individual.*" And a writer of equal ability in the London Encyclopedia pursues the subject in a vigorous and satirical vein as follows. "Let the mathematics be encouraged and patronized. Let them be cultivated to the fullest extent, even with considerable waste of mental power and loss of money, to discover the north-west passage in the polar regions of fluxionary creation; to find out some new calculus, whether differential or integral. Were there no probable, or even possible results, as to such a mixed, impure, vulgar entity, as utility, in contending with practicability, and penetrating to a high mathematical latitude of discovery; were it merely for the sake of the contention of mind, or to have it proved how far the algebraic analyst can go up out of sight in some new-invented infinitesimal balloon; in short if the mathematical progression were a thousand miles ahead of any practical purpose or advantage whatever, we would not be for terminating its career. We have spare hands enough, and spare heads too; and as all cannot find useful employment, it is better perhaps that they should be out of the way of idleness and mischief, by digging mathematical holes and filling them up again, or in perpetual motion to discover new methods of contention of mind, new calculuses, new analyses, new fluxions, new infinitesimals, to rival and supersede the old ones, ad infinitum. Only let us know, if possible, what the mathematics are about, and wherein their infinite quantity of excellence consists."

On this point I may remark further, that quite recently in some of the most valuable of the English scientific magazines, I have observed articles from able professors and distinguished mathematicians, gravely discussing the question of the relative value of *three times nothing*, and *twice nothing;* 0×3, and 0×2. I have not the magazines before me, but I recollect that certain quantities had been carried through an algebraical process, duly invested with the signs of *plus* and *minus*, in which it was argued that the zeros used in the operation became invested also with a positive and negative character, and that the two zeros in the equation, which resulted from the operation, had acquired some sort of an infinitesimal value, *and really had a ratio to each other as three to two.*

When learned professors find themselves driven to such conclusions by their received principles of a science, it would seem to be high time for them to go back to first principles, and see whether there is not something wrong in the very foundations of that science.

But thus it must ever be, while men attempt to reason about *nothing* instead of *something;* whether it be by an algebraical process to establish the value of a cipher, which is absolutely without value, or by geometrical demonstrations to fix the value of lines which are assumed to be entirely and absolutely without breadth or thickness. The value and the results of such labors are well portrayed by Lord Bacon, in reference to the "Schoolmen," before his time, who, he says, "shut up in the cells of monasteries and colleges, and knowing little history, either of nature or time, did out of no great quantity of matter, and infinite agitation of wit, spin out unto us those laborious webs of learning which are extant in their books. For the wit and mind of man, if it work upon *matter*, which is the contemplation of the creatures of God, worketh according to the *stuff*, and is limited thereby. But if it work upon itself, as the spider worketh his web, then it is endless, and brings forth indeed cobwebs of learning, admirable for the fineness of thread and work, but of no substance or profit."

SECTION II.

THE COMMON VIEW OF GEOMETRY.

GEOMETERS have always felt embarrassed by their definitions of lines and surfaces. The explanations and illustrations of these definitions by Robert Simson, the distinguished Professor of Mathematics in the University of Glasgow, are the most elaborate, and generally deemed the clearest and most satisfactory of any that have been given. They are copied and adopted by Professor Playfair, and many other geometers. And yet these very illustrations of Professor Simson, in which he endeavors to prove that a line has no breadth, and that a surface has no thickness, embody a fallacy which entirely destroys the validity of his conclusions. I am not willing to make this remark, however, without adding, that I think the very high reputation acquired by Professor Simson was most eminently deserved, and that science is greatly indebted to him for the zeal with which he devoted a good portion of his life to recover and restore to purity some of the lost and mutilated works of the Greek geometers.

I will endeavor to explain the reasoning of Professor Simson, upon these definitions as briefly as possible. He remarks, "It is necessary to consider a solid, that is, a magnitude which has length, breadth and thickness, in order to understand aright the definitions of a point, line, and superficies; for these all arise from a solid, and exist in it. The boundary, or boundaries, which contain a solid, are called superficies; or the boundary which is *common* to two solids which are contiguous, or which divides one solid into two contiguous parts, is called a superficies." To illustrate the argument more briefly, if not more clearly, than is done by Professor Simson's diagram, we will suppose two perfect cubes, or dice, with faces geometrically contiguous, or in perfect contact. We will call one cube A, and the other B. The boundary, where these cubes meet, is common to them both, and, says the Professor, "is therefore in the one, as well as in the other solid, called a superficies, [or surface,] and has no thickness." For, proceeds the argument, if the surface, which is thus common to the two solids A and B, have any thickness, it must be a part of the thickness of A, or a part of B, or a part of each of them. But it cannot be a part of A; because if A be removed, the surface of B still remains as it was. Nor can this common surface be a part of B; because if B be removed, the surface of A remains as it was. Therefore he comes to the conclusion, that the two solids being geometrically in contact, and the common surface between them being no part of either solid, it can have no thickness.

By precisely the same course of reasoning, he argues that a line has no breadth. If we suppose a surface or plane to be divided by a geometrical line, and call the two parts of the plane C and D, he considers the geometrical line the common boundary alike of each part, and affirms that this line can have no breadth. For, says he, if the line have any breadth, it must be a part of the plane C, or of the plane D, or a part of each of them. But it cannot be a part of C, because if C be removed the line still remains as the boundary of D. Nor can it be a part of D, because if D be removed, the line still remains as the boundary of C. Therefore, considering the edges of the two planes as in geometrical contact, and the boundary line between them being no part of either plane, he comes to the conclusion that the line is absolutely without breadth.

To see the fallacy of this conclusion, or rather of the premises on which it rests, it seems only to be necessary to look at both solids at the same time, the one which is removed as well as the one which remains unmoved. The argument is, if A be removed from B, the common surface between them still remains as it was, the surface of B. But what becomes of poor A in this predicament? Is it sent off into the world without any surface to its back? Have we not as good reason to say that the common surface goes with A, as to say it remains with B? It cannot of course be pretended that A, after being removed, has no surface at all upon that side where it had been in contact with B. And if the two solids while in contact, have but one common surface between them, how shall it be decided which of the two retains the surface when they are separated? Has B, in consequence of remaining unmoved, a stronger attraction for the common surface, than is possessed by A, which is removed? To prevent the possibility of any imaginary advantage being possessed by B in this respect, let the two solids be separated by removing both equally, the one to the right and the other to the left. Which of the two solids, in that case, shall retain the surface that was inherent in them both, and common to them both while in contact? Perhaps it may then be said that the surface is divided between them, each taking its part. If so, and it be still contended that their common surface, when in contact, had *no thickness*, it would seem to follow that the surface of each, when separated, must be *half the thickness of nothing*.

The truth is, the premises in the case are all wrong, and the conclusions, to use again the words of Bacon, can only lead to those "cobwebs of learning, admirable for the fineness of thread and work, but of no substance or profit."

Now, I entirely deny that the two solids, when in contact, have but one surface which is common to them both. On the contrary, I affirm, that each has its own surface entirely distinct and separate from the other; and not only distinct and separate, but that the surface of each occupies a position and place entirely different from the other; and also that each surface has a thickness as definite and as measurable as the solid itself. But before proceeding to describe these surfaces and define their position and value, it may be profitable to pass to a consideration of other topics, which are requisite to afford a clear and distinct view of the whole matter.

SECTION III.

THE TRUE VIEW OF GEOMETRY.

I WOULD call Geometry, the science of magnitude, which measures and compares *extension* and *forms*. And in order to start aright, and scatter light and not darkness in our path, it is important for the mind to obtain a clear view of the *nature* of the extensions and magnitudes which we measure and compare.

The great philosopher of antiquity, whom Dr. Barrow calls "the most subtle and very learned Aristotle," makes this very important remark, viz.—"Mathematicians do neither want nor use infinite magnitude, but *take as much as they please*, when they are minded to terminate it." And this is really the true foundation of all geometry. Our measures of extension or magnitude are not, and cannot be, drawn from infinity. They have no proportion or relation to space infinitely extended or infinitely diminished. Our standards of measure are made by ourselves, from material substances which can be reached and comprehended by our senses, and are consequently *finite*. And by these measures, we never measure positive magnitude, or positive space, as a definite portion of *all* space, but only measure and compare magnitudes or spaces, that are relative and proportional to the standards which we have ourselves made and adopted.

All that is measured in space, may be called quantity or magnitude, and is always and only relative to the standard which we have *assumed* for the measurement. Our measures are noted and designated by numbers; and the standard of every kind of measure must of necessity be unity, or *one*.

In making or determining any standard of measure, in the words of Aristotle, "we take as much magnitude from the infinite space as we please, and then terminate it." For instance, we take the common breadth of a man's thumb, and agree to call it *one inch*. And twelve of these being about equal to the length of a man's foot, we agree to call twelve inches *one foot*. And in this, and similar ways, all measures are established. When a standard of measure is established, we can apply it to any quantity or magnitude which we can reach or comprehend, and tell whether such magnitude is less or greater than our measure, or how many times our measure must be repeated to equal the magnitude. But it must not be forgotten that this gives us no knowledge of the nature or value of *positive* quantity in infinite space. The human intellect is not capable of fathoming or comprehending positive or absolute quantity. No *finite* quantity is of itself great or small, but only so by comparison, or relative to some other quantity. The mind cannot conceive a magnitude so small, but there may be another still smaller, nor yet one so large, but that a larger one may still lie beyond it. Absolute quantity or magnitude can be comprehended by Him alone, whose instrument it is;

"To *Him*, no high, no low, no great, no small,
He fills, he bounds, connects, and equals all."

We cannot obtain the idea of *extension* or *form*, except from *material* substances. Could our minds exist in entirely free space, void of all matter, they could know nothing of extension or form. Infinite space would be one uniform thing; an unbroken, invariable *unit*. It is therefore impossible that such a thing as a *line* or a *surface* can have existence in nature, unless it is formed of some material substance, which occupies a portion of space. And if a line and a surface in nature are of necessity formed of material substances, and of necessity occupy a portion of space, they must with equal necessity possess breadth and thickness; and they must with equal necessity possess *extension* in every direction from their center. Now, the measurement of extension is precisely the object of geometry; and lines without breadth and surfaces without thickness are imaginary things, of which this perfect and exact science can take no cognizance. How vain, therefore, are all those speculations, where these airy nothings are attempted to be forced upon geometry and mingled with its pure demonstrations. And with how much force does the language of Bacon apply here—"For the wit and mind of man, if it work upon *matter*, which is the contemplation of the creatures of God, worketh according to the *stuff*, and is limited thereby. But if it work upon itself, as the spider worketh his web, then it is endless."

I think it clear that every thing which can come within the reach of geometry, must have extension;

must have magnitude; must occupy a portion of space; must have *extension in every direction from its center.*

Now what is the instrument, and the name of the instrument, with which geometry always works? The instrument, by which it measures all magnitudes, is a limited magnitude of a definite form; and the name of that magnitude, that instrument, is unit, or *one.* One square, one cube, one circle, one sphere, one triangle, one tetrahedron, &c., are in their nature units of magnitude, having definite but different forms. The square form, or rather the cubic form, is the one which has been universally adopted, and is undoubtedly the most convenient in practice, for the measurement of magnitudes of all forms. And when other forms have to be measured, they require to be geometrically decomposed, so to speak, and reduced to the cubic form, in order to express their value in magnitude; that is, to express the number of *cubic* units which they contain. It is seen, then, that the cubic unit is the proper instrument of geometry, wherewith it accomplishes all its wonderful work. The cubic unit is its starting point, its first stepping-stone. It has already been seen, when we put this instrument into the hands of geometry to work with, how we fix or determine its size or quantity. "We take as much magnitude as we please," and call it *one;* and geometry does all the rest. It takes the instrument given it, and applies it in a thousand ways, to all definite magnitudes, and among all definite forms, and returns to us an exact account of its labors, with every thing perfectly measured, nothing remaining over and nothing falling short. If we follow the footsteps of geometry by the light of these principles, it will not lead us into a dark labyrinth from which we cannot escape; and while it unfolds to us the beautiful relations and harmonies of all forms, we shall not, like Plato, "lose the real fruits of our labors by considering of forms as absolutely *abstracted from matter*."

Geometers say, there are *three kinds* of quantity in geometry—lines, surfaces, and solids; and that these quantities are not homogeneous; that they are different in their *nature*, each having its own peculiar unit; and that, as quantities or magnitudes, they cannot be measures of each other.

I say there is but *one kind* of quantity in geometry, and that lines, surfaces, and solids are all of the *same nature*, having identically the same unit, and of course are always perfect measures of each other, both arithmetically and geometrically. They are measures of each other in numbers, and they are measures of each other in quantities or magnitudes.

The nature of a *point* in geometry is rightly given in the books. It has position, but not magnitude. It is not a *thing* which geometry recognizes as a measure of any magnitude, or a constituent part of any magnitude. It forms no part of a line, or a surface, or a solid; but is simply *an index of place, or position* of lines, surfaces, and solids. If we bisect a line, which is a positive magnitude, the place of bisection, where the two halves of the line meet each other and are in contact, we designate by calling it a point; thus really accomplishing the poetic paradox of giving "to airy nothing a local habitation and a name."

But when we come to lines and surfaces, geometry may be said to have already stepped upon *solid ground* for it is then really on the ground of *solids*. A mathematical line is not a filmy, airy thread, reasoned down to infinity—to an imaginary *nothing;* but it is a real magnitude, a positive quantity, used to measure and compare positive quantities. It has already been indicated that a *unit* is the name or representative of any assumed magnitude to which it is applied. The definite size of the unit may be infinitely varied, as the magnitudes or quantities in nature or space are infinitely various. "We take as much as we please," and call it *one*. The unit not only represents a magnitude, but it represents a magnitude of *a definite form*. And, as already stated, the form universally adopted in the world as the standard of measure is the *cubic* form. Since the unit always represents a definite magnitude—and a magnitude from its very nature has *an extension in every direction from its center*—the unit is necessarily the representative of something that is extended in every direction from its center. Therefore the unit means not only one in length, but one also in breadth, and one in thickness. One inch, for instance, in pure geometry or mathematics is always one *cubic* inch; but when the object in any process is only to measure a line, or extension in one direction, it is necessary to use in the measurement only one dimension of the unit—that is, the linear edge of the cube; and this we apply along the line, repeating it, till the measure is completed. And in this operation, having no use whatever for the breadth or thickness of the unit, geometers have fallen into the error of regarding a line as length without breadth.

In like manner, when the object is to measure a surface, or extension in two directions only, length and breadth, it is necessary to use in the measurement only two dimensions of the unit—viz., its length and breadth. That is, we use *one face* of the cube, which is a simple square, and this we apply to the quantity or area to be measured, a sufficient number of times to complete the measurement. And in this operation, having no use whatever for the thickness of the unit, geometers have fallen into another error —that of regarding a surface as length and breadth without thickness.

But in both cases, the unit we have been using is the representative of a magnitude, and a magnitude of a *definite form and value;* and the unit never changes its form or value, because in measuring a line we disregard its breadth, nor because in measuring a surface we disregard its thickness. In every unit of the line, and in every unit of the surface, the perfect cube has an implied existence;—that is, the unit, wherever and however employed, possesses in its own right the value of the perfect cube, and is capable of vindicating its claim to that value, in every diagram and geometrical demonstration that can be presented.

It follows that a mathematical line is made up of a succession of single and equal units; and therefore a mathematical line has always a breadth of ONE. Also, that a mathematical surface is made up of a succession of single lines, and therefore a mathematical surface has always *a thickness of* ONE.

These conclusions are stated with boldness ar without hesitation, because they are abundantly pable of being proved in all geometrical demonstrations, and are proved in a great variety of demonstrations in the following pages. The very simplicity of these conclusions—that all lines have a breadth of one, and all surfaces have a thickness of one, may strike some as an argument against their validity. But I regard it as an argument, if any argument at all were needed, decidedly in favor of their truth; for all the operations and reasons of nature, when we once get at them, are found to be very simple. Sir Isaac Newton, who probably looked more widely and deeply into the works of nature than any other philosopher has hitherto done, remarks that "Nature is pleased with simplicity, and affects not the pomp of superfluous causes."

SECTION IV.

COMMENSURABLE AND INCOMMENSURABLE QUANTITIES.

Strictly speaking, there are no quantities or magnitudes in nature that are incommensurable. We can bind and tie up portions of magnitude or quantity by the units we ourselves make and limit, so that while bound by these fetters they cannot measure each other. But all magnitudes and quantities, whether of matter or space, are in their own absolute natures commensurable. Whether *matter* is infinitely divisible, or whether possible division at last terminates in ultimate particles or atoms, is a question on which the most acute thinkers have not been agreed. It is probably a question beyond the reach of the human intellect. Sir Isaac Newton, in speaking of the properties of bodies or solids, remarks as follows:

"The extension, hardness, impenetrability, mobility, and *vis inertiæ* of the whole, result from the extension, hardness, impenetrability, mobility, and *vires inertiæ* of the parts; and thence we conclude the least particles of all bodies to be also all extended, and hard, and impenetrable, and movable, and endowed with their proper *vires inertiæ;* and this is the foundation of all philosophy. Moreover, that the divided but contiguous particles of bodies may be separated from one another, is matter of observation; and, in the particles that remain undivided, our minds are able to distinguish yet lesser parts, as is mathematically demonstrated. But whether the parts so distinguished, and not yet divided, may *by the powers of nature* be actually divided and separated from one another, we cannot certainly determine. Yet, had we the proof of but one experiment that any undivided particle, in breaking a hard and solid body, suffered a division, we might, by virtue of this rule, conclude that the undivided as well as the divided particles may be divided and actually separated to infinity."

But whether matter is infinitely divisible or not, in either case the common measure of two quantities of matter will be reached if the division of them be continued far enough. For if matter be infinitely divisible, there is an infinite number of divisions in which to seek the common measure. Or if the possible division of matter terminates in ultimate particles, which are indivisible, the common measure of two quantities made up of those particles, if not reached before, will certainly be found at last in the ultimate particle.

Whatever may be the fact in nature with regard to the infinite divisibility of matter, there seems to be as much reason to believe *space* infinitely divisible, as there is to believe it infinitely extended. And the laws of geometry apply to all space as well as to all matter. Space is both divisible and extended, beyond the reach of the human intellect. The mind can reach no terminus in either direction. To us, therefore, space is both infinitely divisible and infinitely extended. The profound and subtle Aristotle affirmed, that "whatsoever is *continued* is divisible into parts, again divisible." By "continued" is here meant, whatever has extension, as extension of matter or space, or extension of time or duration. He also tells us that "Plato does therefore make two kinds of infinites, because he thinks there is an infinite procedure both in augmentation and diminution."

With regard to incommensurable quantities, therefore, which have always been so troublesome and perplexing to mathematicians from Euclid's time down to the present day, it may be one step toward getting over these difficulties to know that no such quantities really exist in nature. They are the creations of the mathematicians themselves in tying up quantities into indivisible units, and then attempting to measure the constituent parts of a unit by the whole unit, while thus bound up and indivisible; or attempting to measure a unit of one size by a unit of another size, while both are bound hand and foot, limited and indivisible. And here is seen the reason, for instance, why the diagonal of any square is incommensurable with the side of the square; and why the diagonal of a cube is incommensurable with the linear edge of the cube. They are all relative and constituent parts of the unit, and while thus bound, they form an indivisible whole;—that whole is a perfect cube. Its linear edge is 1, its face is 1, and its solidity or whole body is 1. And whether the positive size or breadth of that cube equal the thousandth part of a hair, or a thousand times the breadth of the earth, it is still, as a unit, one indivisible thing. The diagonal of one of its façes is the square root of 2, and is expressed in numbers by 1.4142+; and the diagonal of the cube, or of the whole unit, is the square root of 3, which is 1.732+; and neither of these expressions can be measured by 1, because the quantities they represent, while bound in their present form, are nothing but the constituent and indivisible parts of the cube or unit; and the cube cannot be applied to its own parts and measure itself. It cannot measure its own diagonal, or the diagonals of its faces. Therefore these diagonals are not commensurable with the side of the square or face, which is 1, or with the whole cube, which is 1. But though, while inherent in the cube, and parts of it, they cannot be commensurable with the cube, yet the very limitation imposed upon them, while thus bound, makes them the roots of other quantities, which are commensurable with the cube. The diagonal of the cube is the square root, or root of a square, whose quantity is just three times as large as the cube; and the diagonal of the face is the square root, or root of a square, whose quantity is just twice as large as the cube.

Here, too, is seen the reason why the solution of the celebrated problem, styled "the duplication of the cube," is in its very nature absolutely *impossible*. This famous problem has been a puzzle to mathematicians, equal to that of the quadrature of the circle, for two thousand years; and is said to have been first proposed by the Oracle of Apollo at Delphos. If so, it proves the Oracle to have been an expert geometer in the received principles of the science, or that he had one of the most expert geometers of antiquity to whisper in his ear. The story is this: While a plague was raging at Athens, the Oracle of Apollo at Delphos was consulted to know when and how the progress of the pestilence could be stayed. The Oracle gave for answer, that the plague should cease when Apollo's altar, which was in the form of a cube, should be doubled. This answer was probably a safer one for the Oracle than even he himself was aware; for as the Omniscient ruler of the world alone could tell when the plague should cease, so none but the Omnipotent geometer of the universe had the power to solve the problem proposed. To man, who has no means of solving it, but by the use of numbers, the principles already laid down show the solution to be an utter impossibility.

The meaning of the question proposed was this: Apollo's altar being supposed to be a perfect cube, and its dimensions exactly given—to find the exact dimensions of another perfect cube containing just double the solidity or bulk of Apollo's altar. Two such cubes may unquestionably exist in nature; but we have no power to measure one by the other, so as to compare exactly their contents; which I think must be evident, if we recur again to the origin of the *unit*, its nature and value, and its uses in geometry. It has already been shown that we can only measure magnitudes by a limited magnitude of some definite form; and that the form universally adopted in the world, as the standard of measure, is the cube; that being the most convenient in practice. Now, let us suppose, at a point in space, a quantity of matter or space exists, infinitely small, or as small as the imagination can reach, but in the form of a perfect cube.

And if we suppose this cube to increase in magnitude, *to grow*, as it were, with a uniform increase of extension in every direction from its center, till it becomes a cube infinitely extended, or as large as the imagination can reach; it is manifest that it will have been a perfect cube at every point of progress during the augmentation, and that in this way the whole may be said to contain an infinite series of perfect cubes, from magnitude infinitely diminished to magnitude infinitely extended. Here, then, is the infinite material, out of which we are to make our units; and, to recur again to the idea of Aristotle, "we take as much as we please," and call it *one*. Suppose we arrest the growing cube at that point where it has exactly reached one cubic inch, and call it one. We have then fixed and determined a standard of measure; we have made our unit; we have tied up our quantity or magnitude into one indivisible thing, which, even though it may be nothing but empty space, is still, in the hands of geometry, harder than adamant, forever invariable in quantity and forever invariable in form. To this quantity or unit we give the name of one cubic inch. Now, suppose the great Geometer of the universe to arrest the growing cube at that point where its quantity or bulk is just double the cubic inch, and to require of us its measurement. We apply our standard of measure, the unit we have made, to this new cube, and endeavor to get a comparison; but we find no agreement. The linear edge of our unit is too short, and its face is both too short and too narrow. If we apply the linear edge of our unit twice, we find no agreement there, for its double is too long for the linear edge of the new cube. And since our unit is one indivisible thing, of a fixed form, it is manifest that it can never reveal to us exactly the quantity or bulk of a magnitude, to which it can show no equality or agreement when applied to it. No cube, therefore, can be an exact measure of another cube of a different size, till it reaches one exactly eight times its own bulk. For if we take our cubic inch and place it on a table, and place another cubic inch by its side, face meeting face, we then have a line of two inches. Our unit can measure that line, because it agrees with its fellow-unit as well as with itself. If we place two more units by the side of this line, we have a square of four inches, and our unit can measure this square, because it agrees with each of the units in it. If we place four more units upon the top of this square, we then arrive at another perfect cube containing eight of the original units. Now, if we apply our unit to the new cube, we find that its linear edge, applied twice, exactly measures the linear edge of the new cube, and the face applied four times, exactly measures the face of the new cube. And we are delighted with the certainty of the truth revealed, that one perfect cube is just eight times as large as the other. Our unit can measure this new cube, because it agrees with each of the eight units of which it is composed. But as no cube can possibly be composed of units of the same size, and containing any number of units between one and eight, no cube can be a measure of another cube that is double its quantity, or triple its quantity, or any larger quantity, till it reaches the cube which is just eight times as large as itself. And hence the reason why the double of Apollo's altar could not be perfectly given.

The next perfect cube in size, which can possibly be composed of units, must of necessity contain twenty-seven units. For if we increase the linear edge beyond two, the next perfect measure must be three; and still retaining the cubic inch as the unit, we shall have a line of three inches. Placing three such lines side by side we obtain a square, containing nine inches. And three such squares, placed in succession one above another, will form a perfect cube, containing twenty-seven inches or units. This cube can be perfectly measured by the unit, because the unit agrees with every single unit of which it is composed.

In like manner, if we seek for the next perfect cube which can possibly be composed of units, we find it to contain sixty-four units. Four units in a line make the linear edge. Four such lines side by side make a square of sixteen units; and four such squares, placed in succession one upon another, make a perfect cube of sixty-four units. In like manner, if we take a linear edge of five units, we shall find that the next perfect cube which can possibly be composed of units, contains a hundred and twenty-five units. Again, if we take the next perfect linear edge, which must be six, we shall find that the next perfect cube which can possibly be composed of units, contains two hundred and sixteen units. And by the same process, taking seven units for a linear edge, we shall find that the next perfect cube which can possibly be composed of units, contains three hundred and forty-three units.

And taking eight units for a linear edge, we find the next possible perfect cube composed of units, contains five hundred and twelve units. And with nine units for a linear edge, we find that the next possible perfect cube composed of units, contains seven hundred and twenty-nine units. And in like manner the next possible perfect cube must contain a thousand units, having ten for its linear edge.

Though mathematicians talk about "the three roots of a cube, one real and two imaginary," I cannot possibly conceive of a cube having more than one root, and that is its linear edge. The "two imaginary roots" I presume must be the result of some very acute algebraical process, so searching in its operation as to discover the difference in the value of "*three times nothing, and twice nothing.*" And perhaps the imaginary roots may, like the *zeros,* have a ratio to each other as 3 to 2.

But it is the "real root" of the cube that we are now dealing with. And as it is not possible to have more than nine perfect cubes composed of units, till we arrive at one thousand, so there are but nine numbers from 1 up to 1000, of which a perfect cube root can be found. These nine numbers are—1; 8; 27; 64; 125; 216; 343; 512; and 729.

The same reason, which limits the number of perfect cubes and perfect cube roots, also limits the number of *perfect squares* to thirty-one, which can possibly be composed of units under one thousand. So there are but thirty-one numbers under a thousand, which can possibly have a perfect square root. The smallest of these numbers is 1, and the largest is 961. And there are but *two* numbers under one thousand which can have both a perfect cube root and a perfect square root; and these numbers are 64, and 729.

From these considerations it must be obvious that all *surd* quantities in mathematics are the necessary results of the fixed and invariable *form* of the assumed unit, and do not arise from any incommensurable quantities actually existing in nature. But though we are not able to obtain the perfect root of these quantities, it is well known that by fractions, or assuming *smaller units* which shall have an exact proportion to the given unit, we may obtain an approximate root of all numbers whatsoever, and carry the approximation as near to the true root as we please, or as far as we can handle numbers and comprehend their value; but, go as far as we may, we can never reach the perfect root, and therefore cannot in this operation entirely satisfy the requirements of geometry, which is content with nothing short of perfect agreement.

It is interesting to observe that all the perfect squares which can possibly be formed of units, are formed by the successive additions of all the *odd* numbers from one, upward. Thus the first perfect square is 1, and its root is 1. Add to 1 the next odd number, 3, and it makes 4, which is the next perfect square, whose root is 2. Add the next odd number, 5, to 4, and it makes 9, which is the next perfect square, whose root is 3. Add to 9 the next odd number, 7, and it makes 16, which is the next perfect square, whose root is 4. And so on, as in the following table, embracing all the perfect squares which can possibly be formed of units under one thousand.

Successive Sums of the odd Numbers.	*Perfect Squares.*	*Perfect Roots.*
1	1	1
1 + 3	4	2
1 + 3 + 5	9	3
1 + 3 + 5 + 7	16	4
1 + 3 + 5 + 7 + 9	25	5
1 + 3 + 5 + 7 + 9 + 11	36	6
36 + 13	49	7
49 + 15	64	8
64 + 17	81	9
81 + 19	100	10
100 + 21	121	11
121 + 23	144	12
144 + 25	169	13
169 + 27	196	14
196 + 29	225	15
225 + 31	256	16
256 + 33	289	17
289 + 35	324	18
324 + 37	361	19
361 + 39	400	20
400 + 41	441	21
441 + 43	484	22
484 + 45	529	23
529 + 47	576	24
576 + 49	625	25
625 + 51	676	26
676 + 53	729	27
729 + 55	784	28
784 + 57	841	29
841 + 59	900	30
900 + 61	961	31

The necessity of the results exhibited in the preceding table will more clearly appear from an examination of the annexed diagram.

We will take the small square in the corner of the diagram, marked 1, for a unit. It is then a fixed quantity both in magnitude and *form*. It is 1 in length, and 1 in breadth, and is really also 1 in thickness. But we now entirely disregard the thickness, because we are considering it as a plane figure, in which extension is measured or considered only in

1	3	5	7	9
3	3	5	7	9
5	5	5	7	9
7	7	7	7	9
9	9	9	9	9

length and breadth. This unit, which we have assumed, is an indivisible thing, invariable in form. It is 1 square. This instrument is just what we have made it; it can never be anything else; and we must work with it as it is, or with other units of precisely the same size and form. Now, from inspection or trial, it is obvious that the next square which can possibly be formed of such units, or measured by such units, must be made by adding three more units to the first square, as is done in the diagram by adding the three units or squares marked 3. It is then seen to be true on the diagram, as it was in numbers in the preceding table, that the two first odd numbers, 1 and 3, added together, make a perfect square, containing four units. On further inspection, it will appear that the next square, which can possibly be formed of such units, must be made by adding five more units, as those marked 5 in the diagram. We then get another perfect square, containing nine units. This square is made up, both in magnitude and numbers, by adding together the three first odd numbers, 1 unit, and 3 units, and 5 units. In like manner, the next square, which can possibly be formed of such units, must be made by adding seven more units, as those marked 7 in the diagram. This gives a perfect square, containing sixteen units, made up of the first four odd numbers, or of the squares which the numbers represent, 1 unit, 3 units, 5 units, and 7 units. And by adding nine more units, the next odd number, as is done by adding those marked 9 in the diagram, we have another perfect square, containing twenty-five units. This square has a perfect root, which is 5, for the length of its side is 5 units. In like manner perfect squares and perfect roots may be shown by diagram to agree with numbers, as in the table, as far as we choose to carry them.

From these considerations and many others which will subsequently appear, it will be manifest, that in handling abstract numbers we are in reality handling *positive magnitudes;* and that numbers in their own essence are nothing but *signs* of those magnitudes, while absolute geometrical quantities having extension are truly *the things signified.*

It is worthy of remark here also, that the difference of the squares of any two consecutive numbers, from one upward, is an odd number, and this odd number equals the sum of the two consecutive numbers which are the roots of the squares. Thus, take the two consecutive numbers 2 and 3. The square of 2 is 4, and the square of 3 is 9; and the difference between 4 and 9 is 5, and that equals the sum of 2 and 3.

Take the consecutive numbers 3 and 4. The square of 3 is 9, and the square of 4 is 16; and the difference between 9 and 16 is 7, and that is the sum of 3 and 4.

Take the consecutive numbers 4 and 5. The square of 4 is 16, and the square of 5 is 25; and the difference between 16 and 25 is 9, and that is the sum of 4 and 5. And so on, for all consecutive numbers.

SECTION V.

CIRCUMFERENCE, DIAMETER, AND AREA OF PLANE FIGURES.

Plane figures are those in which extension is measured or considered in two directions only, length and breadth, disregarding entirely their thickness. The area, or quantity of extension of a plane figure, is determined and measured by a diameter, a line passing through the center of the figure, and its circumference, a line outside of the figure and inclosing it. And these lines, as well as all other lines, have always a breadth of one.

By a necessary law of numbers and quantities, twice the square root of any given quantity or number, less than 4, is greater than the given quantity. And twice the square root of any given quantity or number, greater than 4, is less than the given quantity. But if 4 is the given quantity, twice the square root is equal to the given quantity. And here we arrive at a general, simple, and beautiful law, by which diameter controls the relation between area and circumference, and by which circumference controls the relation between area and diameter. This law, in general terms, is as follows:—If diameter be 1, the area equals one-fourth the circumference. If diameter be 2, the area equals two-fourths, or one-half the circumference. If diameter be 3, the area equals three-fourths the circumference. If diameter be 4, the area equals the circumference. And if diameter be greater than 4, circumference is less than area. This general law applies to all regular plane figures, the circle, equilateral triangle, square, pentagon, hexagon, and all regular polygons of any number of sides.

But, in all these cases, the law requires that the diameter, used in the measurement, shall be *the diameter of the inscribed circle.* In the equilateral triangle, if the inscribed circle (touching each side of the triangle) has a diameter of 1, then the area of the triangle equals one-fourth of its circumference. If the diameter of the inscribed circle be 2, the area of the triangle will equal one-half its circumference. If the diameter be 3, the area will equal three-fourths of the circumference. And if the diameter of the inscribed circle be 4, the area of the triangle will just equal its circumference. It will just equal the circumference if computed in numbers, and just equal it if measured in space or geometrical quantity. For the line of circumference having a breadth of 1, it occupies and fills a portion of space as truly as the area of the triangle. And when the diameter of the inscribed circle is 4, the quantity of space occupied by the line of circumference of the triangle, is just equal to the quantity of space or area of the triangle itself.

So also in the square, the pentagon, hexagon, or any regular polygon, if the inscribed circle (touching each side of the figure,) has a diameter of 1, the area of the figure equals one-fourth of its circumference. If the inscribed circle has a diameter of 2, the area of the figure equals one-half its circumference. If diameter is 3, the area equals three-fourths of the circumference. If the diameter of the inscribed circle is 4, the area of the figure equals its circumference. And if diameter is greater than 4, circumference is less than area.

By the same general law, and in the same manner, circumference, when given or assumed, controls the relation between area and diameter. In the circle, triangle, square, pentagon, hexagon, and all regular polygons of any number of sides, if circumference be 1, area equals one-fourth of the diameter. If circumference be 2, area equals one half the diameter. If circumference be 3, area equals three-fourths of the diameter. If circumference be 4, area and diameter are equal. And if circumference be greater than 4, diameter is less than area.

But here also the law requires, that in all these cases, the diameter, used in the computation, shall be the diameter of the inscribed circle. Thus it seems that circumference and diameter, in a certain sense, cross each other at the point of 4. It is a universal law of all plane figures, if circumference is less than 4, diameter is greater than area. If circumference is greater than 4, diameter is less than area. And on the other hand, if diameter is less than 4, circumference is greater than area; and if diameter is greater than 4, circumference is less than area.

After tracing these general laws of circumference, diameter, and area thus far, I perceived that they have a still more general and wider application than has been already laid down; for they apply, not only to all *regular* plane figures, but also to all *irregular* rectilineal figures, provided their lines of circumfer ence are so drawn that an inscribed circle shall touch every side. Thus, we may have plane figures of three, or four, or a dozen, or twenty sides, and all the sides irregular or unequal in length, and yet if these lines are so arranged that an inscribed circle shall touch every side of the figure, the magic power of the circle holds them all to the same general law. If the diameter of the circle is 1, the area of the figure is one-fourth its circumference. If diameter is 2, area is one-half the circumference. If diameter is 3, area is three-fourths the circumference; and if diameter is 4, area and circumference are equal.

If the circumference of the figure is 1, its area is one-fourth the diameter of the inscribed circle; if circumference is 2, area is one-half the diameter; if circumference is 3, area is three-fourths the diameter; and if circumference is 4, the area of the figure is equal to the diameter of the inscribed circle.

SECTION VI.

DIAMETER, SOLIDITY, AND SURFACE OF SOLID FIGURES.

Solid figures are those in which extension is measured or considered in three directions—length, breadth, and height, or thickness.

Plane figures really have extension from their centers to every point of their circumferences; but as they are all measured by being brought into squares, and squares are considered as having extension in length and breadth only, therefore all plane figures are said to be those in which extension is measured in two directions, length and breadth.

So all solid figures really have extension in every direction from their centers to their surfaces; but as they are all measured by being brought into the form of a cube, and a cube is considered as having extension in length, breadth, and thickness only, therefore all solid figures are said to be those which have three dimensions, or in which extension is measured in three directions, length, breadth and thickness.

The term *solidity*, in geometry, simply means *bulk*, or the amount or quantity of extension which is to be measured. So that, in geometry, a cubic inch of air, or even a cubic inch of empty space, has exactly the same solidity as a cubic inch of gold.

The terms *surface* and *superficies*, which are used indiscriminately in geometry as synonymous, require some particular consideration. As we have given thickness to surface, and made it a solid body, or body having extension, it becomes more important to find a place for it and define its locality, than it would be, if it were the airy nothing which it has heretofore been considered. In the view taken of surface by Professor Simson and other geometers, it is considered as inherent in the solid, but existing at the extreme limits of the solid. In the cube, for instance, they make the surface to be identically the same thing as the six faces of the cube. But face and surface are different words, and literally and truly have different meanings. Face is a very proper word to represent the extreme limit of the cube; but *sur*-face, or *super*-facies, literally and properly means that which is *upon* the face. And that is precisely what is wanted in a term to express the true meaning and locality of surface. So that we find the right and proper term already in use, though always used heretofore without its true and legitimate meaning being attached to it. The surface of a cube, therefore, is a quantity or magnitude lying *upon* and exactly covering each face of the cube, and always having a thickness of *one*. And the same definition of surface applies to all solids, of whatever form. The surface covers all the faces with a thickness of one.

Now, as in plane figures the diameter of an inscribed circle holds all rectilineal figures to one general and simple law, so in solid figures the diameter of an inscribed sphere holds all solids with plane surfaces to a similar law, equally general and equally simple.

The only difference in the two cases is this; the point of equality in plane figures is 4, while the corresponding point of equality in solid figures is 6. And the reason of this will be manifest, if we look again for a moment at the cubic unit, by which all magnitudes, both plane and solid, are really measured. Let us take a cubic inch and place it upon a table, and call it a unit, or one inch. First, if we consider it as a plane figure, its extension is measured in two directions, length and breadth, and we call it one *square* inch. It has four sides, or faces turned outward horizontally, and its circumference must cover these four faces. Its circumference therefore is 4; which will more distinctly appear when we come to the demonstrations in Part Second. If we regard the cubic inch as a solid figure instead of a plane, we must then consider its extension in three directions, length, breadth and thickness; and instead of circumference it must be provided with a surface. Then, to the four faces, which are turned outward horizontally we must add the face at the top and the face at the bottom, making six faces, which must be covered by the surface. The surface therefore is 6. The same identical unit, which, as a plane figure, has a circumference of 4, as a solid figure, has a surface of 6. And as circumference and diameter, in governing the area of plane figures, make a point of equality at 4, so surface and diameter, in governing the solidity of solid figures, make a point of equality at 6. Accordingly we find, if the diameter of a sphere is 1, its solidity is equal to one-sixth of its surface. If its diameter is 2, its solidity equals two-sixths or one-third its surface. If its diameter is 3, its solidity equals three-sixths or one-half its surface. If diameter is 4, solidity equals four-sixths or two-thirds of the surface. If diameter is 5, solidity equals five-sixths of the surface. If diameter is 6, solidity equals the surface.

On the other hand, if the surface of a sphere is 1, solidity equals one-sixth of the diameter. If the surface is 2, solidity equals two-sixths or one-third of the diameter. If the surface is 3, solidity equals one-half the diameter. If the surface is 4, solidity equals two-thirds of the diameter. If the surface is 5, solidity equals five-sixths of the diameter. If the surface is 6, solidity and diameter are equal.

This same general law which thus governs the solidity of the sphere, applies also to all solids with plane surfaces. But the law requires that the diameter used shall be the diameter of an inscribed sphere; that is, a sphere which shall touch every plane constituting the surface of the solid. Thus in the tetrahedron, the solid bounded by four plane equilateral triangles, if the diameter of the inscribed sphere is 1, the contents or solidity of the tetrahedron equals one-sixth of its surface. If the diameter of the inscribed sphere is 2, the solidity of the tetrahedron equals one-third its surface. If the diameter is 3, solidity equals one-half the surface. If the diameter is 4, solidity equals two-thirds of the surface. If the diameter is 5, solidity equals five-sixths of the surface. If the diameter is 6, the solidity of the tetrahedron and the surface of the tetrahedron are equal. On the other hand, if the surface of the tetrahedron is 1, its solidity equals one-sixth of the diameter of its inscribed sphere. If the surface of the tetrahedron is 2, its solidity is one-third the diameter of the inscribed sphere. If the surface is 3, solidity is one-half the diameter. If surface is 4, solidity is two-thirds the diameter. If surface is 5, solidity is five-sixths of the diameter. If surface is 6, solidity and diameter are equal.

And so the hexaedron or cube, the octaedron, the dodecaedron, and icosaedron, are all perfectly and rigorously bound by their inscribed sphere to the same general law.

SECTION VII.

NUMBERS ARE NOTHING BUT SIGNS OF QUANTITIES OR MAGNITUDES.

All geometrical quantities or magnitudes may be said to have a possible existence everywhere and always, throughout all space; that is, in any part of infinite space any possible magnitude may be assumed to exist, and the quantity of space thus assumed has a permanent existence; it is the same before we consider it, and the same afterward. It is not so with mathematical numbers. They are not things existing of themselves. They have no place or being in nature till we apply them to quantities as *signs* or *names*, by which we may express our knowledge or ideas of those quantities. Numbers therefore always being nothing but signs, and quantities or magnitudes always being the things signified, it follows of necessity that abstract numbers can do nothing whatever of themselves; but all they seem to do is really done by the quantities they represent. It is not the sign, but the thing signified, which truly performs the operation in every mathematical process. The number does not make or limit the value of the quantity, but the quantity stamps its own value, as it were, upon the number, making it the representative of the quantity. If we assume any abstract number first, say the number 1, it can give us no knowledge or idea whatever, for it has no meaning. We know not whether it may mean 1 egg, or 1 apple, or 1 mountain, or 1 anything else. But if we first assume some definite quantity or magnitude, say one cubic inch, the quantity is a real thing, making itself known to our senses, and giving us an idea of its amount of extension, without any sign or name being attached to it. If we then apply the number 1 to it, as the sign or name to represent it, the quantity stamps its value upon the number 1, so that it cannot mean 1 egg, or 1 apple, or 1 mountain, but must of necessity mean 1 cubic inch, and nothing else.

Mathematical numbers therefore must always represent magnitudes. Dr. Barrow, the distinguished and learned predecessor of Sir Isaac Newton in the mathematical chair at Cambridge, also took this view of numbers. But I have found no other writer who has considered them in the same light. "I am convinced," says Dr. Barrow, "that number really differs nothing from what is called continued quantity; but is only formed to express and declare it. And consequently, that arithmetic and geometry are not conversant about different matters, but do both equally demonstrate properties common to one and the same subject. And very many and very great improvements will appear to be derived from hence upon the republic of mathematics."

Again, he says, "There is neither any general axiom nor particular conclusion agreeing with geometry, but what by the same reason also agrees with arithmetic. And on the other hand, nothing can be affirmed, concluded, or demonstrated, concerning numbers, which may not in like manner be accommodated to magnitudes. Whence accrues a remarkable light and vast improvement to both sciences."

And yet again, says this very acute and philosophical thinker, "No geometrical argument is of force, which agrees not exactly with an arithmetical calculus. Also all true conclusions and lawful demonstrations in geometry may be illustrated and confirmed by help of an arithmetical calculus."

And again he adds, "I say that a mathematical number has no existence proper to itself, and really distinct from the magnitude it denominates."

Thus it is evident that Dr. Barrow arrived at the clear conviction, that mathematical number always represents *magnitude;* and to me it seems wonderful that he did not go a step further, and discover *how much* magnitude, that is, *how much relative magnitude,* every number truly represents. But the progress of every mind, clothed in the flesh, has its limits, as if infinite Wisdom had said to it, thus far shalt thou go, but no farther.

As all geometrical quantities or magnitudes may be said to have a possible existence everywhere and always, throughout all space, so all geometrical *forms* may be said to have a possible existence everywhere and always, throughout all matter and all space. That is, all geometrical forms are truly inherent and have a possible existence in every assumed quantity of matter, and in every assumed portion of space. This idea is also clearly and elegantly expressed by Dr. Barrow, as follows: "All imaginable geometrical figures are really inherent in every particle of matter; I say really inherent in fact, and to the utmost perfection, though not apparent to the sense; just as the effigies of Cesar lies hid under the unhewn marble, and is no new thing made by the statuary, but only is discovered and brought to sight by his workmanship, by removing the parts of matter which involve and overshadow it. Which made Michael Angelus, the most famous carver, say, that sculpture was nothing else but a purgation from things superfluous; for take all that is superfluous from the wood or stone, and the rest will be the figure you intend."

In the fact, that the intimate and inseparable relation existing between numbers, magnitudes, and forms, has not been clearly understood, may be found the principal cause of the difficulties and mysteries which have always attended the investigation of many mathematical subjects. From a want of this knowledge, Lord Bacon, in classing the sciences, seemed to be puzzled and in doubt, whether to place mathematics with the *physical* or *metaphysical* sciences. He, however, came to the conclusion that it was "more agreeable to the nature of things, and to the light of order, to place it as a branch of metaphysic; for the subject of it being quantity, not quantity indefinite, which is but relative, but quantity determined, or proportionable, it appeareth to be one of the essential forms of things, causative in nature of a number of effects."

The want of this knowledge, also, was undoubtedly the principal cause of the divisions and disputes between some of the celebrated schools of antiquity. Lord Bacon says, "In the schools of Democritus and Pythagoras, the one did ascribe *figure* [form] to the first seeds of things, and the other did suppose *numbers* to be the principles and originals of things." These schools might undoubtedly have shaken hands and united, had they but known, that numbers are nothing but *signs* of magnitudes and forms, and forever follow magnitudes and forms, as the shadow follows the substance, and can no more exist without magnitudes and forms than the shadow can exist without the substance which it represents.

The want of this knowledge also caused a strange confusion of terms among the ancients, which could not fail to throw a mist over their subjects of discussion, and lead to endless disputes. "For," says Dr. Barrow, "Aristotle improperly makes the Greek *to poson* to be the common genus of both *multitude*, as it is numerable, and *magnitude*, as it is measurable; which word properly signifies *quotity*, and only respects *number*. And, on the contrary, *quantitas*, by which word the Latins used to express the *to poson* of Aristotle, denotes *magnitude* only, and cannot properly be referred to *number*; which defect of a common name to magnitude and multitude is perhaps an argument that the ancients had no common perception of them."

Numbers are entirely blind with regard to absolute or positive magnitudes, for they know no difference between *great* and *small*, and can only recognize *relative* quantities of *definite forms*.

If we divide the circumference of any square, however great or however small, by the circumference of its inscribed circle, it will always produce the *same* quotient—viz., 1.273+, and this, too, whether the square contain an area of one inch or ten thousand inches;—and the *square root* of this quotient, thus obtained from all squares, is the diameter of a circle whose area equals 1 square.

So also, if we divide the area of any square whatever by the area of its inscribed circle, we always obtain the same quotient as when we divide the circumference of the square by the circumference of the circle—viz., 1.273+.

If we divide the diagonal of any square, however great or however small, by the side of the same square, we always obtain the same quotient, which is the square root of 2—viz., 1.4142+.

If we divide the surface of any cube, however great or however small, by the surface of its inscribed sphere, we always obtain one and the same quotient, viz., 1.9098+; and the *cube root* of this quotient is the diameter of a sphere whose solidity equals 1 cube. So also, if we divide the solidity of any cube by the solidity of its inscribed sphere, we obtain always the same quotient—viz., 1.9098+.

SECTION VIII.

IDENTITY OF PURE AND MIXED MATHEMATICS.

"The speculation was excellent," says Lord Bacon, "in Parmenides and Plato, although but a speculation in them, that all things by scale did ascend to unity. So then always that knowledge is worthiest, which is charged with least multiplicity."

In nearly all our works on mathematical subjects, we are told that mathematics are of two kinds, *pure* and *mixed*. If there be in reality such a distinction, founded in the nature and principles of the science, it should undoubtedly be recognized, and the boundaries of the two branches of mathematics clearly defined, so that the principles which belong exclusively to the one branch may not be confounded with those which apply to the other. But if it be merely a distinction without a difference; if all the principles of the one are identically the same with the principles of the other, is it not high time that so useless an absurdity were expunged from our books of science and instruction, and our mathematics clothed with such habiliments of simplicity as their principles will allow?

Pure mathematics, they tell us, considers and measures *abstract quantities*. And mixed mathematics considers and measures quantities as they exist in *material bodies*. Thus, if a mathematician considers and calculates the surface of a cube whose linear edge is

2, he finds it to be 24; and this is called a process of *pure* mathematics. If a carpenter makes a box, whose linear edge is 2 feet, the length, breadth, and height, all being the same, and measures and calculates the number of square feet of boards used in the construction, he finds it to be 24; and this is called a process of *mixed* mathematics. But the laws of computation are precisely the same in the two cases, nor is it possible to conceive any principle involved in the one, which does not apply equally to the other. In the cube, the linear edge is 2, which being squared makes one face of the cube to be 4, and there being 6 faces, 4 repeated six times, or multiplied by 6, makes all the faces or the whole surface to be 24. So in the box, the linear edge is 2 feet, which squared, or multiplied into itself, makes one side or one face of the box to be 4 feet, and this multiplied by 6, the number of sides or faces of the box, makes the whole surface 24 feet.

The truth is, the 2 used by the mathematician, in the abstract operation, as truly represents a *quantity*, or magnitude, as the 2 used by the carpenter. The only difference is, the quantity or size of the unit used by the carpenter is *defined*, and we know what it is, while the unit used by the mathematician is not defined, but left unlimited, and may mean a cube of any magnitude whatsoever, from magnitude infinitely diminished, to magnitude infinitely extended. And all the laws, principles, and results of abstract numbers in any process carried on by the mathematician, agree precisely with the laws, principles and results of the numbers used by the mechanic or the natural philosopher in their dealings with material bodies. One set of these numbers is certainly no more *pure* than the other, nor is one set any more *mixed* than the other. Why then should they be distinguished by different names, as though they possessed different natures, or different powers, or could work with different degrees of purity?

The absurdity of this division of mathematics has been observed and condemned by several writers of high authority, though they had not discovered the true nature and value of the unit. The author of the able article on mathematics in the London Encyclopedia, remarks on this subject as follows:

"Mathematics are commonly distinguished into pure and mixed. Pure mathematics, it is said, considers quantity abstractedly; and mixed mathematics treats of magnitudes as subsisting in material bodies, and consequently are interwoven everywhere with physical considerations. This is one of the objectionable distinctions, positions, and definitions, too frequently to be met with in connection with a science which boasts of accuracy and certainty. The notion of quantity abstractedly, or separately from material bodies and physical considerations, is manifestly absurd; for where or how can quantity exist, or be conceived of as existing, but in some material body? We might philosophize about color, form, or shape, solidity, fluidity, elasticity, gravity, &c., &c., abstractedly from material bodies, and physical considerations, [as we long indeed attempted,] and call this pure philosophy; but it would be a pure fiction of the brain

—a mere absurdity. If pure mathematics really consisted in such abstractions, they might be defined the science of non-existents. But even in the most imaginary quantity of the most absolute abstraction, the imagination [to say nothing of the understanding] of the purest or most speculative mathematician must have something of the nature of *materia firma* to rest upon."

To show that the process of reasoning and handling numbers is precisely the same in mixed mathematics as in the pure, a writer in the Edinburgh Encyclopedia gives the following illustration.

"The geometer, who reasons on the comparative weights of a globe of brass and a cube of water, is perfectly indifferent whether, in any particular globe presented to his view, the microscope may not discover superficial irregularities, or whether the instruments used for taking specific gravities be susceptible of mathematical precision. He reasons on them as creatures of his own imagination, agreeing in their forms and qualities with certain arbitrary definitions, and is fully aware, that in so far as bodies are to be found in nature which conform to those definitions, in so far only will his conclusions apply to them."

The truly philosophical and always correct Dr. Barrow remarks as follows: "In reality, those which are called mixed, or concrete mathematical sciences, are rather so many examples only of geometry, than so many distinct sciences separate from it; for when once they are disrobed of particular circumstances, and their own fundamental and principal hypotheses come to be admitted, they become purely geometrical." And again the same learned author remarks: "There is no reason why the doctrine of generals should be separated from the consideration of particulars, since the former entirely includes and primarily respects the latter. Whence it is altogether amiss to disjoin geodesy from geometry; for the multitude of sensible things, to which number and magnitude may be applied, is too diffusive to be circumscribed within these limits; and consequently if this supposition be admitted, the field of mathematics will become far too wide and extensive."

Sir Isaac Newton, whose ideas were not only always clear, but also clearly expressed, throws a distinct light on this subject as follows: "The ancients considered mechanics in a two-fold respect: as rational, which proceeds accurately by demonstration; and practical. To practical mechanics all the manual arts belong, from which mechanics took its name. But as artificers do not work with perfect accuracy, it comes to pass that mechanics is so distinguished from geometry, that what is perfectly accurate is called geometrical; what is less so, is called mechanical. *But the error is not in the art, but in the artificers.* He that works with less accuracy is an imperfect mechanic; and if any could work with perfect accuracy, he would be the most perfect mechanic of all; for the description [construction] of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn. * * * * Therefore *geometry is founded in mechanical practice,* and is nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measuring."

Pythagoras and his followers divided mathematics into *four* kinds; two of which they called pure and primary—viz., arithmetic and geometry; and the other two, mixed and secondary—viz., music and astronomy. "Thus did the Pythagoreans of old," says Dr. Barrow, "divide mathematics—I suppose because they had not yet applied themselves to the other parts—such as optics, mechanics, &c."

On the whole, there does not seem to be any good reason why mathematics may not and ought not to be further simplified by banishing from our books the imaginary distinction between the pure and the mixed;—for as geometry really has but one kind of quantity, and does not furnish one kind of quantity for a line, and another kind for a surface, and still another kind for a solid—so mathematics truly deals in but one kind of numbers, which are immutable in their nature and invariable in their laws, and are therefore always precisely the same things, whether in the hands of the mathematician, or the mechanic, or the natural philosopher.

SECTION IX.

THE REASON WHY THE SCIENCE OF GEOMETRY HAS NOT HITHERTO MADE ANY ESSENTIAL PROGRESS IN MODERN TIMES.

It is a remarkable fact, that, while the world has been making wonderful progress in arts and sciences, and all departments of knowledge, during two or three centuries past—new arts and new sciences being continually discovered and carried to great perfection, and old ones improved wherever improvement seemed possible—it surely is a remarkable fact, and worthy of special attention, that geometry, the most important of the sciences, though resting on an essential fundamental error, has been left till the present day, standing almost precisely where Euclid and Archimedes left it two thousand years ago.

There is some reason to suppose that the ancient Egyptians attained to a high knowledge of geometry, though we have no positive evidence of the fact. Thales is said to have brought the science from Egypt into Greece; and Gale, as quoted in Rees' Encyclopedia, says, "The Egyptians used geometrical figures, not only to express the generations, mutations, and destructions of bodies, but the manner, attributes, &c., of the Spirit of the Universe, who, diffusing himself from the center of his unity, through infinite concentric circles, pervades all bodies, and fills all space.

But of all other figures, they most affected the *circle* and the *triangle*." This last remark is strong presumptive evidence that the Egyptians had a high knowledge of geometrical figures; for the circle and equilateral triangle sustain a very remarkable relation to each other, being the two extreme limits of what may be regarded an infinite series. That is, in all the infinite variety of regular plane figures having the same length of circumference, the equilateral triangle contains *the least possible area* of the whole series, and the circle contains *the greatest possible area* of the whole series. There would seem to be a reason, therefore, why the Egyptians should "most affect the circle and triangle," if they had a deep knowledge of geometrical figures.

Pythagoras is said to have first discovered, that of all regular plane figures, having the same length of circumference, the circle contains the largest amount of area; and of all regular solids, having the same extent of surface, the sphere contains the greatest bulk or amount of solidity. Mr. Parker shows in his reasonings and demonstrations, that if a circle and equilateral triangle have an equal amount of area, the diameter of the circle and the perpendicular of the triangle are in *opposite duplicate ratio to each other*.

But when we pass from the Egyptians to the Greeks, among whom Dr. Barrow enumerates "the wonderful Pythagoras, the sagacious Democritus, the divine Plato, the most subtle and very learned Aristotle, men whom every age has hitherto acknowledged and deservedly honored as the greatest philosophers, the ringleaders of the arts," we are not left in the dark as to the progress of geometrical science, for we find it cultivated to a degree that sheds a luster over the Grecian name, unequaled in that department of knowledge among any other people in any age of the world. And yet the Greeks possessed but a very imperfect arithmetic and inconvenient and defective methods of notation, while the moderns possess an arithmetic of great perfection, and a method of decimal notation, of wonderful practical facility and unfailing accuracy.

It becomes then a question of great interest, and still greater importance, to know why geometry made such progress and rose to such high perfection among the Greeks, with all their disadvantages, and why among the moderns, with vastly superior facilities, it has hitherto remained essentially upon the same level where the Greeks left it.

The Greek method of investigating geometrical subjects was with rule and compasses, and by diagrams of lines and circles. The modern method, since the time of Descartes, has been principally by the algebraical process. And to this different method of handling geometrical subjects I cannot but think must be attributed the striking difference in the results, so injurious to modern science. As I have but little practical acquaintance with algebra, and judge of it mostly from general principles, I should express this opinion with great diffidence, or perhaps not at all, if I did not find it supported by some of the highest and best authorities. It seems due to the importance of the subject, that a few of the opinions alluded to, should be presented here for consideration.

Algebra is a very condensed, short-hand method of handling mathematical subjects, and is undoubtedly very convenient and very useful in many things; but it was probably a great error to apply it to geometry, whereby that pure and clear-sighted science, walking the earth with free and unerring footsteps, became, as it were, blindfolded, embarrassed, and stayed in her progress.

The Edinburgh Encyclopedia says: "The essential character of algebra consists in this—that when all the quantities concerned in any inquiry to which it is applied, are denoted by general symbols, the results of operations do not, like those of arithmetic and geometry, give the individual values of the quantities sought, but only show what are the *arithmetical* or *geometrical* operations, which *ought to be performed* on the original given quantities in order to determine their values."

Professor Simson, of Glasgow, became so thoroughly impressed with the superior accuracy and utility of the Greek geometrical methods over the modern algebraic, that he devoted almost his whole life in labors to revive the methods of the ancients, and to restore and repair some of their lost and mutilated works. And we are told by the London Encyclopedia, that Professor Simson "came at last to consider algebraic analysis as little better than a kind of mechanical knack, in which we proceed without ideas of any kind, and obtain a result without meaning, and therefore without any conviction of its truth."

Bishop Berkley, one of the most acute and subtle reasoners of modern times, "maintains, in the Analyst, that the differential calculus, new analysis, or doctrine of fluxions, is *inaccurate* in its principles, and that if it ever lead to true conclusions, it is from an accidental compensation of errors, that cannot be supposed always to take place." The writer in the London Encyclopedia, from whom this paragraph is quoted, adds: "No one who knows Berkley's ability requires to be told that the Analyst is a masterly production; nor can some of his arguments and charges be fairly met."

Laplace, the incomparable Laplace, compares the Greek method with the modern as follows: "The geometrical synthesis has the advantage of never losing sight of its object, and of illuminating the whole path which leads from the first axioms to their last consequences; whereas the algebraic analysis soon causes us to forget the principal object in order to occupy us with abstract combinations."

Sir Isaac Newton himself, who invented the method of fluxions, and carried the algebraic analysis to the highest pitch of power and refinement, left his testimony and the weight of his great example in favor of the Greek geometry. When but a youth, and just entering upon his collegiate studies, a biographer says that, "Regarding the propositions contained in Euclid as self-evident truths, he passed rapidly over this ancient system—*a step which he afterward much regretted*—and mastered, without further preparatory study, the Analytical Geometry of Descartes." In after life, he "censured the handling of geometrical subjects by algebraical calculations, and the maturest opinions he expressed were additionally in favor of the geometrical method." So decided was his opinion on this point, that it is said of him, "he thought that a mathematical proposition ought not to be made public, or was not fit to be seen, till invested in a synthetic dress." And we know that he demonstrated and published to the world his own sublime discoveries by the methods of the Greek geometers.

Upon such testimony, is it not fair to conclude that he, who is pursuing geometrical subjects by the methods of algebra, is like one groping blindfolded to hunt for gems among pebbles?—for as the one may pass over a thousand gems without seeing them, and if he chance to get one in his hand, cannot understand its value till he takes it to the light; so the other is forever reaching in the dark; and, if by chance he grasps a truth, is unable to tell what it is, till he borrows the light of arithmetic or geometry to reveal it. But let him descend into the deep caverns of geometry with the Greek rule and compass in his hand, guided by the perfect modern numerical notation, which the Greeks did not possess, and he carries a torch before him, which lights up his entire pathway, and the rich and bright gems of truth on all sides come flashing upon his gladdened sight from every crag and corner.

Let the teachers of the world give this important question a fair hearing; and if this shall prove to be the right view of it, let them wrest geometry out of the hands of algebra, strip the bandage from her eyes, and let her walk forth again upon the earth with unclouded vision. Then shall she brush away the cobwebs and dust of modern abstractions, and, clothed in a garment of new light and beauty, shall stand before the world more perfect and more comely than in the days of her Grecian youth. Then shall she carry forward her high mission to elevate the condition of man, by teaching him God's everlasting truths. Then shall her dignity and divine importance be vindicated, perhaps even to justify the assertion of Plato concerning the probable employment of Deity, that "He geometrizes continually."

SECTION X.

CONCLUDING REFLECTION.

It must be manifest from what has been already presented, that geometry and mathematics are not such hard and incomprehensible matters as the world has generally regarded them; and that their perplexing difficulties and forbidding and endless labors are not so much to be found in the nature of the sciences themselves, as in the errors which have hitherto been lying at their very foundation. The mathematics have justly been regarded by the wise in all ages as the best of all disciplines. They were considered by Pythagoras, as "the first step toward wisdom." Dr. Barrow in describing the importance of the science, says, "If the fancy be unstable and fluctuating, it is as it were poised by this ballast, and steadied by this anchor; if the wit be blunt it is sharpened upon this whetstone; if luxuriant, it is pared by this knife; if headstrong, it is restrained by this bridle; and if dull, it is roused by this spur."

Says Lord Bacon, "Men do not sufficiently understand the excellent use of the pure mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they sharpen it; if too wandering they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect it maketh a quick eye, and a body ready to put itself into all postures; so in the mathematics, that use which is collateral and intervenient, is no less worthy than that which is principal and intended."

I most earnestly desire, therefore, to do something to simplify the study of geometry, the real foundation of all mathematics; something to make it not only a delight to the student of the University, but a welcome guest in every common school, and a cherished visitor at every family fire-side; something by which the benefits of its admirable discipline may become more widely diffused, and its beauties and harmonies more generally enjoyed. The world should no longer be afraid to come in contact with the works of geometers and mathematicians, "the sight of whose writings," says Dr. Barrow, "everywhere shining with the rays of geometrical diagrams, the unskillful in these things are afraid of." The true principles of geometry are so simple, that he that runs may read, and a child can understand them. I desire, therefore, that none may feel deterred from reading these pages, or examining the succeeding demonstrations, from an apprehension that their knowledge of such subjects is too limited to enable them to understand them. Let them bring to the labor a little patient and persevering thought, and examine each step with some vigor of attention, and they will be surprised at the light resting on every diagram, and charmed by the beauty, simplicity, and harmony, with which an endless variety of forms blend and yield obedience to a few general and simple laws.

Geometry should always precede arithmetic, or rather go hand in hand with it, in a system of education. As soon as a child had learned to count his ten fingers I would begin to teach him geometry; for, as it is the most simple and perfect of all sciences, so it is the most readily comprehended if properly taught. Through geometry he should learn all his arithmetic. Then would he find the dark and puzzling labyrinths of numbers to lighten up at every step of his progress. Then would the toilsome and blind path of arithmetic become a bright and pleasant road, and her mystic and vague expressions would open to him full of clear and beautiful meaning. Then would he see and comprehend what is meant by those perplexing, enigmatical things, the square root, and the cube root. Then would the boy "with shining morning face," no longer be seen "creeping like snail, unwillingly to school," but tripping with a light heart, and singing for joy.

Should the present treatise contribute anything toward bringing about such results, it will furnish a true response to what I trust has been the leading stimulus to carry me through the labor of its preparation; —for the end and aim of all knowledge should be to *do good*—to elevate both the giver and the receiver. Saint Paul beautifully said, "Though I have the gift of prophecy, and understand *all mysteries* and *all knowledge*, and though I have all faith, so that I could remove mountains, and have not *charity*, I am nothing." And the same sentiment is so finely expanded and so happily expressed by Lord Bacon in the following passage, that I am sure its quotation cannot fail to give pleasure and profit to the reader:

"The greatest error of all is the mistaking or misplacing the last or farthest end of knowledge;—for men have entered into a desire of learning and knowledge, sometimes upon a natural curiosity and inquisitive appetite; sometimes to entertain their minds with variety and delight; sometimes for ornament and reputation; and sometimes to enable them to victory of wit and contradiction; and most times for lucre and profession; and seldom sincerely to give a true account of their gift of reason, to the benefit and use of men; as if there were sought in knowledge a couch whereupon to rest a searching and restless spirit; or a terrace, for a wandering and variable mind to walk up and down with a fair prospect; or a tower of state, for a proud mind to raise itself upon; or a fort or commanding ground, for strife and contention; or a shop for profit and sale; *and not a rich storehouse, for the glory of the Creator, and the relief of man's estate.*"

PART SECOND.

DEMONSTRATIONS IN GEOMETRY.

Remark.—Before entering upon this part of our work, it may be well to apprise the critical reader, that it has not been deemed at all necessary to adopt the rule apparently followed by Euclid, viz., never to suppose anything done, till the manner of doing it has been shown or explained. The rule was a very safe one, in the early progress of the science, to prevent the possibility of error, or the danger of resting on unwarrantable assumptions; but it also led to much unnecessary labor and tedious prolixity. The rule which I have rather endeavored to follow in these demonstrations, is, to give under each proposition *all that is necessary* to produce *perfect conviction of the truth stated*, and not to encumber the demonstration with anything more. So that if it should appear to the critical geometer, that links are sometimes omitted which he may think ought to be brought into the chain of reasoning, he may understand the reason of the omission. He may sometimes miss the repetition of an axiom or a well-known and established principle of geometry, which might have served to lengthen out a demonstration, but would not make the truth any more apparent. So also in these demonstrations many figures are required to be drawn when no rule or mode of drawing them has been given. But, says Sir Isaac Newton, "geometry does not teach us to draw these figures, but requires them to be drawn." The construction of them is entirely a mechanical operation.

DEFINITIONS.

1. *Numbers* are the signs or representatives of things, or of whatever has existence.

2. *Arithmetic* is the science of numbers, and regards things only as they are numerable, or may be counted.

3. *Geometry* is the science of magnitude, and measures and compares *extension* and *forms*.

4. Arithmetic has but one *language* or mode of expression, which is by numbers.

5. Geometry has two languages or modes of expression, one by numbers, and one by material substances, or pictures representing material substances.

6. *The unit* in arithmetic is the sign or representative of anything considered as one and indivisible, without regard to form or magnitude.

7. The unit *in geometry* is the sign or representative of any assumed magnitude, considered as one and indivisible, and in the *form of a cube*. A unit in geometry, therefore, is always one in length, one in breadth, and one in thickness.

8. A unit in geometry may be of any positive magnitude, from magnitude infinitely diminished, to magnitude infinitely extended.

9. *A straight line* is composed of a succession of single and equal units. A line therefore always has a breadth of *one*.

10. A line, or length, is *measured* by the application of one dimension only of the unit, viz., its linear edge.

11. *A surface* or plane is composed of a succession of single lines. A surface therefore, always has a *thickness* of *one*.

12. *Plane figures, or forms*, are those in which extension is measured in two directions only, length and breadth, without regard to thickness.

13. The elements of plane figures consist of area, perimeter, circumference, and diameter.

14. *The area* of a plane figure is the quantity of extension or space inclosed by its circumference; and is measured by the application of two dimensions of the unit, its length and breadth.

15. *The perimeter* of a plane figure is the distance around it, measured upon the extreme limits of the figure.

16. *The circumference* of a plane figure is a line, or lines, touching and inclosing it, having a breadth equal to *one*, and a length equal to the perimeter of the figure.

17. *The diameter* of a plane figure is the diameter of its inscribed circle.

18. *A circle* is a plane figure, which has an equal extension in every direction from its center to its circumference.

19. *The diameter of a circle* is a straight line passing through its center, and extending in length to the extreme limits of the circle.

20. A circle is said to be *inscribed* in any plane figure, when the circle touches every side or line of the circumference of the figure; and *circumscribed* when the circumference of the circle touches every corner or angle of the figure.

21. A plane figure is said to be *circumscribed* about a circle, when every side of its circumference touches the circle; and *inscribed* when every corner or angle touches the circumference of the circle.

22. *The base* of a plane figure is the side on which it is supposed to rest, when considered in a vertical position.

23. *The perpendicular*, or height, of a plane figure is the shortest distance from any point in the base to a line drawn *parallel* to the base and resting on the highest point of the figure.

24. *Parallel lines* are those which everywhere preserve an equal distance between them. Parallel lines therefore can never meet each other, however far they may be produced.

25. *Solid figures*, or bodies, are those in which extension is measured in three directions, length, breadth, and thickness.

26. The elements of solid figures consist of solidity, face, surface, and diameter.

27. *The solidity* of a solid, in geometry, is the quantity of extension, or the amount of bulk, inclosed by the surface. The solidity is measured by the application of the unit in its three dimensions, length, breadth, and thickness.

28. *The faces* of a solid are the planes by which its extension is terminated.

29. *The surface*, [*super*-facies,] of a solid is the sum of all the planes supposed to perfectly cover all its faces, and everywhere having a thickness of *one*.

30. *The diameter* of a solid, with plane faces, is the diameter of its inscribed sphere.

31. *A sphere* is a solid figure which has an equal extension in every direction from its center to its surface. Its surface therefore is a perfect curve, everywhere returning into itself.

32. *The diameter of a sphere* is a straight line passing through its center, and extending in length to the extreme limits of the sphere.

33. A sphere is said to be *inscribed* in a solid with plane faces, when the sphere touches every plane of the surface; and *circumscribed* when the surface of the sphere touches every solid angle.

34. A solid is said to be *circumscribed* about a sphere, when every plane of its surface touches the sphere; and *inscribed* when every solid angle touches the surface of the sphere.

Lines are of two kinds, straight and curved.

35. *A straight line* is one which never changes its direction in any part of its length. A straight line therefore is the shortest distance between two points.

36. *A curved line* is one which continually changes its direction in every part of its length.

37. The straight line is a measure of extension in one direction only.

38. The curved line is a measure of extension in every possible direction.

39. Straight lines are used in the composition of diameters, circumferences, and surfaces.

40. Curved lines are used only for circumferences and surfaces.

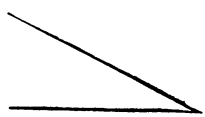
41. A plane surface is composed of straight lines.

42. A curved surface is composed of curved lines.

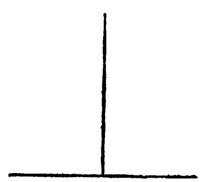
43. *The radius* of a circle is a straight line drawn from the center to the circumference;—and because a circle has an equal extension in every direction from the center to the circumference, every radius of the circle is equal to every other radius of the same circle.

The radius of a sphere is a straight line from the center to the surface;—and because a sphere has an equal extension in every direction from the center to the surface, every radius of a sphere is equal to every other radius of the same sphere.

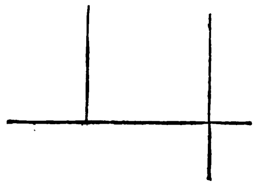
44. *An angle* is the opening between two lines which meet each other at a point.



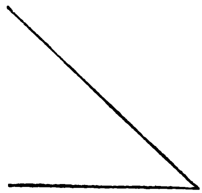
45. One straight line is *perpendicular* to another, when the angles on each side of the perpendicular are equal to each other.



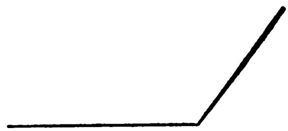
46. Angles made by lines meeting each other perpendicularly, or crossing each other perpendicularly, are called *right angles.*



47. *An acute angle* is one that is smaller or sharper than a right angle.

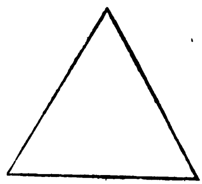


48. *An obtuse angle* is one that is larger or more open than a right angle.

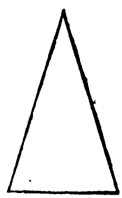


49. *A triangle* is a plane figure inclosed by three straight lines of circumference.

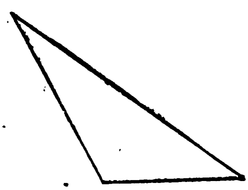
50. *An equilateral triangle* is one whose three sides are all of equal length.



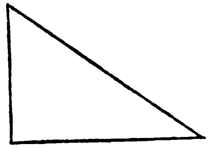
51. *An isosceles triangle* is one which has two sides equal to each other.



52. *A scalene triangle* is one whose three sides are all unequal in length.

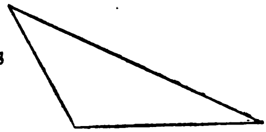


53. *A right angled* triangle has one right angle.



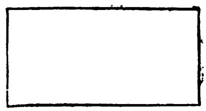
54. *The hypothenuse* is the longest side of a right angled triangle, or the side opposite to the right angle.

55. *An obtuse angled* triangle has one obtuse angle.

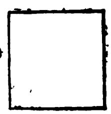


56. All other triangles have three acute angles, and are called *acute angled.*

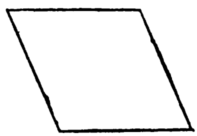
57. *A rectangle* is a plane figure having four sides, and four right angles.



58. *A square* is a rectangle whose four sides are equal to each other.



59. *A rhombus* is a plane figure, having four equal sides, and two obtuse and two acute angles.



60. *A parallelogram* is a plane figure with four sides, having the opposite sides parallel to each other. And because parallel lines everywhere preserve an equal distance between them, the opposite sides of parallelograms are equal.

The rectangle, the square, and the rhombus, are species of parallelograms.

61. *Equal plane figures* are such as contain an equal amount of area; also such as, being supposed applied to each other, would manifestly coincide in their whole extent.

62. *Equal solid figures* are such as contain an equal amount of solidity or bulk.

63. *Similar plane figures* are such as have the same number of angles, and each angle in the one equal to an angle in the other, and the sides adjacent to any angle in the one *proportional* to the sides adjacent to the equal angle in the other. And such proportional sides in two similar figures are called *homologous* sides.

64. *Similar solid figures* are such as have an equal number of *similar* plane faces. The angles in the one solid being respectively equal to the angles in the other.

65. *The diagonal* of a parallelogram, or of a cube, or of an octahedron, is a straight line passing through the center and extending to two opposite angles.

66. As an angle in a plane figure is formed by the meeting of lines, which constitute part of a circumference, so an angle in a solid figure, [commonly called a solid angle,] is formed by the meeting of planes, which constitute part of a surface.

67. *A polygon* is the general name applied to plane figures with any number of sides.

68. *A regular polygon* is one which has all its sides equal to each other, and all its angles equal to each other. If either the sides or the angles are unequal, the polygon is called irregular.

Polygons with but few sides are generally designated by names expressive of the number of their sides or angles. A pentagon has five sides, a hexagon has six, a heptagon seven, an octagon eight, a nonagon nine, a decagon ten, an undecagon eleven, a dodecagon twelve, &c.

69. *A cylinder* is a regular round solid, having a plane circle for its base, a plane circle for its top or side opposite and parallel to the base, and having every point of its curve surface in the circumference of a circle equal to the base, and also in a straight line perpendicular to the base.

70. *A cone* is a regular round solid, having a plane circle for its base, a point for its top or apex, and having every point of its curve surface in the circumference of a circle parallel to the base, and also in a straight line extending from the apex to the perimeter of the base.

REMARK.—The last definition applies strictly only to the *right* cone, in which the apex is in the perpendicular from the center of the base. If the cone is *oblique,* sections *parallel* to the base would not be perfect circles.

71. *An arc of a circle* is any part of its circumference.

72. *The chord* of an arc is a straight line joining the two extremities of the arc.

73. *The segment* of a circle is the part of the area cut off by a chord, or the part inclosed by an arc and its chord.

74. *An axiom* is a self-evident truth, or one so manifest that it cannot be made more clear by any demonstration; such as,

First. Magnitudes which are equal to the same thing, are equal to each other.

Second. Magnitudes which are double, triple, &c., of the same, or of equal magnitudes, are equal to each other.

Third. Magnitudes which are each one-half, one-third, &c., of the same or of equal magnitudes, are equal to each other.

Fourth. If equals be added to, or taken from equals, the results will be equal.

Fifth. The whole is greater than a part.

Sixth. The whole is equal to the sum of all its parts.

75. *A theorem* is a truth which is made manifest by a course of reasoning; and that course of reasoning is called a *demonstration.*

76. *A problem* presents some operation to be performed.

77. *A proposition* is a general term applied either to a theorem or a problem.

78. *A corollary* is an obvious truth, resulting from a demonstration.

SIGNS.

This sign, + [called *plus*, or more], when placed after a number, or series of numbers, denotes that something more is to be added in order to complete the perfect quantity intended to be represented by the number or numbers.

This sign, — [called *minus*, or less], when placed after a number, denotes that something must be subtracted from the number in order to make it represent the perfect quantity intended.

This sign, =, placed between two numbers or quantities, denotes that they are equal.

This, ×, placed between two numbers or quantities, denotes that they are to be multiplied, the one by the other.

This, ÷, placed between two numbers or quantities, denotes that the one is to be divided by the other.

This, $\sqrt{}$, placed before a number, denotes that the square root is to be extracted.

This, $\sqrt[3]{}$, denotes that the cube root is to be extracted.

RULES.

1. To obtain the area of any rectangle or parallelogram; multiply the base by the perpendicular height.

2. To obtain the area of any triangle; multiply half the base by the perpendicular height.

3. To obtain the side of an equilateral triangle, when the perpendicular is given; square the perpendicular, add to the square one-third of the square, and then extract the square root.

4. To obtain the perpendicular of an equilateral triangle, when the side is given; square the side, take three-fourths of the square, and then extract the square root.

5. To obtain the area of any plane figure from the circumference and diameter; multiply half the circumference by half the diameter, (diameter always being the diameter of the inscribed circle.)

6. To obtain the circumference of any plane figure from the area and diameter; multiply the area by 4, and divide by the diameter.

7. To obtain the diameter of any plane figure from the area and circumference; multiply the area by 4, and divide by the circumference.

8. To obtain the circumference of any circle from the diameter; multiply the diameter by the circumference of a circle whose diameter is 1, viz., by 3.14159+.

9. To obtain the surface of any sphere whose diameter is given; first obtain the circumference by the last rule, and then multiply the circumference of the sphere by its diameter for the surface.

10. To obtain the solidity of any sphere; multiply one-third of the surface by the radius, or half the diameter.

11. To obtain the solidity of a regular pyramid; multiply the area of the base by one-third of the perpendicular height.

12. To obtain the solidity of a right cone; multiply the area of the base by one-third of the perpendicular height.

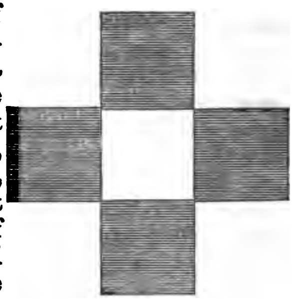
13. To obtain the curve surface of a right cone; multiply the perimeter of the base by one-half the side, or half the slant height of the cone.

14. To obtain the solidity of a cylinder; multiply the area of the base by the perpendicular height.

15. To obtain the curve surface of a cylinder; multiply the perimeter of the base by the perpendicular height.

PROPOSITION I.

The circumference of any magnitude, assumed as a geometrical unit, is a quantity four times as large as the assumed magnitude; and it is impossible that it should be anything greater or anything less. And the surface of a magnitude, assumed as a geometrical unit, is a quantity six times as large as the assumed unit; and it is impossible that it should be anything greater or anything less.



Let the light square, in the center of the diagram, be assumed as a geometrical unit, [Definition 7.] It is then a fixed magnitude, indivisible, invariable in quantity, and invariable in form. It may be assumed of any positive size, from magnitude infinitely diminished, to magnitude infinitely extended. But whatever may be the positive size of the magnitude assumed, when it is fixed and bound up into *one*, and made the standard of measure, it is a unit in every particular. Its solidity is 1, its diameter is 1, its length is 1, its breadth is 1, and its thickness is 1. This unit is the only instrument with which geometry works. In the hands of geometry, this little unit, of simple and perfect form, becomes the magician's wand, decomposing all magnitudes and all forms, and moulding them into quantities of its own perfect likeness, thus furnishing the material with which geometry constructs all its perfect works, and out of which it manufactures all its diameters, its areas, its circumferences, its solidities, and its surfaces.

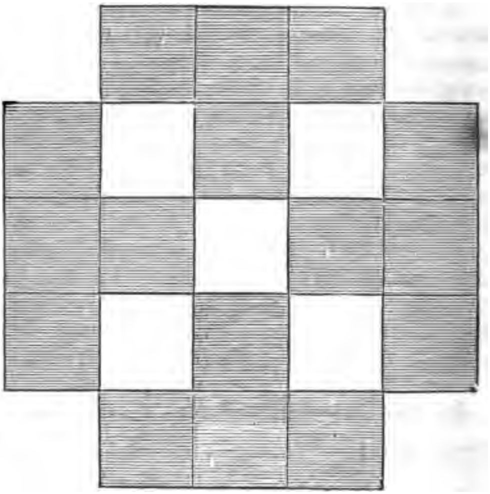
Now let geometry commence with the white unit in the center of the diagram, and supply it with a circumference; that is, surround it with something which shall touch and entirely cover the four faces of the unit that look in a horizontal direction. It is the business of geometry to make this circumference of such form and such magnitude, that it can perfectly measure it and give an exact account of it. The only possible way in which geometry can accomplish this, is by applying four magnitudes, like the four shaded squares in the diagram, of exactly the same form and size as the assumed unit. Because, in the first place, it has no other materials to work with but just such units; and in the second place, if it had other materials of different form or size, they would not answer the purpose. If they were smaller, it is manifest they would not entirely cover the four faces of the unit, and therefore would not form a perfect circumference. And if they were larger, it is manifest they would crowd each other from their position of *contact* with the unit, and therefore would not in that case form a perfect circumference. And again, since geometry has but one standard of measure, the indivisible unit, invariable in form and invariable in size, if a circumference could any way be patched up from materials of different form or size, it would be utterly impossible for geometry to measure it and render an exact account of it; for geometry can measure nothing which does not perfectly agree with the assumed unit. Therefore it is impossible that the circumference of a magnitude, assumed as a geometrical unit, should be anything greater or anything less than four times the quantity of the assumed unit.

If the unit be regarded as a *solid*, instead of a plane figure, it then has a solidity instead of an area, and must be entirely inclosed by a *surface*, instead of only being surrounded by a circumference. In that case, there are two more faces to be covered, one at the top and one at the bottom of the unit, making six faces requiring a surface. It is the business of geometry to make a surface for this unit, and to make it of such dimensions that she can measure it and give an exact account of it. The only possible way that she can accomplish this, is to apply six magnitudes of exactly the same form and size as the assumed unit; for the same reasons and necessities govern in this case, that governed in the formation of the circumference. If they were less magnitudes, they could not entirely cover the faces of the unit, and if they were greater, they would crowd each other from their position of contact with the unit, so that they could not form a perfect surface, entirely inclosing it and everywhere touching it. And if they were either greater or less, or of a different form, geometry could not measure them by the unit. It is therefore impossible that the surface of a magnitude, assumed as a geometrical unit, should be anything greater or anything less than six times the quantity of the assumed magnitude.

What is true of the unit of any determined size or magnitude, is true of units of every possible size; for numbers know no difference between small and great, and nature, in her divisibilities and extensions, as Sir Isaac Newton has well remarked, "is not confined to any bounds."

It may appear to the student, before he has wel[illegible] on the matter, that in order to complete the circ[illegible] e square, as represented in the diagram, there shou[illegible] more units to fill the vacant spaces at the four corners of the diagram. But examination will show, that the moment four more such units are added we have a new square to deal with. It is no longer a square with a single unit for its diameter, but a square with three units for diameter, and requiring for its circumference a line of three units on each of its four sides, making twelve for circumference, and leaving again a vacant space at each of the four corners, as in the annexed diagram.

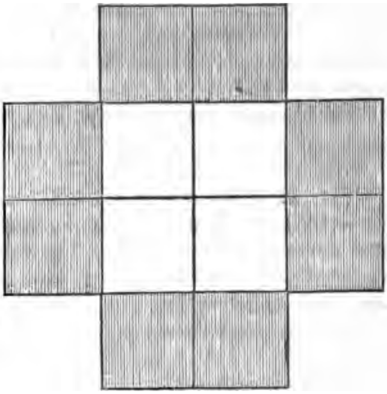
Again, if these four vacant spaces at the corners were filled up by placing a unit in each of them, the figure would no longer be a square with three units for diameter, nine for area, and twelve for circumference, but a new square with five units for diameter, twenty-five for area, and requiring twenty more units, five on each side, for a new circumference.



PROPOSITION II.

In the square whose diameter equals two units, the area equals half its circumference.

The principles already established render but little more necessary, for the proof or illustration of this proposition, than to present the diagram for inspection.

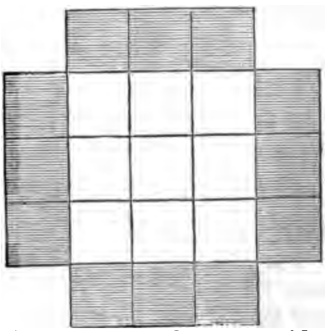


Let the light square in the center be composed of four units. Its diameter then is two units, its area four, and its shaded circumference eight. The area therefore equals half the circumference.

In the arithmetical calculation of the same square, the diameter or side of the square is represented by 2, which being squared, or multiplied by itself, produces 4, for the area. And the side of the square being 2, and there being 4 sides, 4 multiplied by 2 produces 8, for the circumference. Area is 4, and circumference 8. Area therefore equals half the circumference in numbers, as well as in magnitude or quantity of extension.

PROPOSITION III.

In the square whose diameter is three units, the area equals three-fourths of its circumference.

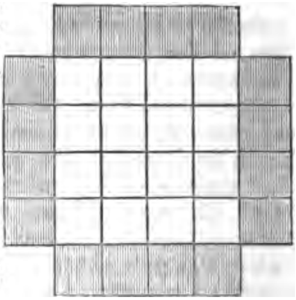


In the diagram, the white square is seen to have a diameter of three units, and an area of nine units. The shaded circumference is seen to consist of twelve units. And nine is three-fourths of twelve; therefore the area equals three-fourths of the circumference.

In the arithmetical calculation of the same square, the diameter or side of the square is 3, and 3 squared or multiplied by itself gives 9 for the area. And 3, one side, multiplied by 4, the number of sides, gives 12 for circumference. Area therefore equals three-fourths of the circumference, in numbers as well as in magnitude.

PROPOSITION IV.

In the square whose diameter is four units, the area equals the circumference.



The white square in the diagram is seen to have four units for diameter and sixteen units for area.. The shaded circumference is also seen to consist of sixteen units. The area therefore equals the circumference.

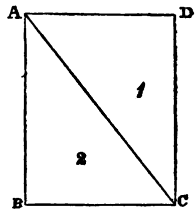
In the arithmetical calculation of the same square, the side is 4, which being multiplied by itself produces 16 for area. And 4 for one side, being multiplied by 4, the number of sides, gives 16 for circumference. The area therefore equals the circumference in numbers as well as in magnitude or geometrical quantity.

REMARK.—Mathematicians have always been able to tell us that a square, whose diameter or side is 4, has an area of 16, and a circumference of 16. But while they have always seen this perfect agreement in numbers, they have always denied that there could be any agreement in geometrical quantity or magnitude between area and circumference, because of the fundamental error in which they have always rested in supposing that a mathematical line has no breadth.

Before proceeding to demonstrate the general principles of area, circumference, and diameter, applied to other forms besides the square, it will be necessary to present two or three preliminary demonstrations, especially for the help of readers who are not already familiar with the established truths of geometry.

PROPOSITION V.

The diagonal of a rectangle divides the area into two equal parts.

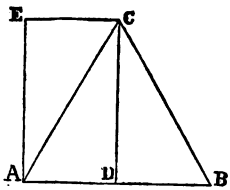


The rectangle ABCD is divided by the diagonal AC into two triangles, 1 and 2. And, because the opposite sides of a rectangle or parallelogram are equal, the sides of one triangle are severally equal to the sides of the other triangle. AD is equal to BC, DC is equal to AB, and AC forms the third side of both triangles. If, therefore, we suppose the triangle 1 to be removed and turned round, so as to bring the angle D upon the angle B, both being right angles, the side AD would coincide with BC, and the side DC would coincide with AB, and the two triangles would manifestly coincide in their whole extent, and would therefore be equal, by Definition 61.

The same proposition is true with regard to all parallelograms, as well as rectangles.

PROPOSITION VI.

The area of a triangle equals the area of a rectangle of the same height and half the base of the triangle.



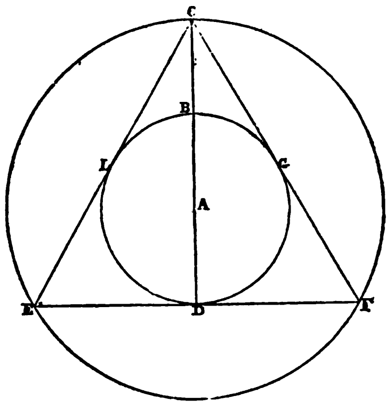
Let ABC be an equilateral or isosceles triangle. CD drawn from the vertex perpendicularly to the base AB will divide the base into two equal parts, and also divide the area of the triangle into two equal parts. From C draw CE parallel and equal to AD, and join AE. Then ADEC will be a rectangle of the same height and half the base of the triangle ABC. But the right angled triangle ADC is half the triangle ABC. And, by the last proposition, the same triangle ADC is half the rectangle ADEC. Therefore the whole triangle ABC is equal to the whole rectangle ADEC, which is of the same height and half the base of the triangle.

The same proposition is true with regard to all parallelograms, as well as rectangles.

Corollary.—If a triangle and a rectangle have the same or equal bases, and the same or equal heights, the area of the triangle equals half the rectangle. And the same is true with regard to triangles and parallelograms.

PROPOSITION VII.

The diameter of a circle inscribed in an equilateral triangle equals two-thirds of the perpendicular of the triangle.



From the center A, with the radius AB, equal 1, [in this diagram the unit is half an inch,] draw the circle BLD. With the radius AC, equal 2, or double AB, draw the circle CEF. From the point C draw the straight line CE touching the inner circle at L, and draw the straight line CF touching the inner circle at G. Then will the straight line joining E and F also touch the inner circle at D, and CEF will be an equilateral triangle, because none but an equilateral triangle can have an inscribed and a circumscribed circle drawn from the same center. By construction the radius AB is 1, therefore the radius AD is 1, [Def. 43.] AC is double AB, therefore BC is 1. And DB, the diameter of the circle equals 2, and DC, the perpendicular of the triangle, equals 3. Therefore the diameter of a circle inscribed in an equilateral triangle equals two-thirds of the perpendicular of the triangle.

Corollary.—From this demonstration it follows, that the perpendicular of an equilateral triangle inscribed in a circle is three-fourths of the diameter of the circle. For AC, the radius of the outer circle, is 2; therefore the diameter of the circle is 4. But the perpendicular CD of the inscribed triangle is 3, and is therefore three-fourths of the diameter of the circle.

Remarks.—We may now proceed to apply the laws of area, circumference, and diameter to the equilateral triangle, and other forms both regular and irregular.

The student will find great convenience and facility, in constructing and demonstrating diagrams, by assum-

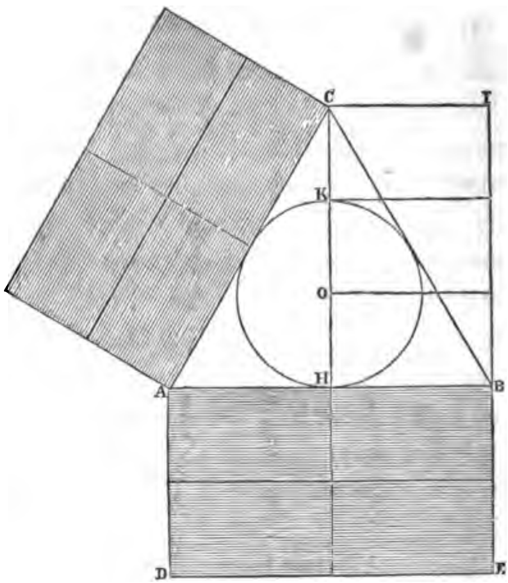
ing units of definite measures, as given on the rule or scale which he uses, and which he can take in his compasses and transfer to paper. As the rules are generally marked in inches and parts of inches, the unit adopted in most of the following diagrams will be found to be one inch, or where that would make the diagram too large, half an inch, or a third, or a quarter, is sometimes taken for unit.

It may perhaps appear to the reader, in perusing the following pages, that more demonstrations are given, than were needed, to illustrate and demonstrate a few general and simple principles. But when it is remembered that these principles are in direct opposition to principles laid down and maintained in all works on geometry from Euclid's time to the present day, some latitude and diffusiveness in this particular may well be excused. For as Dr. Barrow has very justly remarked, "no diligence or solicitude should be thought too much, which is spent in establishing the first principles of sciences. It is far better that many demonstrations be redundant, than for one to seem defective."

But I feel that there is another argument in favor of these diffuse and familiar demonstrations. By holding a principle up to view in three or four different lights, by a different dress, in several different demonstrations, the reader becomes master of it with less labor and fatigue, and with more profit and delight. And thus the general reader, as well as the professed student, may be led on by an easy and pleasant pathway into the broad and beautiful fields of geometry, where he may gather the rich fruits of its discipline and enjoy its perfect and delightful harmonies.

PROPOSITION VIII.

In the equilateral triangle, whose diameter is *one*, the area equals one-fourth of the circumference.



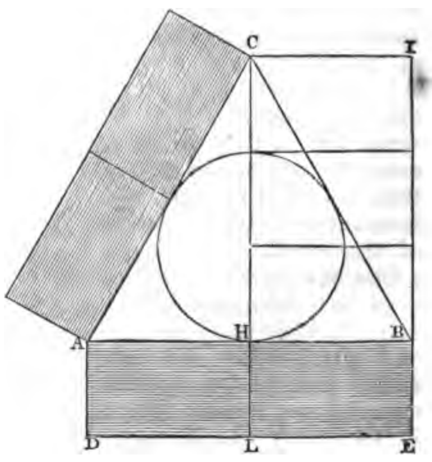
Let the diameter of the circle, HK, be *one*, (one inch.) Then the diameter of the equilateral triangle circumscribed about the circle is also one, [Def. 17.] Let the diameter of the circle coincide with the perpendicular of the triangle, CH, then will the base AB be equally divided at H; and the rectangle HBIC, will equal the area of the triangle, [Prop. 6.] Then because the radius of the circle, HO, is half the diameter, HK, and the diameter, HK, is two-thirds the perpendicular, CH, [Prop. 7,] CH is divided into three equal parts at the points O and K, and lines drawn from these points parallel to HB manifestly divide the rectangle HBIC into three equal rectangles, each having a length equal to HB, and a breadth equal to the radius, HO, or half of one.

On AB construct the rectangle ABDE, having a breadth, AD, equal to one. Then if this be divided into four equal rectangles, each will have a length equal to HB and a breadth equal to half of one, and they will, therefore, be severally equal to the three rectangles contained in HBIC. But the rectangle ABDE is one-third of the circumference of the triangle, for its length, AB, is one-third of the perimeter, and its breadth, AD, is emto *one*. [Def. 16.] Therefore the three sides, or whole circumference of the triangle, must be equal to twelve such rectangles as the four contained in ABDE. It has been shown that the area of the triangle is equal to three such rectangles, and three is one-fourth of twelve. Therefore in the equilateral triangle, whose diameter is one, the area equals one-fourth of the circumference, agreeably to the proposition.

Arithmetical calculation. In the arithmetical calculation of the same triangle, in decimal numbers, the perpendicular is 1.5. (Once and a-half, or once and five-tenths of the diameter of the circle.) To obtain the side of an equilateral triangle, add one-third to the square of the perpendicular, and extract the square root. Therefore, 1.5, squared, produces 2.25. A third of this is, .75, which added to 2.25 produces 3, and the square root of 3, viz., 1.732+, is the side of the triangle, or one-third of the circumference, and multiplied by 3, makes the whole circumference, 5.196+. The area of an equilateral triangle is obtained by multiplying the perpendicular by half the base or side. Half of 1.732+ is .866+, and this multiplied by 1.5 produces for area 1.299+, and this area multiplied by 4, produces 5.196+, equal the circumference. Therefore the area is one-fourth the circumference in numbers, as well as in magnitude or geometrical quantity.

PROPOSITION IX.

In the equilateral triangle whose diameter is *two*, the area equals half the circumference.

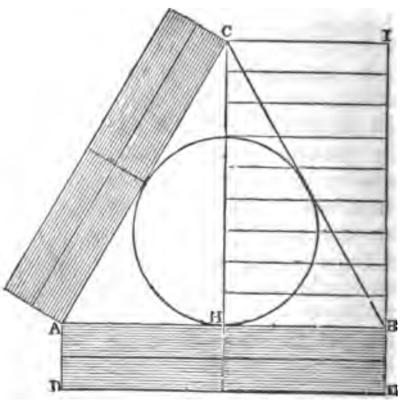


In the present diagram, which is the same as the last, except that the line of circumference is but half the breadth of the last, the unit is taken at half an inch. The radius of the circle therefore is 1, the diameter of the circle and of the triangle is 2, and the perpendicular of the triangle is 3. By the proofs exhibited in the last proposition, the rectangle HBIC equals the area of the triangle ABC. And this area in the rectangle HBIC is divided, as in the last proposition, into three smaller rectangles, equal to each other. The shaded line of circumference on one side of the triangle, contains two such rectangles, its length being equal to one-third of the perimeter of the triangle, and its breadth equal to 1. It is manifest, therefore, that the whole circumference would equal six such rectangles. And since the area is equal to three, the area of the triangle whose diameter is two, equals half its circumference, agreeably to the proposition.

Arithmetical calculation of the same triangle.—The perpendicular is 3, and 3 squared makes 9. One-third of the square, or 3, added to the square makes 12. The side of the triangle therefore is the square root of 12, viz., 3.464+; and this, multiplied by 3, gives the three sides or circumference of the triangle, viz., 10.392+. Half the side or base [half of 3.464+] is 1.732+, which being multiplied by the perpendicular, 3, produces 5.196+ for area. And 5.196+ multiplied by 2 produces 10.392+, equal the circumference. Therefore the area of an equilateral triangle with two for diameter, equals half its circumference in numbers, as well as in magnitude or geometrical quantity.

PROPOSITION X.

In the equilateral triangle whose diameter is *three*, the area equals three-fourths of the circumference.

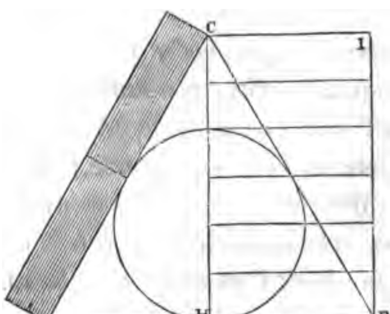


In the present diagram, the diameter of the circle being three units, the perpendicular of the triangle, CH, must be four and a half units, [Prop. vii.] Therefore if CH be divided into nine equal parts, each part will equal half the unit, and six of the parts will be contained in the diameter of the circle. From these points of division in CH let lines be drawn parallel to HB, and it is manifest they will divide the rectangle HBIC into nine rectangles, equal to each other. Now let AD equal one-third the diameter of the circle, that is, equal the unit. If AD be divided in the center, each part would equal half the unit, and AB being twice the length of HB, it is manifest that the rectangle ADEB contains four rectangles, each equal to each of the nine rectangles contained in HBIC. But the *four* rectangles constitute the shaded line of circumference on one side of the triangle, having a breadth of 1, and a length equal to the side of the triangle. Therefore the three sides, or the whole circumference, must be composed of twelve such rectangles. The *nine* rectangles in HBIC equal the area of the triangle ABC [Prop. vi.], and nine is three-fourths of twelve. Therefore the area of an equilateral triangle, whose diameter is three, equals three-fourths of its circumference, agreeably to the proposition.

Arithmetical calculation.—The perpendicular of the triangle being 4.5, the square of it is 20.25, and one-third of the square is 6.75, which added to 20.25, makes 27. The square root of 27, viz., 5,196+, therefore equals one side of the triangle, and multiplied by 3 produces 15.588+ for circumference. Half the base, or half of one side, viz., 2.598+, multiplied by 4.5, the perpendicular, produces 11.691+ for area, and this is three-fourths of 15.588+ ; for the last number divided by 4 and multiplied by 3 produces 11.691+. Therefore the area of an equilateral triangle, whose diameter is 3, equals three-fourths of its circumference in numbers, as well as in magnitude or geometrical quantity.

PROPOSITION XI.

In the equilateral triangle, whose diameter is *four*, the area equals the circumference.



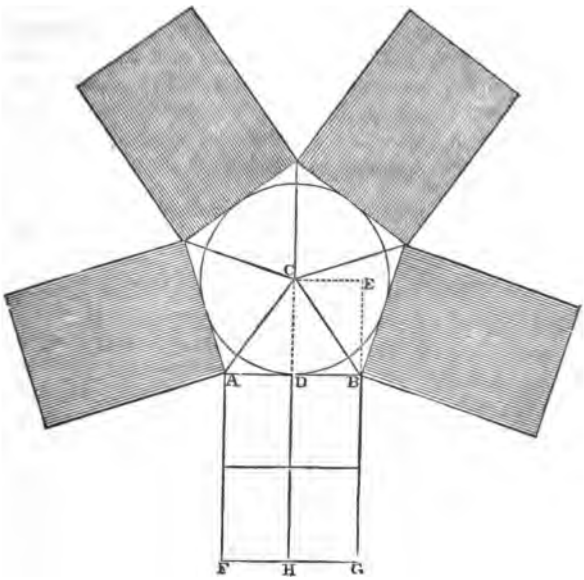
In this diagram the circle and the triangle are still of the same *positive* size as in the preceding, that is, the positive length of the diameter of the circle is one inch; but one inch is not the *unit* of the calculation. The value of the unit has been diminished in each successive triangle, to avoid the necessity of making the diagrams of larger size. The unit in the first triangle was one inch, in the second it was half-an-inch, in the third it was a third of an inch, and, in the present, it is the fourth part of an inch. And the diameter of the circle containing four units, the perpendicular of the triangle, CH, must contain six units, [Prop. vii.] Therefore let the perpendicular be divided into six equal parts, and lines be drawn parallel to HB, dividing the rectangle HBIC into six equal rectangles, which will together equal the area of the triangle ABC, [Prop. vi.] The shaded line of circumference being drawn with a breadth equal to 1, it is manifest that the rectangle ADEB, which constitutes the circumference on one side of the triangle, contains two rectangles, severally equal to those contained in HBIC. Therefore the whole circumference would be composed of six such rectangles; and the area of the equilateral triangle, whose diameter is 4, is therefore equal to its circumference, agreeably to the proposition.

Arithmetical calculation.—The perpendicular of the triangle being 6, the square of it is 36, and a third of the square added, makes 48. Therefore the square root of 48, viz., 6.928+, equals one side of the triangle; and multiplied by 3, produces 20.784+ for the three sides or whole circumference. And half of one side, that is, half the base, 3.464, multiplied by 6, the perpendicular, produces 20.784+ for area, which is the same that was obtained for circumference. Therefore the area of an equilateral triangle, whose diameter is 4, is equal to its circumference in numbers, as well as in magnitude or geometrical quantity.

REMARK.—Thus it has been seen in the last four propositions, that in equilateral triangles, as well as squares, there is a perfect agreement in *geometrical quantity* between the area of the triangle and its line of circumference. It is seen also that the law of agreement is precisely the same in the triangle as in the square—viz., if diameter is 1 area equals one-fourth of the circumference; if diameter is 2, area equals half the circumference; if diameter is 3, area equals three-fourths of the circumference; and if diameter is 4, area and circumference are equal. We shall soon see how this same simple and beautiful law extends to all other forms.

PROPOSITION XII.

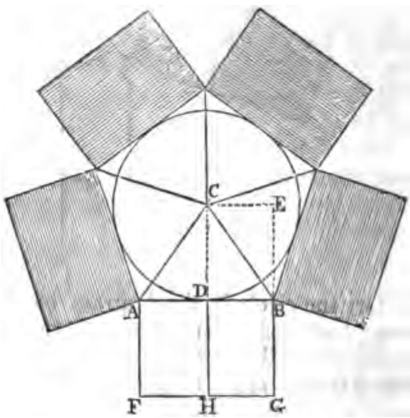
In the regular pentagon, whose diameter is *one*, the area equals one-fourth of the circumference.



Let the circle in the diagram be described with a diameter of one, [one inch,] and draw around it a regular pentagon. The diameter of the pentagon will then be one, [Definition 17.] Because it is a *regular* pentagon; the sides are all equal, and the angles are all equal, [Definition 68.] Therefore it is manifest that lines drawn from the center to the five angles of the pentagon divide the area into five equal isosceles triangles; for the base of each triangle is one side of the pentagon, as AB, and the height or perpendicular of each triangle is the radius of the circle, as CD. The triangle ABC therefore is one-fifth of the area of the pentagon; and let DBCE be a rectangle, and the triangle and rectangle are equal to each other, [Prop. 6.] The area of the pentagon, therefore, is equal to five such rectangles as DBCE. Let the radius of the circle CD be produced to H, making DH equal to one—that is, equal to the diameter of the circle, or twice CD, and draw the rectangle DBHG. This rectangle then will manifestly be equal to two such rectangles as DBCE, for it has the same base and double the height. And ABFG is manifestly composed of four such rectangles. But ABFG is one-fifth of the circumference of the pentagon, for it has the length of one-fifth of the perimeter, AB, and a breadth of *one*, DH, [Definition 16.] Therefore the whole circumference would be composed of twenty such rectangles as DBCE. The area equals five such rectangles, and five is one-fourth of twenty; therefore in the regular pentagon whose diameter is one, the area equals one-fourth of the circumference, agreeably to the proposition.

PROPOSITION XIII.

In the regular pentagon, whose diameter is *two*, the area equals one-half the circumference.

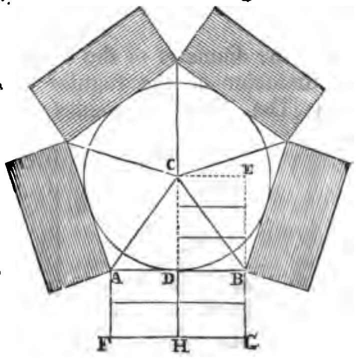


The circle and pentagon in the present diagram being the same as in the last, if we take the unit at half an inch, diameter will be two, CD, the radius, will be one, and DH, made equal to CD, will be the breadth of the line of circumference. As was shown in the last proposition, the area of the pentagon is equal to five such rectangles as DBCE. The two rectangles contained in ABFG are each equal to the rectangle DBCE, for by construction they each have an equal base and equal height. But ABFG is one-fifth of the circumference of the pentagon; therefore the whole circumference must be composed of ten such rectangles. And the area of the pentagon being equal to five such rectangles, the area is equal to half the circumference, agreeably to the proposition.

PROPOSITION XIV.

In the regular pentagon, whose diameter is *three*, the area equals three-fourths of the circumference.

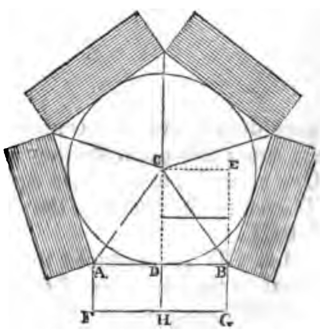
In the present diagram, diameter being three, the radius CD is one and a half. Therefore if DH be made equal to one, it will equal two-thirds of CD. Then let CD be divided into three equal parts, and DH into two equal parts, and it is manifest that the three rectangles drawn in DBCE, and the four rectangles drawn in ABFG are severally equal to each other, for by construction they all have equal bases and equal heights. But the area of the pentagon is equal to five such rectangles as DBCE; therefore it is equal to fifteen of the smaller rectangles. And the circumference of the pentagon is equal to five such rectangles as ABFG; therefore it is equal to twenty of the smaller rectangles. Fifteen is three-fourths of twenty, therefore the area of the pentagon, whose diameter is three, equals three-fourths of its circumference, agreeably to the proposition.



PROPOSITION XV.

In the regular pentagon, whose diameter is *four*, the area equals the circumference.

In the present diagram, diameter being four, the radius, CD, is two. If CD be produced to H, making DH equal to one, the breadth of the line of circumference, it is manifest that the two rectangles drawn in DBCE are severally equal to the two rectangles drawn in ABFG, for they all have equal bases and equal heights by construction. But DBCE equals one-fifth of the area of the pentagon; therefore the area of the pentagon equals ten such rectangles as the two

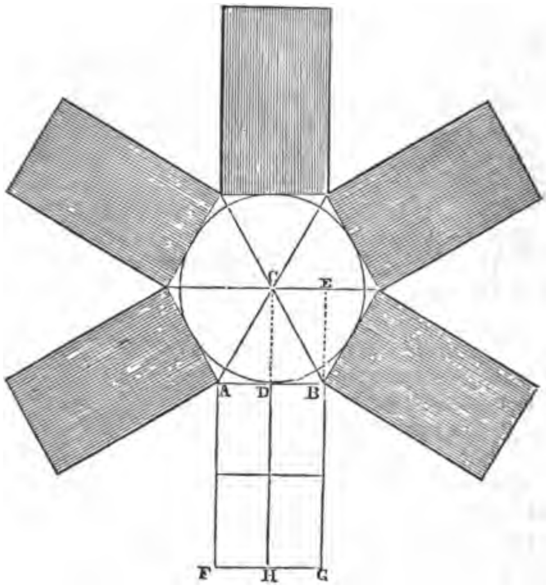


contained in DBCE. ABFG is one-fifth of the circumference of the pentagon, therefore the circumference equals ten such rectangles as the two contained in ABFG. Therefore the area of the pentagon, whose diameter is four, equals its circumference, agreeably to the proposition.

PROPOSITION XVI.

In the regular hexagon, whose diameter is *one*, the area equals one-fourth of the circumference.

Let the diameter of the circle be unit, or one, [one inch ;] then the diameter of the regular hexagon, circumscribed about it, is one ; [Def. 17.] The sides of the hexagon being equal, [Def. 68,]



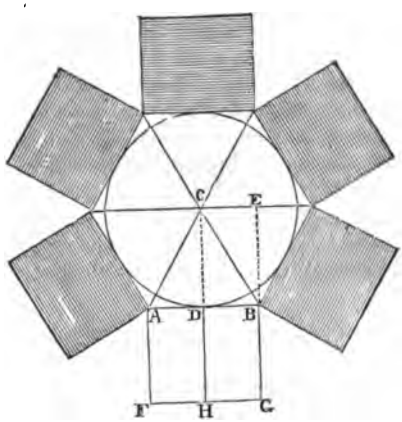
lines drawn from the center to the six angles divide the area into six equal triangles, for they all have equal bases and equal heights. One triangle, as ABC, is equal to the rectangle, DBCE, [Prop. 6.] Therefore the area of the hexagon equals six such rectangles as DBCE.

Let CD be produced to H, making DH equal to one, that is, equal to the diameter of the circle. Then DH will be double CD, and AB being double DB, it is manifest that the rectangle ABFG

contains four rectangles, each equal to DBCE, for they have equal bases and equal heights by construction. But ABFG is one-sixth of the circumference of the hexagon, having a length of one-sixth of the perimeter, AB, and a breadth, DH, equal to one, [Def. 16.] Therefore the whole circumference must be composed of twenty-four such rectangles as the four contained in ABFG. And the area of the hexagon being equal to six such rectangles, therefore the area of a hexagon, whose diameter is one, equals one-fourth of its circumference, agreeably to the proposition.

PROPOSITION XVII.

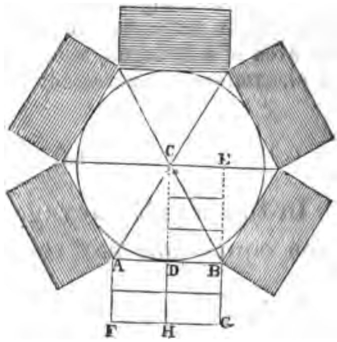
In the regular hexagon, whose diameter is *two*, the area equals one-half the circumference.



Diameter being two, the radius, CD, equals one, and DH or AF, made equal to one, is the breadth of the line of circumference. Lines drawn from the center to the angles of the hexagon divide the area into six equal and equilateral triangles. One of these triangles, ABC, [Prop. 6.] is equal to the rectangle BDCE. Therefore the whole area of the hexagon must be equal to six such rectangles. The two rectangles contained in ABFG are each equal to BDCE, for by construction they have equal bases and equal heights. The rectangle ABFG is one-sixth of the circumference of the hexagon, for the length, AB, is one-sixth of the perimeter, and the breadth, AF, is *one*. Therefore the whole circumference must be equal to twelve such rectangles as the two contained in ABFG. The area is equal to six such rectangles. Therefore in the regular hexagon whose diameter is two, the area equals one-half the circumference, agreeably to the proposition.

PROPOSITION XVIII.

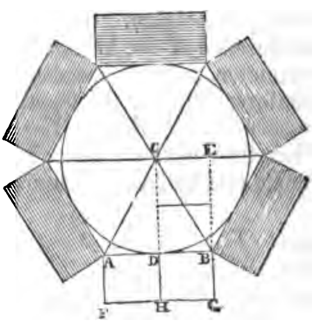
In the regular hexagon, whose diameter is *three*, the area equals three-fourths of the circumference.



In this diagram, the diameter being three, the radius, CD, equals one and a-half. And DH being drawn equal to the breadth of the line of circumference, and therefore equal to one, is consequently equal to two-thirds of CD. Therefore let CD be divided into three equal parts, and DH into two equal parts, and let the small rectangles be completed, and it is manifest that the four rectangles contained in the rectangle ABFG are severally equal to each of the three rectangles contained in DBCE. But DBCE equals one-sixth of the area of the hexagon; therefore the whole area of the hexagon must equal eighteen such rectangles as the three contained in DBCE. And ABFG is one-sixth of the circumference; therefore the whole circumference must be composed of twenty-four such rectangles as the four contained in ABFG. Eighteen is three-fourths of twenty-four, therefore the area of a hexagon, whose diameter is three, equals three-fourths of its circumference, agreeably to the proposition.

PROPOSITION XIX.

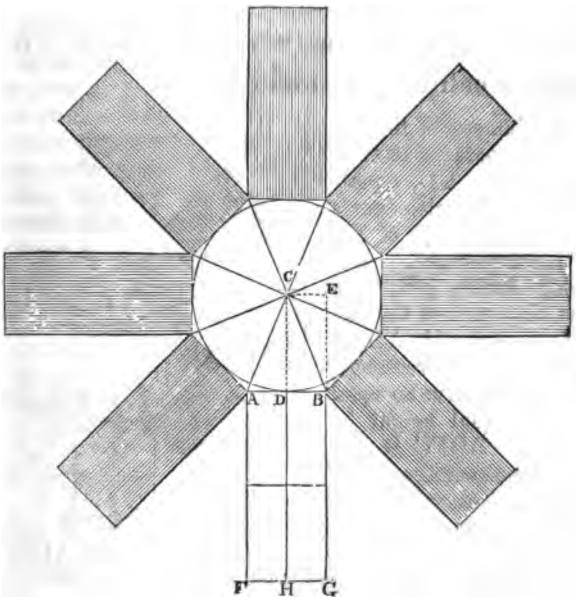
In the regular hexagon, whose diameter is *four*, the area equals the circumference.



In the present diagram, the diameter is divided into four units, therefore CD, the radius, is two, and DH, the breadth of the line of circumference, is made equal to one. CD being double of DH, let CD be divided in the center, and the small rectangles completed. Then it is manifest that the two rectangles contained in ABFG are severally equal to the two contained in DBCE, for they all have equal bases and equal heights by construction. But DBCE is equal to one-sixth of the area of the hexagon; therefore the whole area of the hexagon is equal to twelve such rectangles as the two contained in DBCE. And ABFG is one-sixth of the circumference of the hexagon; therefore the whole circumference must be composed of twelve such rectangles as the two contained in ABFG. Area and circumference, each being equal to twelve such rectangles, are equal to each other. Therefore the area of a hexagon, whose diameter is four, is equal to its circumference, agreeably to the proposition.

PROPOSITION XX.

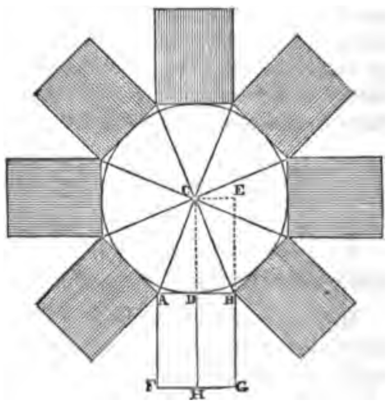
In the regular octagon, whose diameter is *one*, the area equals one-fourth of the circumference.



Take the diameter of the circle for unit, and then the diameter of the octagon is one, [Def. 17.] And the radius, CD, is half of one. Let CD be produced to H, making DH equal to one, or double CD; and AB being double DB, it is manifest that the rectangle DBCE is equal to each of the four rectangles contained in the rectangle ABFG, for they have equal bases and equal heights by construction. Lines drawn from the center to each of the eight angles of the octagon, divide the area into eight equal triangles, for the base of each triangle is an equal side of the octagon, and the height or perpendicular of each triangle equals the radius CD. One of the triangles, ABC, is equal to the rectangle DBCE, [Prop. 6.] Therefore the area of the octagon equals eight such rectangles as DBCE. AB being one-eighth of the perimeter of the octagon, and DH being equal to one, the rectangle ABFG constitutes one-eighth of the circumference, [Def. 16.] Therefore the whole circumference must be composed of thirty-two such rectangles as the four contained in ABFG. And the area being equal to eight such rectangles, therefore the area of an octagon, whose diameter is one, is equal to one-fourth of its circumference, agreeably to the proposition.

PROPOSITION XXI.

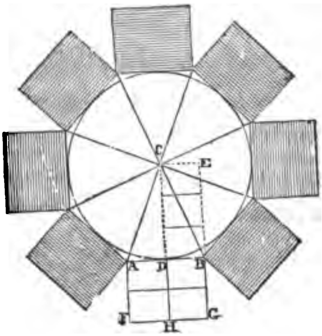
In the regular octagon, whose diameter is *two*, the area equals half the circumference.



Let the diameter of the circle and of the octagon be two; then the radius, CD, equals one. DH or AF also equals one, being the breadth of the line of circumference. And AB being double DB, it is manifest that each of the two rectangles contained in ABFG is equal to the rectangle DBCE, for they have an equal breadth and equal height by construction. ABFG is one-eighth of the circumference of the octagon; therefore the whole circumference must be equal to sixteen such rectangles as the two contained in ABFG. The area of the octagon equals eight such rectangles as BDCE, [Prop. 6.] Therefore the area of an octagon, whose diameter is two, equals one-half its circumference, agreeably to the proposition.

PROPOSITION XXII.

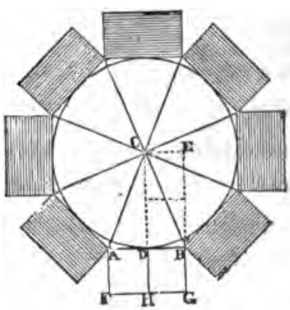
In the regular octagon, whose diameter is *three*, the area equals three-fourths of the circumference.



In the present diagram, diameter being three, the radius CD is one and a half, and DH, the breadth of the line of circumference, is one, and therefore equal to two-thirds of CD. Let CD be divided into three equal parts, and DH into two equal parts, and the small rectangles drawn upon the equal parts will be equal, having equal bases and equal heights. As was shown in the two last propositions, the area of the octagon is equal to eight such rectangles as DBCE, therefore it must be equal to twenty-four such rectangles as the three contained in DBCE. And ABFG being one-eighth of the circumference of the octagon, the whole circumference must be equal to thirty-two such rectangles as the four contained in ABFG. Twenty-four is three-fourths of thirty-two, therefore the area of an octagon, whose diameter is three, equals three-fourths of its circumference, agreeably to the proposition.

PROPOSITION XXIII.

In the regular octagon, whose diameter is *four*, the area equals the circumference.



In this diagram, diameter is divided into four units; therefore the radius CD is two, and DH, the breadth of the line of circumference, is one. CD being divided into two equal parts, and AB being double DB, it is manifest that the two rectangles contained in ABFG are equal to the two rectangles in DBCE, for they all have equal bases and equal heights. As already shown in the preceding propositions, the area of the octagon is equal to eight such rectangles as

DBCE; it is therefore equal to sixteen such rectangles as the two contained in DBCE. The rectangle ABFG is one-eighth of the circumference of the octagon, therefore the whole circumference must be equal to sixteen such rectangles as the two contained in ABFG. Therefore the area of an octagon, whose diameter is four, is equal in geometrical quantity to its circumference, agreeably to the proposition.

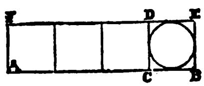
REMARK.—In all the preceding propositions, diameter has been the given or assumed quantity, and it has been seen to control the relation between area and circumference by one simple and uniform law, whatever may be the form or figure in which the area is presented. We shall now see that when circumference is the given or assumed quantity, it controls the relation between area and diameter by precisely the same simple and uniform law, so rigidly enforced by diameter in the preceding cases.

PROPOSITION XXIV.

In the square, whose circumference is *one*, the area equals one-fourth of the diameter.

REMARKS.—The terms of the proposition may seem to require a few words of explanation, to make them appear consistent with the principles already laid down. Since the unit in geometry always has the form of a cube, and when once assumed, is always indivisible, and invariable in form or size, the question may arise in the mind of the student, how can *one* constitute a circumference, or inclose area? It is manifest that *the* one, assumed as the unit, cannot inclose area. But we may assume smaller units, a certain number of which shall together be equal in size and value to the first assumed unit, and out of these smaller units we may form a circumference that will inclose an amount of area having a certain and fixed relation to the first assumed unit. When therefore we speak of a square whose circumference is *one*, it simply means a square whose circumference is equal in quantity to an assumed unit, that is, a square whose side is equal to the fourth part of an assumed unit.

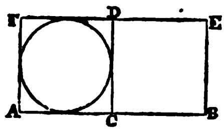
Now, in pursuing the proposition, we will assume the same quantity for the unit which we have been using in the previous demonstrations, viz., one inch. In comparing diameter and area, we have nothing whatever to do with circumference, except to consider its extension simply in length, for it is this length which controls the relation between diameter and area.



Therefore let AB be *one* [one inch]. Let it be divided into four equal parts, [which simply means, let four smaller units be assumed, each of which shall be equal to a fourth part of AB,] and out of these four parts form the square BCDE. Then the square BCDE, having a circumference which is equal in quantity to AB, may be said to have a circumference of *one*, and by the terms of the proposition, the area of this square must equal one-fourth of its diameter. The diameter of the square is the diameter of its inscribed circle, [Def. 17.] And this diameter must be measured by a perfect *line*, whose breadth is fixed by the assumed unit. The diameter, passing through the center of the circle, must extend in *length* to the extreme limits of the circle, and will therefore in length be equal to BE or AF, [Def. 19.] But the breadth of the diameter is *one*, that is, equal to the length of AB; for a line is always one in breadth, and varies not, whether it be to measure a quarter of an inch or a thousand inches. Therefore the rectangle ABEF is the diameter of the square BCDE. And AB being divided into four equal parts, the square BCDE is manifestly a fourth part of the rectangle ABEF, for it has the same height, and a fourth part of the base of the rectangle. Therefore, in the square whose circumference is *one*, the area equals one-fourth of the diameter, agreeably to the proposition.

PROPOSITION XXV.

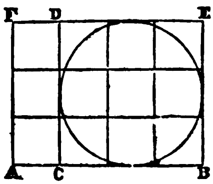
In the square, whose circumference is *two*, the area equals one half the diameter.



Let AB be one, [one inch.] Divide it into two equal parts at C, and on AC construct the square ACDF. Then each side of the square will be half of one, and the circumference of the square ACDF equals two. On CB also construct the square CBDE, which will also have a circumference of two, and the two squares will be equal to each other. The whole rectangle ABEF is the diameter of the square ACDF, for the breadth of this rectangle, AB, is *one*, and the length, AF or CD, extends to the height or extreme limits of the inscribed circle, [Definitions 17 and 19.] But this rectangle ABEF, which constitutes the diameter, is double the area of the square ACDF. Therefore in the square whose circumference is two, the area equals one-half the diameter, agreeably to the proposition.

PROPOSITION XXVI.

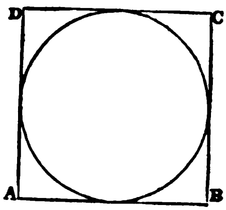
In the square, whose circumference is three, the area equals three-fourths of the diameter.



Again, let AB be one. Divide it into four equal parts, and on three of these parts CB, construct the square BCDE. Each side of this square is composed of three parts, each equal to a quarter of BA, making twelve quarters. Twelve quarters equal three units, therefore the circumference of the square BCDE equals three. BE is the length of the diameter, and AB is the breadth of the diameter; therefore the rectangle ABEF is the diameter of the circle and of the square BCDE. Divide the rectangle ABEF into equal squares on the four equal parts of AB, and the rectangle ABEF, or the diameter, will be seen to contain twelve of these equal squares. The square BCDE will be seen to contain nine of these small equal squares; and nine is three-fourths of twelve. Therefore in the square, whose circumference is three, the area equals three-fourths of the diameter, agreeably to the proposition.

PROPOSITION XXVII.

In the square, whose circumference is *four*, the area equals the diameter.



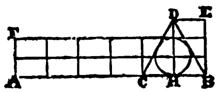
Again, let AB be one, and upon it construct the square ABCD. Each side of the square is equal to one, [Def. 58,] therefore the circumference is equal to four. The length of the diameter is BC, which now equals one, and the breadth of the diameter is AB, which is one; therefore the diameter is one in length and one in breadth, or one square. But the area also is one square. Therefore in the square, whose circumference is four, the area equals the diameter, agreeably to the proposition.

COROLLARY.—The diameter of a circle of *one* diameter is precisely equal to the circumscribed square of the same circle, [Definitions 19 and 21.]

REMARK.—The student may now be prepared to see why, in numbers, 1 can never be made anything but 1 by any operation performed upon it. If 2 be squared, it becomes 4, and if it be cubed it becomes 8. But 1 squared, is still 1, and 1 cubed is still 1, and the root of 1 is still 1, and sometimes to the no small embarrassment and perplexity of the student or mathematician. The mystery and difficulty seem to vanish when we understand that 1 represents a *magnitude*, which is 1 in every particular; a magnitude which is 1 in length, 1 in breadth, and 1 in thickness; a magnitude whose diameter is 1, whose area is 1, and whose solidity is 1; a magnitude whose square is 1, and whose cube is 1, and whose square root is 1, and whose cube root is 1. As the unit is *one* in all these particulars, the number which represents each and all of these particulars must always be 1, and can never be made anything else.

PROPOSITION XXVIII.

In the equilateral triangle, whose circumference is *one*, the area equals one-fourth of the diameter.



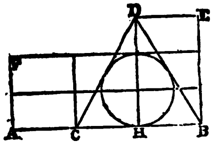
Let AB be the unit, [one inch.] Divide AB into six equal parts, and on two of the parts, CB, equal to one-third of AB, erect the equilateral triangle BCD. Each side of the triangle being equal to a third of AB, or a third of *one*, the whole circumference of the triangle is equal to one. And DH being the perpendicular of the triangle BCD, the area of the triangle is equal to the rectangle BHDE, [Proposition 6.] The rectangle BHDE is divided into three equal smaller rectangles by lines drawn parallel to the base, one through the center of the circle and one touching its upper limit, because the diameter or height of the circle is equal to two-thirds of the perpendicular of the triangle, [Proposition 7.] And these two lines being produced parallel to AB and equal in length to AB, and perpendiculars being drawn from the several points of division in AB, it is manifest that the whole figure will be divided into small rectangles equal to each other. The length of the diameter of the circle and of the triangle is AF, and the breadth of the diameter is AB, therefore the diameter is seen to contain twelve of the small equal rectangles. And BHDE, which equals the area of the triangle, contains three such rectangles. And three is one-fourth of twelve. Therefore the area of an equilateral triangle, whose circumference is one, equals one-fourth of its diameter, agreeably to the proposition.

Arithmetical calculation of the same triangle. The side of the triangle being a third of 1, the decimal expression is .333333+, which being squared gives .111111+. Three-fourths of this square is .0833333+, the square root of which is .288675+. And this last number is the perpendicular of the triangle. But the diameter of a circle inscribed in an equilateral triangle is two-thirds the perpendicular. Therefore the diameter of the circle, and of the triangle, is two-thirds of .288675+, viz., .19245+; and one-fourth of this last number, by the proposition, must equal the area. Multiply the perpendicular, .288675+, by half the base or side, viz., .166666+, and it gives for the area of the triangle .048112+, and this multiplied by 4 gives .1924+, equal to the diameter.

Therefore in the equilateral triangle whose circumference is 1, the area equals one-fourth of the diameter in numbers as well as in geometrical quantity.

PROPOSITION XXIX.

In the equilateral triangle, whose circumference is *two*, the area equals one-half the diameter.

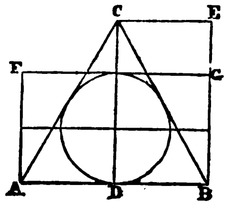


Let AB be the unit, divided into three equal parts at C and H. On CB erect the equilateral triangle BCD. Each side of the triangle equals two-thirds of one, therefore the whole circumference equals six-thirds of one, or equals two. The area of the triangle BCD equals the rectangle BHDE, [Proposition 6.] And the rectangle BHDE is divided as in the last proposition into three equal rectangles. The diameter has a length, AF, equal to two-thirds of DH, and a breadth, AB, equal to one. Therefore by lines drawn as in the last proposition, the diameter is seen to be divided into six rectangles, each similar and equal to the three rectangles in BHDE. Area is equal to three rectangles, and diameter to six. Therefore in the equilateral triangle, whose circumference is *two*, the area equals half the diameter, agreeably to the proposition.

Arithmetical calculation of the same triangle. The side of the triangle being two-thirds of 1, is .666666+, which being squared is .444444+. And three-fourths of the square is .333333+, the square root of which is .57735+, and this is the perpendicular of the triangle. And the perpendicular multiplied by half the base, viz., .33333+ gives for area .19245+. And two-thirds of the perpendicular, which is the diameter of the circle, gives .3849+, which is double .19245+. Therefore in the equilateral triangle, whose circumference is two, the area equals one-half the diameter in numbers as well as in geometrical quantity.

PROPOSITION XXX.

In the equilateral triangle, whose circumference is *three*, the area equals three-fourths of the diameter.

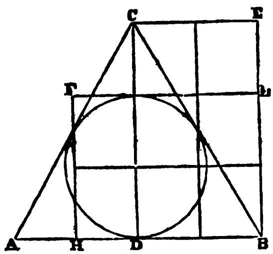


Take AB equal to one, and upon it construct the equilateral triangle ABC. Each side of the triangle then is one, and the circumference is 3. The area of the triangle is equal to the rectangle BDCE, [Prop. vi.] AF is the length of the diameter of the circle, and AB the breadth, therefore the rectangle BAFG is the diameter, and it is seen to be divided into four equal rectangles. And BDCE, which is equal to the area, is divided into three similar equal rectangles. Therefore, in the equilateral triangle whose circumference is three, the area equals three-fourths of the diameter, agreeably to the proposition.

Arithmetical calculation of the same triangle.—The side of the triangle being 1, the square of the side is 1, and three-quarters of the square is .75. Therefore the square root of the decimal .75, viz., .8660254+, is the perpendicular of the triangle. And two-thirds of the perpendicular, viz., .5773502+, is the diameter, [Prop. vii.] Now for the area, multiply the perpendicular, .8660254+, by half the base, that is, by half of 1, or the decimal .5, and it gives .4330127+, and this last sum is three-fourths of .5773502+, which equals the diameter. Therefore in the equilateral triangle, whose circumference is three, the area equals three-fourths of the diameter in numbers, as well as in geometrical quantity.

PROPOSITION XXXI.

In the equilateral triangle, whose circumference is *four*, the area equals the diameter.

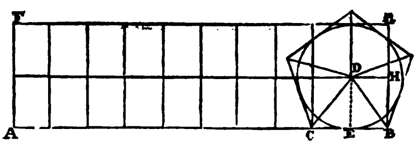


Take AB equal to one and one-third. Divide AB into four equal parts, and HB, being three of the parts, will be equal to one, or the unit. On AB erect the equilateral triangle ABC; then each side of the triangle being one and one-third, the whole circumference will be four. The area of the triangle equals the rectangle BDCE, [Prop. vi.] And the diameter of the triangle and the circle is the rectangle BHFL, having a breadth of one, HB, and extending in length to the extreme limits of the circle, [Definitions 17 and 19.] HB is divided into three equal parts, and CD is also divided into three equal parts at the center and circumference of the circle. Therefore, lines drawn through these points of division, divide the whole figure into small rectangles equal to each other. BDCE, equal to area, contains six of these rectangles; and BHFL, the diameter, also contains six of these rectangles. Therefore, in the equilateral triangle, whose circumference is four, the area equals the diameter, agreeably to the proposition.

Arithmetical calculation of the same triangle.—The side of the triangle being one and one-third, viz., 1.333333+, the square of the side is 1.777777+; and three-quarters of the square is 1.3333333+, [equal to the side.] The square root of this last sum, viz., 1.1547+, is the perpendicular of the triangle; and two-thirds of the perpendicular, viz., .7698+ is the diameter, [Prop. vii.] The perpendicular, 1.1547+, multiplied by half the base, .666666+, gives also .7698+ for area. Therefore in the equilateral triangle, whose circumference is four, the area equals the diameter in numbers, as well as in geometrical quantity.

PROPOSITION XXXII.

In the regular pentagon, whose circumference is *one*, the area equals one fourth part of the diameter.

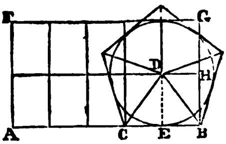


Let AB be the unit. [In this instance the unit is taken at a length of two inches instead of one inch, in order to give sufficient distinctness to the diagram.] Then if AB be one, the side of a regular pentagon, whose circumference is one, must be one-fifth of AB. Therefore divide AB into ten equal parts, and upon two of those parts, CB, construct a regular pentagon. The pentagon will then have a circumference equal to one. The diameter of the pentagon and of the inscribed circle is the rectangle ABFG, for AB is the breadth of the diameter, being one, and AF or BG is the length of the diameter, as they extend the line of diameter in length to the extreme limits of the circle. AB being divided into ten equal parts, perpendiculars drawn from those points of division, and a line drawn parallel to AB through the center of the circle, manifestly divide the rectangle ABFG into twenty equal rectangles. The area of the pentagon is divided into five equal triangles by the lines drawn from the center to the five angles. One triangle, BCD, is equal to one rectangle, BEDH, [Prop. vi.] Therefore the five triangles, or the whole area of the pentagon, is equal to five of the rectangles.

And five is one-fourth of twenty; therefore in the regular pentagon, whose circumference is one, the area equals one-fourth of the diameter, agreeably to the proposition.

PROPOSITION XXXIII.

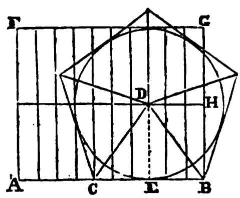
In the regular pentagon, whose circumference is *two*, the area equals one-half the diameter.



Let AB be one. [In this diagram it is again one inch.] Divide AB into five equal parts, and on two of the parts, CB, construct a regular pentagon. The circumference of the pentagon will then be equal to ten such parts, or two units. The diameter is the rectangle ABFG, and it is seen to be divided, as was shown in the last diagram, into ten equal rectangles. The area of the pentagon is divided into five equal triangles, the base of each triangle being a side of the pentagon. And one triangle, BCD, is equal to one rectangle, BEDH, of the same height and half the base of the triangle, [Prop. 6.] Therefore the five triangles, or the whole area, is equal to five of the rectangles, of which the diameter contains ten. Therefore in the regular pentagon, whose circumference is two, the area equals half the diameter, agreeably to the proposition.

PROPOSITION XXXIV.

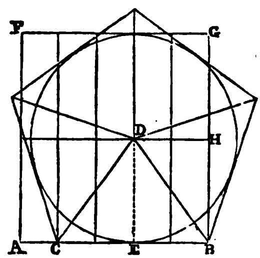
In the regular pentagon, whose circumference is *three*, the area equals three-fourths of the diameter.



Let AB be one. Divide it into ten equal parts, and on six of the parts, CB, construct a regular pentagon. Each side of the pentagon then being equal to six-tenths of one, the five sides, or whole circumference, will be equal to thirty-tenths, or *three* units. AB being one, and AF the height of the circle, the rectangle ABFG constitutes the diameter. And by the equal divisions of AB, and the line through the center of the circle parallel to AB, the diameter is seen to be divided into twenty equal rectangles. The area of the pentagon is divided into five equal triangles, and one triangle, BCD, is equal to three rectangles, contained in BEDH, [Prop. 6.] Therefore the five triangles, or the whole area, must be equal to fifteen of the rectangles; and fifteen is three-fourths of twenty. Therefore in the regular pentagon, whose circumference is three, the area equals three-fourths of the diameter, agreeably to the proposition.

PROPOSITION XXXV.

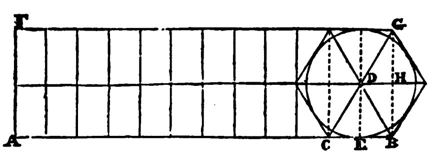
In the regular pentagon, whose circumference is *four*, the area equals the diameter.



Let AB be one, and divide it into five equal parts. On four of the parts, CB, construct a regular pentagon. Each side of the pentagon will then be equal to four-fifths of one, and the five sides, or whole circumference, will be equal to twenty-fifths of one, or equal to *four* units. AB is the breadth of the diameter, and AF the length, and the rectangle ABFG, which constitutes the diameter, is seen to be divided into ten equal rectangles, having equal bases—viz., one-fifth of AB, and equal heights—viz., the radius of the circle, DE. The area of the pentagon is divided into five equal triangles, one of which, BCD, is equal to two of the rectangles, contained in BEDH, [Prop. 6.] Therefore the five triangles, or the whole area of the pentagon, must be equal to ten such rectangles, and consequently equal to the diameter. Therefore in the regular pentagon, whose circumference is four, the area equals the diameter, agreeably to the proposition.

PROPOSITION XXXVI.

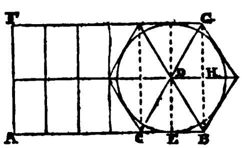
In the regular hexagon, whose circumference is *one*, the area equals one-fourth of the diameter.



Let AB be the unit. [It is taken at the length of two inches, in order to be large enough to give distinctness to the diagram.] Then if AB be one, let it be divided into twelve equal parts, and upon two of the parts, CB, construct a regular hexagon. The circumference of the hexagon will then be equal to AB, for each of the six sides is equal to two of the divisions of AB. The circumference therefore is equal to one. AF is the length of the diameter, being equal to the height of the circle, and AB is the breadth of the diameter because it is the unit. Therefore the rectangle ABFG constitutes the diameter, and it is seen to be divided into twenty-four equal rectangles by the equal divisions of AB, and the line parallel to AB drawn through the center of the circle. The area of the hexagon is divided into six equal and equilateral triangles, the base of each triangle being a side of the hexagon, and its perpendicular the radius of the circle, as DE. One triangle, BCD, is equal to one of the rectangles, BEDH, [Prop. vi.] Therefore the six triangles, or the whole area of the hexagon, must be equal to six of the rectangles. But the diameter is equal to twenty-four such rectangles. Therefore in the regular hexagon whose circumference is one, the area equals one-fourth part of the diameter, agreeably to the proposition.

PROPOSITION XXXVII.

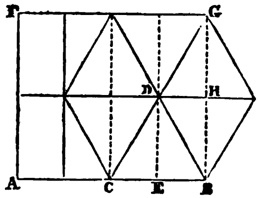
In the regular hexagon, whose circumference is *two*, the area equals one-half the diameter.



Let AB be one, and divide it into six equal parts. On two of the parts, CB, construct a regular hexagon. The six sides, or circumference of the hexagon, will then be equal to twelve such parts, or equal to two units, and circumference will be two. The rectangle ABFG is the diameter, as previously shown, and it is seen to be divided into twelve equal rectangles. The area of the hexagon is divided into six equal triangles. And one triangle, BCD, is equal to one of the rectangles, BEDH. Therefore the whole area of the hexagon is equal to six of the rectangles, and diameter being equal to twelve, area equals half the diameter. Therefore in the regular hexagon, whose circumference is two, the area equals half the diameter, agreeably to the proposition.

PROPOSITION XXXVIII.

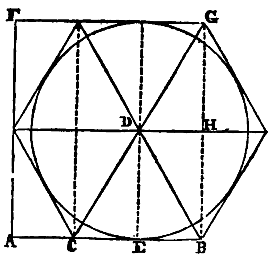
In the regular hexagon, whose circumference is *three*, the area equals three-fourths of the diameter.



Let AB be the unit, divided into four equal parts. On two of the parts, CB, construct a regular hexagon. The circumference of the hexagon is then equal to three, for each of the six sides is equal to half of one, that is, half of AB. The diameter is the rectangle ABFG, as before shown, and it is seen to be divided into eight equal rectangles.—The hexagon is divided into six equal triangles, each triangle having a side of the hexagon for its base. One triangle, BCD, is equal to one of the rectangles, BEDH, [Prop. 6.] Therefore the six triangles, or the area of the hexagon, is equal to six of the rectangles. But diameter is equal to eight rectangles, and six is three-fourths of eight; therefore in the regular hexagon, whose circumference is three, the area equals three-fourths of the diameter, agreeably to the proposition.

PROPOSITION XXXIX.

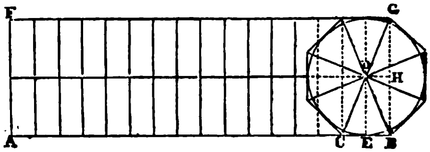
In the regular hexagon, whose circumference is *four*, the area equals the diameter.



Let AB be one, and divide it into three equal parts. On two of the parts, CB, construct a regular hexagon. One side of the hexagon, CB, being two-thirds of one, the six sides, or whole circumference, will be equal to twelve-thirds, or four units. Therefore circumference is four. The diameter has a breadth of one, equal to AB, and a length extending to the height of the circle, equal to AF. Therefore the rectangle ABFG constitutes the diameter, and it is seen to be divided into six equal rectangles. The hexagon is divided into six equal triangles, having the sides of the hexagon for their bases, and one of these triangles, BCD, is equal to one of the rectangles, BEDH, [Prop. 6;] therefore the six triangles of the area are equal to the six rectangles of the diameter. Therefore in the regular hexagon, whose circumference is four, the area equals the diameter, agreeably to the proposition.

PROPOSITION XL.

In the regular octagon, whose circumference is *one*, the area equals one-fourth of the diameter.

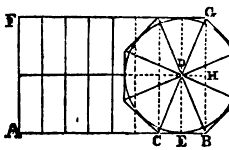


Let AB be the unit, [taken at a length of two inches on account of the smallness of the octagon,] and let AB be divided into sixteen equal parts. Then if two of the parts be made the side of an octagon, the circumference of the octagon will be one, that is, equal to AB. Therefore upon two of the equal parts, CB, construct a regular octagon, and its circumference will be one. The breadth of the diameter is AB, and its length is AF. Therefore the rectangle ABFG constitutes the diameter, and it is seen to be divided into thirty-two equal rectangles; equal, because the base of each is the sixteenth part of the unit AB, and the height of each equals the radius of the circle, DE.

The area of the octagon is divided into eight equal triangles, each having a side of the octagon for a base, and a perpendicular equal to the radius of the circle. One of these triangles, BCD, is equal to one of the rectangles, BEDH, [Prop. 6.] Therefore the eight triangles, or the whole area of the octagon, must be equal to eight rectangles. But the diameter equals thirty-two rectangles, and eight is one-fourth of thirty-two. Therefore in the regular octagon whose circumference is one, the area equals one-fourth of the diameter, agreeably to the proposition.

PROPOSITION XLI.

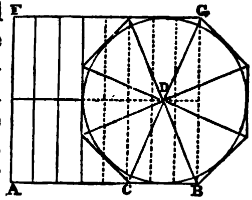
In the regular octagon, whose circumference is *two*, the area equals half the diameter.



Let AB be one, divided into eight equal parts, and on two of the equal parts, CB, construct a regular octagon. Each side of the octagon then is equal to a fourth part of AB, or a fourth of one, and the eight sides therefore equal eight-fourths, or two units; and circumference is therefore two. The rectangle ABFG, which constitutes the diameter, is seen to be divided into sixteen equal rectangles. The area of the octagon is divided into eight equal triangles, one of which, BCD, is equal to one of the rectangles, BEDH, [Prop. 6.] Therefore the eight triangles, or the whole area of the octagon, is equal to eight rectangles. But the diameter is equal to sixteen such rectangles; therefore in the regular octagon, whose circumference is two, the area equals half the diameter, agreeably to the proposition.

PROPOSITION XLII.

In the regular octagon, whose circumference is *three*, the area equals three-fourths of the diameter.

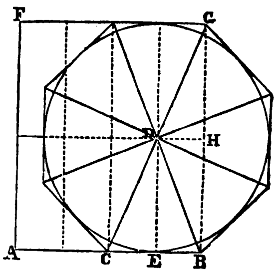


Take AB equal to one, (one inch,) and divide it into eight equal parts. On three of the parts, CB, construct a regular octagon. The circumference of the octagon will then be equal to three; for one side being three-eighths of one, the eight sides will be equal to twenty-four-eighths, or twelve quarters, or six halves, or three wholes or units. Therefore circumference is three.

The rectangle ABFG, which constitutes the diameter, is divided into sixteen equal rectangles by the perpendiculars from the points of division in AB and the line drawn parallel to AB through the center of the circle. The area of the octagon is divided into eight equal triangles, each having a side of the octagon for its base. One of these triangles, BCD, is seen to have the same height or perpendicular as one of the rectangles, and to have a base, CB, equal to the base of three of the rectangles. Therefore the triangle is equal to one rectangle and a-half, [Prop. 6;] and the eight triangles, or the whole area of the octagon, must be equal to twelve rectangles. Diameter is equal to sixteen rectangles, and twelve is three-fourths of sixteen. Therefore in the regular octagon, whose circumference is three, the area equals three-fourths of the diameter, agreeably to the proposition.

PROPOSITION XLIII.

In the regular octagon, whose circumference is *four*, the area equals the diameter.

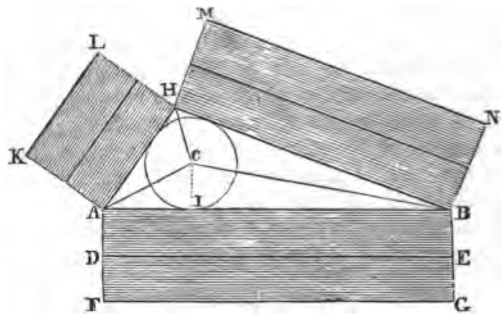


Let AB be one, and divide it into four equal parts. On two of the parts, CB, construct a regular octagon. One side of the octagon, CB, being half of one, two sides will be equal to one, and the whole eight sides equal to four. Therefore the circumference of the octagon is four. The rectangle ABFG is the diameter of the octagon and of the inscribed circle, for it has a breadth of one, AB, and a length, AF, equal to the height or extreme limits of the circle, [Definitions 17 and 19.] The diameter is divided into eight equal rectangles by the perpendiculars drawn from the points of equal divisions in AB, and the line drawn parallel to AB through the center of the circle. The area of the octagon is divided into eight equal triangles, each having a side of the octagon for its base. One of these triangles, BCD, is equal to one of the rectangles, BEDH, [Prop 6;] therefore the eight triangles of the octagon must be equal to the eight rectangles of the diameter. Therefore in the regular octagon, whose circumference is four, the area equals the diameter, agreeably to the proposition.

PROPOSITION XLIV.

In any triangle, regular or irregular, whose diameter is *one*, the area equals one-fourth of the circumference.

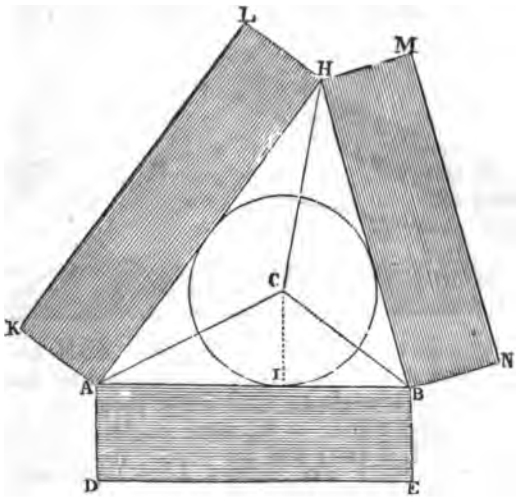
Let the diameter of the circle in the diagram be one. [The unit in this demonstration is taken at half-an-inch.] Now draw around it the triangle ABH, without any regard to the relative length of the sides. The



diameter of the triangle is then one, [Def. 17.] On each side of the triangle construct a rectangle of the breadth of one, or a breadth equal to the diameter of the circle. The three rectangles then constitute the circumference of the triangle, for their whole length equals the perimeter of the triangle, and their breadth is one, [Def. 16.] Lines drawn from the center, C, to each of the angles of the triangle, divide the area into three new triangles. Let each of these triangles be compared separately with that portion of the circumference adjacent to it. Take the triangle ABC. Its base is AB, and its perpendicular or height is CI, the radius of the circle, equal to half of one. The portion of circumference adjacent to this triangle is the rectangle ABFG, having a breadth of one; and if divided into two equal rectangles, AD will be half of one. The triangle ABC and the rectangle ABDE have the same base, AB, and equal heights, CI and AD; therefore the triangle equals half the rectangle ABDE, [Corollary, Prop. 6,] and consequently the triangle ABC equals one-fourth of the rectangle ABFG. In the same manner it may be shown, that the triangle AHC equals one-fourth of the rectangle AHKL, and that the triangle HBC equals one-fourth of the rectangle HBMN. Therefore the three triangles together, equal one-fourth of the three rectangles. Therefore in any triangle, whose diameter is one, without regard to the relative length of the sides, the area equals one-fourth of the circumference, agreeably to the proposition.

PROPOSITION XLV.

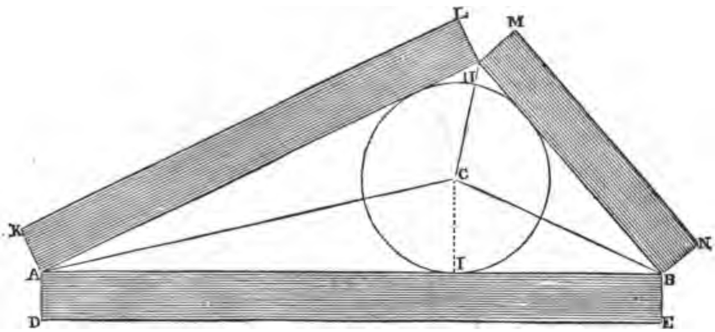
In any triangle whatever, whose diameter is *two*, the area equals one-half of the circumference.



Let the diameter of the circle be two, making the unit half-an-inch. Circumscribe around it the triangle ABH, without regard to the relative length of the sides. Divide the area into three triangles by lines drawn from the center of the circle to the three angles of the triangle ABH. Diameter being two, the radius, CI, is one; therefore the perpendicular or height of the triangle ABC is one. The shaded lines of circumference on each side of the triangle being drawn with a breadth of one, AD is equal to CI. And the rectangle ABDE and the triangle ABC having the same base, AB, and equal heights, AD and CI, the triangle equals half the rectangle. In the same manner it may be shown that the triangle AHC equals half the rectangle AHKL, and that the triangle HBC equals half the rectangle HBMN. Therefore the three triangles together equal half of the three rectangles together. But the three triangles constitute the area of the triangle ABH, and the three rectangles constitute the circumference. Therefore in any triangle whose diameter is two, the area equals one-half the circumference, agreeably to the proposition.

PROPOSITION XLVI.

In any triangle whatever, whose diameter is *four*, the area equals the circumference.

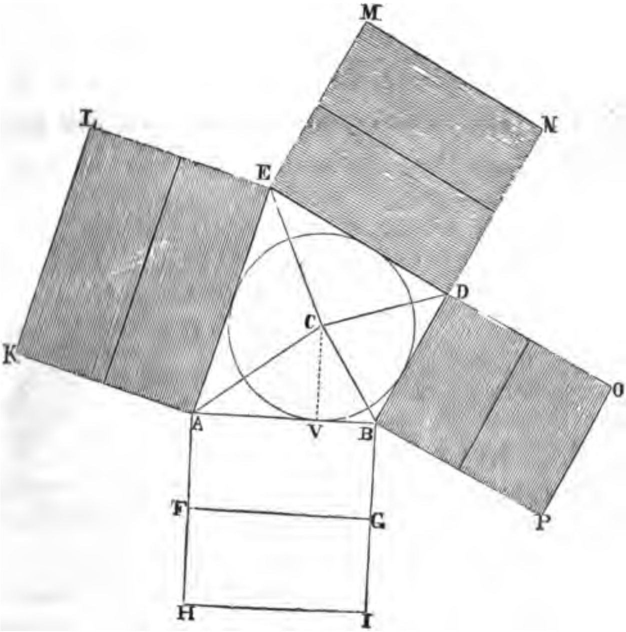


Let the diameter of the circle be four, and ABH be a triangle circumscribed around it without regard to the relative length of the sides. And let the triangle be divided into three new triangles by lines drawn from the center to the three angles. Diameter being four, the radius, CI, is two; and the shaded line of circumference having a breadth of one, AD equals half of CI. Therefore the triangle ABC is equal to the rectangle ABDE; for if AD were equal to CI, the rectangle would be double the triangle, as appeared in the last proposition.

In the same manner it may be shown, that the triangle AHC equals the rectangle AHKL, and that the triangle HBC equals the rectangle HBMN. Therefore the three triangles, which constitute the area of the triangle ABH, are equal to the three rectangles, which constitute the circumference. Therefore in any triangle whatever, whose diameter is four, the area equals the circumference, agreeably to the proposition.

PROPOSITION XLVII.

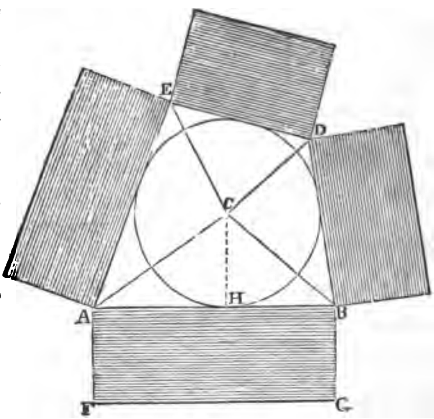
In any quadrilateral, or four-sided figure, regular or irregular, whose diameter is *one*, the area equals one-fourth of the circumference.



Let the diameter of the circle be one, and ABDE a quadrilateral figure circumscribed around it, without any regard to the relative length of the sides. On each of the four sides construct a rectangle, each having a breadth of one. The four rectangles then will constitute the circumference of the figure. Lines drawn from the center of the circle to the four angles divide the area of the figure into four triangles, each having a side of the figure for its base, and the radius of the circle for its height or perpendicular. Then if AH be equally divided at F, the triangle ABC will equal half the rectangle ABFG, for they have the same base, AB, and equal heights, CV and AF, [corollary, Prop. 6.] And ABFG being half of ABHI, the triangle ABC is equal to one-fourth of the rectangle ABHI. In the same manner it may be shown that each of the three remaining triangles is equal to one-fourth of that portion of the circumference adjacent to it. Therefore the four triangles which constitute the area are together equal to one-fourth of the four rectangles which constitute the circumference. Therefore in any quadrilateral figure whatever, whose diameter is one, the area equals one-fourth of the circumference, agreeably to the proposition.

PROPOSITION XLVIII.

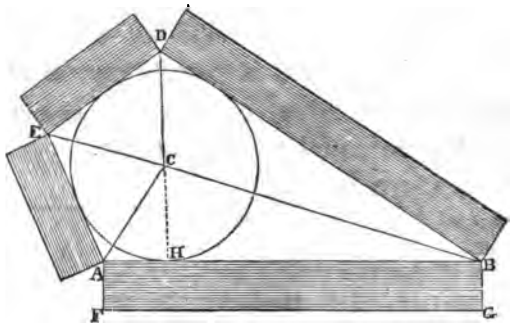
In any quadrilateral figure whatever, whose diameter is *two*, the area equals one-half of the circumference.



Let the diameter of the circle be two, and ABDE a quadrilateral figure circumscribed around it, without any regard to the relative length of the sides. On each of the four sides construct a rectangle, each having a breadth of one. Diameter being two, the radius, CH, is one, and equal to the breadth of the circumference, AF. Therefore, the triangle ABC equals one-half the rectangle ABFG, for they have the same base, AB, and equal perpendiculars, CH and AF. [cor. Prop. 6.] In the same manner it may be shown that the three remaining triangles of the area are equal to half the three remaining rectangles of the circumference. Therefore the whole area is equal to half the circumference. Therefore in any quadrilateral figure whatever, whose diameter is two, the area equals half of the circumference, agreeably to the proposition.

PROPOSITION XLIX.

In any quadrilateral figure whatever, whose diameter is *four*, the area equals the circumference.

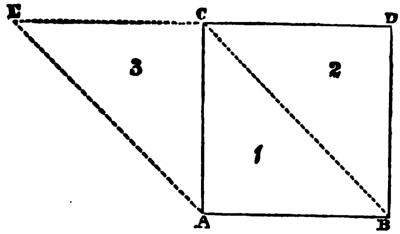


Let the diameter of the circle be four. Then the radius, CH, is two, and the breadth of the line of circumference, AF, is one. Therefore the triangle ABC is equal to the rectangle ABFG, for they have the same base, AB, and the perpendicular of the triangle, CH, is double the perpendicular of the rectangle, [corollary, Prop. 6.] In the same manner it may be shown that the triangle AEC is equal to the rectangle adjacent to it, and that the triangle EDC is equal to the rectangle adjacent to it, and that the triangle DBC is equal to the rectangle adjacent to it. But the four triangles constitute the area, and the four rectangles constitute the circumference of the quadrilateral figure; therefore the whole area is equal to the whole circumference. Therefore in any quadrilateral figure whatever, whose diameter is four, the area equals the circumference, agreeably to the proposition.

Remark.—The demonstrations thus far are deemed sufficient to establish the general law, that in all polygons, regular or irregular, of any number of sides whatever, when diameter is one, area equals one-fourth of the circumference; when diameter is two, area equals one-half the circumference; when diameter is three, area equals three-fourths of the circumference; when diameter is four, area and circumference are equal. And as the law remains in all polygons, even to an infinite number of sides, it manifestly remains when the sides at last vanish and the polygon dissolves into the perfect circle. So that the law necessarily applies to the circle as well as to all plane figures bounded by straight lines.

PROPOSITION L.

A rectangle and a parallelogram of the same base and equal heights, or of equal bases and the same height, are equal to each other.

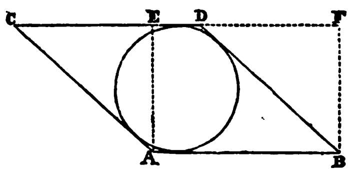


Let ABCD be a rectangle, and ABCE a parallelogram. Then the opposite sides of the two figures are respectively equal to each other—viz., AB to CD, AB to CE, BD to AC, and BC to AE, [Def. 60.] The rectangle ABCD is divided by the diagonal BC into two equal triangles, marked 1 and 2, and the parallelogram ABCE is divided by the diagonal AC into two equal triangles, marked 1 and 3, [Prop. 5.] But the triangle 1 is half of the rectangle, and it is also half of the parallelogram. Therefore the whole rectangle is equal to the whole parallelogram, [Axiom Third.] Both figures have the same base, AB, and the same or equal heights, AC. Therefore a rectangle and parallelogram, of the same base and equal heights, are equal to each other, agreeably to the proposition.

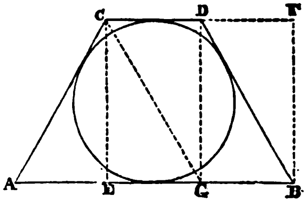
Also, if the diagram be turned the other side up, the figures will be seen to have equal bases, DC, and CE, and the same height, AC, according to the proposition.

PROPOSITION LI.

In any quadrilateral figure, whose circumference is four, the area equals the diameter.



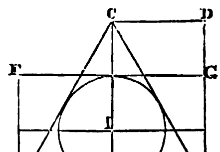
Let ABCD be a parallelogram, whose circumference is four, each side being one. The rectangle ABEF is the diameter of the inscribed circle, and therefore the diameter of the parallelogram, for it has a breadth of one, AB, and a length, AE, or BF, extending to the height of the circle. But the rectangle ABEF is equal to the parallelogram ABCD, [Prop. 50.] Therefore the area of the parallelogram is equal to its diameter, agreeably to the proposition.



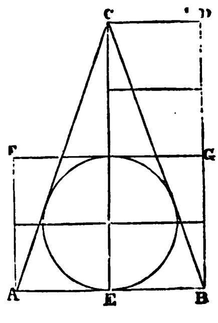
Again: let ABCD be a quadrilateral figure, whose circumference is four. If the circumference be so divided as to make AB equal one and a half, AC and BD each one, and CD equal half of one, and parallel to AB, the figure then can receive an inscribed circle. Take BE equal to one, and the rectangle BECF is the diameter of the circle and of the quadrilateral figure ABCD. Draw the perpendicular DG and the diagonal CG, and the whole diagram is seen to be divided into five right angled triangles, which are equal to each other, for they have equal bases, and equal heights. Four of these triangles are contained in the quadrilateral ABCD, and four are contained in the rectangle BECF. Therefore the area of the quadrilateral, whose circumference is four, equals its diameter, agreeably to the proposition.

PROPOSITION LII.

In all triangles whatever, the whole circumference bears the same proportion to the base as the perpendicular of the triangle bears to the radius of the inscribed circle.

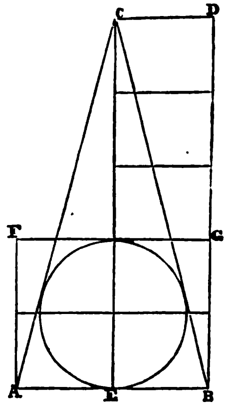


First. Let ABC be a triangle, whose circumference is three [three inches], and the base, AB, one. Draw the perpendicular, CE, and divide it into three equal parts, and one part, EI, will be the radius of the inscribed circle. Circumference is to the base as three to one, and the perpendicular is to radius as three to one, agreeably to the proposition.

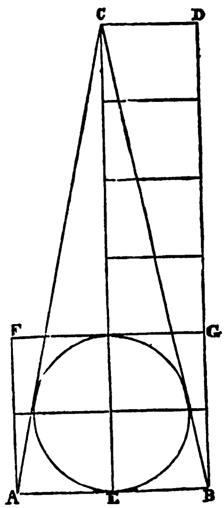


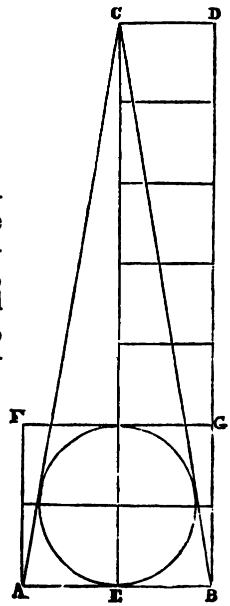
Second. Let ABC be a triangle, whose circumference is four, and base one, making the other two sides together equal to three; and if they are equal, or the triangle is isosceles, the two legs will each be one and a half. Draw the perpendicular CE, and divide it into four equal parts, and one part will be the radius of the inscribed circle. Circumference is to the base as 4 to 1, and the perpendicular is to radius of the inscribed circle as 4 to 1, agreeably to the proposition.

Third. Let the circumference of the triangle be five, and the base, AB, one. Then the sides AC and BC, being equal, will each be two. Draw the perpendicular, CE, and divide it into five equal parts, and one part will be the radius of the inscribed circle. Circumference is to the base as 5 to 1, and the perpendicular is to the radius of the inscribed circle as 5 to 1, agreeably to the proposition.

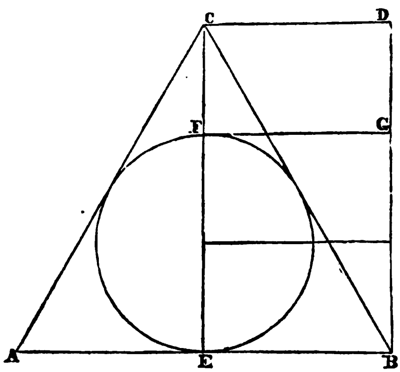


Fourth. Let the circumference of the triangle be six, and the base one. The other two sides will then be two and a half each. Draw the perpendicular, CE, and divide it into six equal parts, and one part will be the radius of the inscribed circle. Circumference is to base as 6 to 1, and perpendicular to radius as 6 to 1, agreeably to the proposition.





Fifth. Let the circumference of the triangle be seven, and the base one, making the other two sides each three. If the perpendicular, CE, be divided into seven equal parts, one part will be the radius of the inscribed circle. Circumference is to base as 7 to 1, and perpendicular to radius as 7 to 1, agreeably to the proposition.

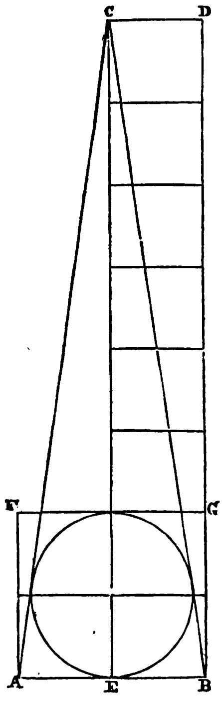


Sixth. Let the circumference of the triangle be six, and the base two. Then if the other two sides are equal, the triangle will be not only isosceles, but equilateral, each side being two. If the perpendicular, CE, be divided into three equal parts, one part will be the radius of the inscribed circle. Circumference is to base as 6 to 2 or 3 to 1, and the perpendicular is to the radius of the circle as 3 to 1, agreeably to the proposition.

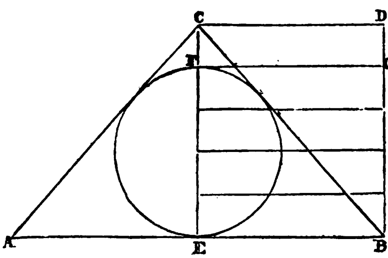
Corollary.—The diameter of a circle inscribed in an equilateral triangle equals two-thirds the perpendicular of the triangle.

Remark.—The truth contained in this corollary was attempted to be proved in the seventh proposition, as the principle was needed in subsequent demonstrations. It is, however, clearly developed as a corollary from the demonstrations under the present proposition, and presents itself with great clearness and simplicity.

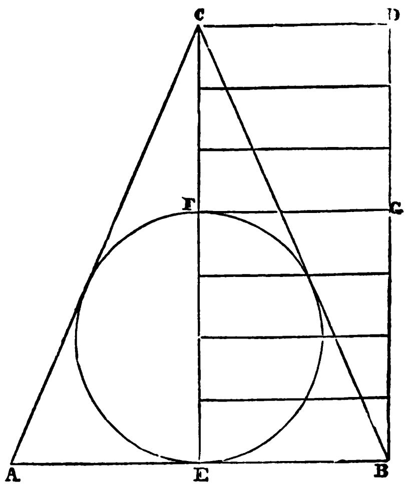
Seventh. Let the circumference of the triangle be eight, and the base one, the other two sides being three and a half each. If the perpendicular, CE, be divided into eight equal parts, one part will be the radius of the inscribed circle. Circumference is to base as 8 to 1, and perpendicular to radius as 8 to 1, agreeably to the proposition.



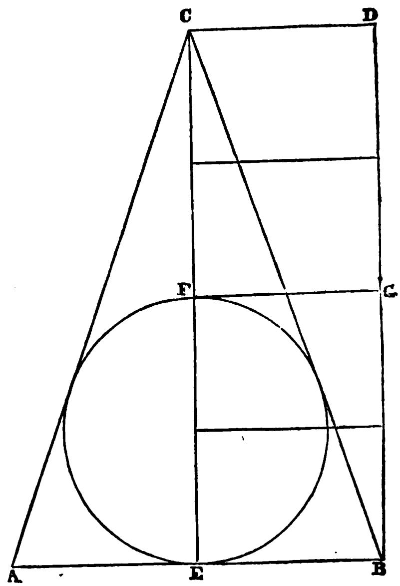
Eighth. Let the circumference of the triangle be five, and the base *two*, making the other two sides one and a half each. If the perpendicular, CE, be divided into five equal parts, two parts will be the radius of the inscribed circle. Circumference is to base as 5 to 2, and perpendicular to radius as 5 to 2, agreeably to the proposition.



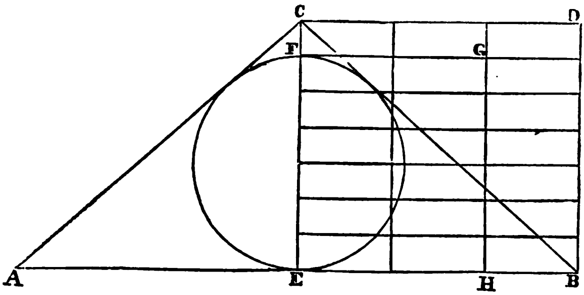
Ninth.—Let the circumference of the triangle be seven, and the base two. Then the other sides will each be two and a half. If the perpendicular CE be divided into seven equal parts, two of the parts will constitute the radius of the inscribed circle. Circumference is to base as 7 to 2, and perpendicular to radius as 7 to 2, agreeably to the proposition.



Tenth.—Let the circumference of the triangle be eight, and the base two, making the other two sides each three. If the perpendicular CE be divided into four equal parts, one part will be the radius of the inscribed circle. Circumference is to base as 8 to 2, or 4 to 1, and perpendicular is to radius as 4 to 1, agreeably to the proposition.

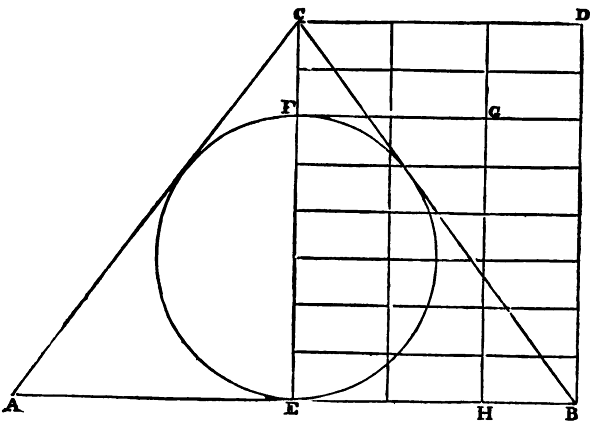


Eleventh.—Let the circumference of the triangle be seven, and the base *three*, making each of the other sides two. If the per-



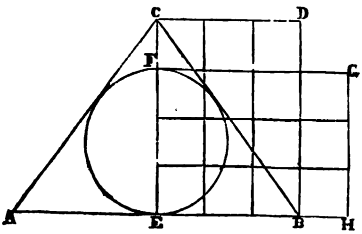
pendicular CE be divided into seven equal parts, three of the parts will constitute the radius of the inscribed circle. Circumference is to the base as 7 to 3, and the perpendicular to the radius as 7 to 3, agreeably to the proposition.

Twelfth.—Let the circumference of the triangle be eight, and the base three, making the other sides two and a half each. If the

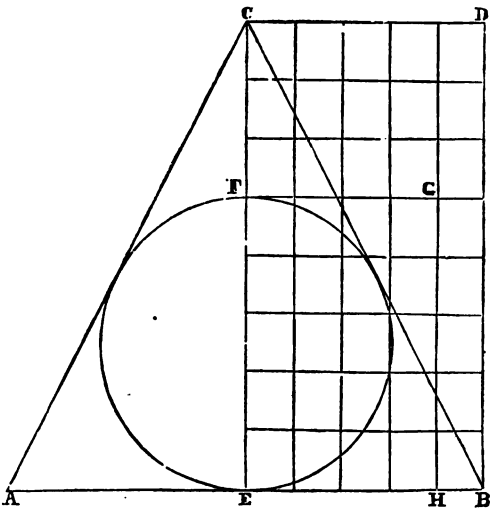


perpendicular CE be divided into eight equal parts, three of the parts will constitute the radius of the inscribed circle. Circumference is to the base as 8 to 3, and perpendicular to radius as 8 o 3, agreeably to the proposition.

Thirteenth.—Let the circumference of the triangle be four, and the base one and a half, making the other sides each one and a quarter. If the perpendicular CE be divided into four equal parts, one part and a half will constitute the radius of the inscribed circle. Circumference is to base as four to one and a half, or as 8 to 3, and perpendicular is to radius as four to one and a half, or as 8 to 3, agreeably to the proposition.

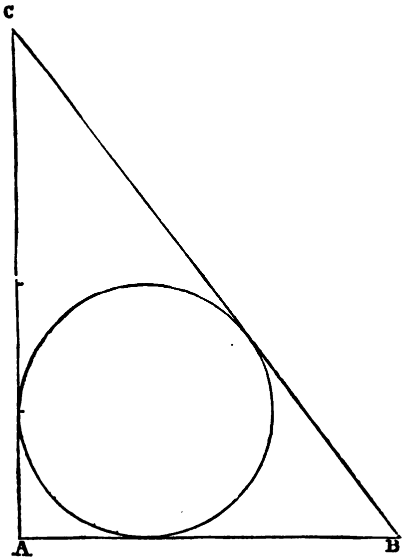


Fourteenth.—Let the circumference of the triangle be eight, and the base two and a-half, making each of the other sides two and three-quarters. If the perpendicular be divided into eight equal parts, two parts and a-half will constitute the radius of the



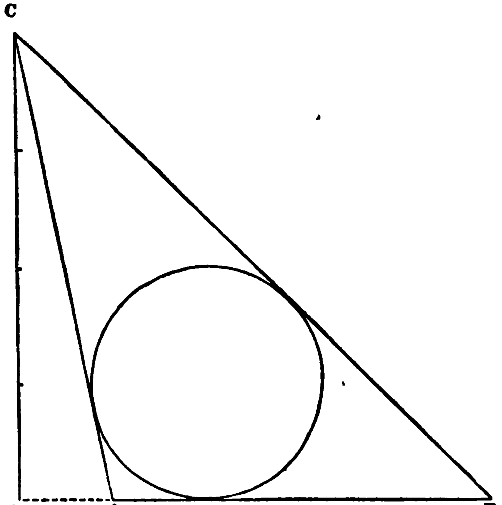
inscribed circle. The circumference of the triangle is to the base as eight to two and a-half, or as 16 to 5, and the perpendicular is to the radius as eight to two and a-half, or as 16 to 5, agreeably to the proposition.

Remark.—Thus far the examples given under this proposition have been isosceles triangles; but the law applies universally to all triangles, as will appear from the following additional examples.

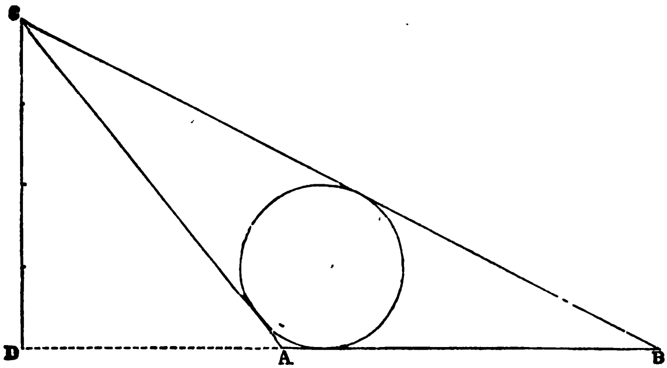


Fifteenth.—Let ABC be a right angled triangle, whose circumference is 8 (inches), and base 2. Then the perpendicular, A C, will be 2.6666+, and the hypothenuse, BC, will be 3.3333+. If the perpendicular be divided into four equal parts, one part will equal the radius of the inscribed circle.

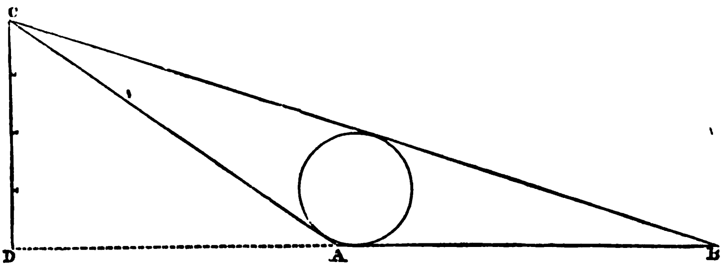
Sixteenth.—Let ABC be an obtuse angled triangle, whose circumference is 8, and base 2. Let AC be two and a-half, and BC three and a-half. On the base produced, draw the perpendicular



DC, and divide it into four equal parts. One part will equal the radius of the inscribed circle.



Seventeenth.—Let the circumference of the triangle ABC be 8, the base 2, AC two and two-tenths, and BC three and eight-tenths. On the base produced, draw the perpendicular CD, and divide it into four equal parts. One part will equal the radius of the inscribed circle.



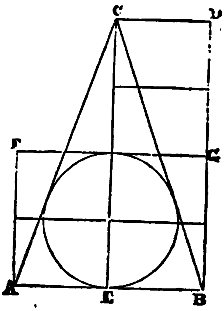
Eighteenth.—Again let circumference be 8, the base 2, AC two and one-tenth, and BC three and nine-tenths. On the base produced, draw the perpendicular CD, and divide it into four equal parts. One part will equal the radius of the inscribed circle.

REMARK.—In each of the last four examples, the circumference of the triangle is to the base as 4 to 1, and the perpendicular is to the radius of the inscribed circle also, as 4 to 1, agreeably to the 52d Proposition.

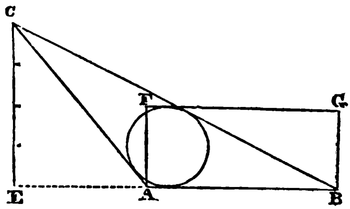
All these demonstrations under the 52d Proposition can be readily tested with the rule and compasses, with sufficient accuracy to establish their truth.

PROPOSITION LIII.

In any triangle, whose circumference is four, the area equals the diameter.



Let the base AB be one, and the sides AC and BC each be one and a half. Then the circumference is four, and the base one, and the radius of the inscribed circle is one-fourth of the perpendicular CE, [Prop. 52.] The rectangle BECD equals the area of the triangle, [Prop. 6,] and this rectangle, by the radius of the circle as a measure, is divided into four equal rectangles. The diameter, BAFG, is also divided into four rectangles equal to each other and equal to the four contained in BECD. Therefore the area equals the diameter, agreeably to the proposition.

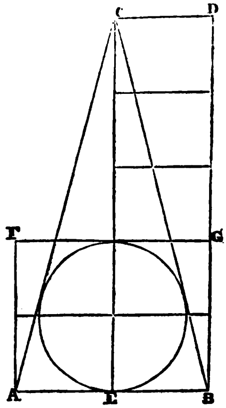


Again: Let the base AB be one inch, the side BC one inch and nine-tenths, and AC one inch and one-tenth. Then the circumference of the triangle ABC will be four. On the base, produced to E, draw the perpendicular CE, and divide it into four equal parts. One part will equal the radius of the inscribed circle, [Prop. 52.] The rectangle BAFG is the diameter, for it has a breadth of one, AB, and a length, AF, equal to the height of the circle. The rectangle and the triangle have the same base, AB, and if they had equal heights, the rectangle would be double the triangle, [Prop. 6.] But the rectangle has just half the height of the triangle, and its area is therefore equal to the triangle. Therefore the area of any triangle, whose circumference is four, equals its diameter, agreeably to the proposition.

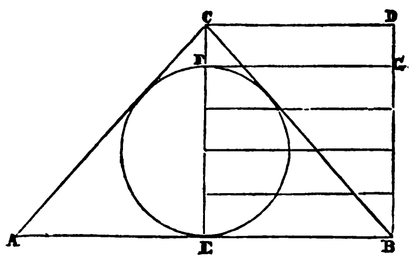
PROPOSITION LIV.

In any triangle, whose circumference is five, the diameter equals four-fifths of the area.

Let the base AB be one, and the sides AC and BC each two. Then the circumference of the triangle ABC will be five; and the radius of the inscribed circle is one-fifth of the perpendicular CE, [Prop. 52.] If CE is divided into five equal parts, it is manifest that the rectangle BECD is divided into five equal rectangles, which are together equal to the triangle ABC, [Prop. 6.] The rectangle BAFG is the diameter, because it has a breadth of one, AB, and a length, AF or BG, extending to the height of the circle. The diameter BAFG is seen to be divided into four equal rectangles, severally equal to the five contained in BECD. Therefore in the triangle whose circumference is five, the diameter equals four-fifths of the area, agreeably to the proposition.

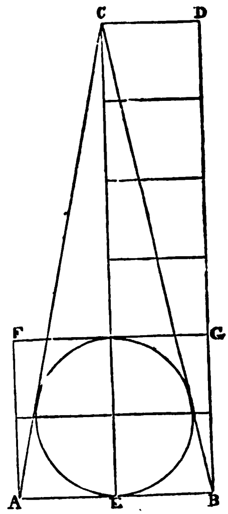


Again: let the base AB be two, and AC and BC each one and a half. Then the circumference of the triangle will be five, and the radius of the inscribed circle is two-fifths of the perpendicular CE, [Prop. 52.] The rectangle BECD, which equals the area of the triangle, is seen to be divided into five equal rectangles, and the diameter, BEFG, contains four of them. Therefore the diameter equals four-fifths of the area, agreeably to the proposition.

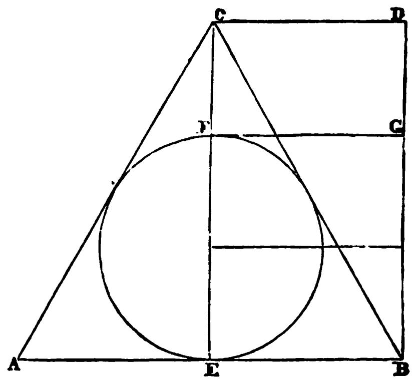


PROPOSITION LV.

In any triangle whose circumference is six, the diameter equals two-thirds of the area.



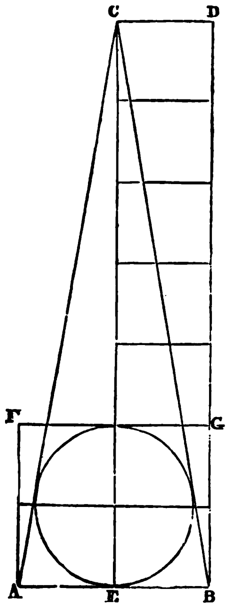
Let the base, AB, be one, and AC and BC each two and a-half. Then the circumference of the triangle ABC will be six, and the radius of the inscribed circle is one-sixth of the perpendicular CE, [Prop. 52.] Therefore if CE be divided into six equal parts, it is manifest that the rectangle BECD is divided into six equal rectangles, which are together equal to the area of the triangle ABC, [Prop. 6.] The rectangle BAFG is the diameter, because it has a breadth of one, AB, and a length, AF or BG, extending to the height of the circle; and it is manifest that BAFG contains four rectangles severally equal to the six contained in BECD. Therefore in any triangle whose circumference is six, the diameter equals two-thirds of the area, agreeably to the proposition.



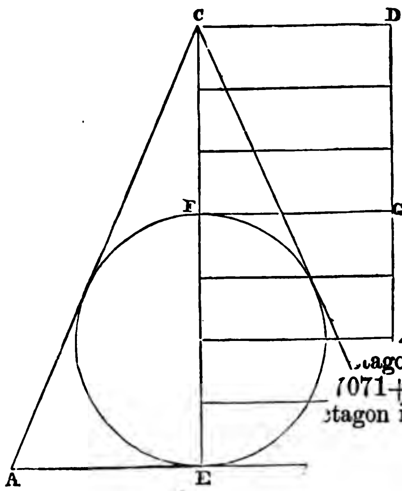
Again.—Let the base AB be two, and AC and BC each two. Then the circumference of the triangle will be six; and the radius of the inscribed circle is two-sixths or one-third of the perpendicular CE. It is manifest that BECD, which equals the area of the triangle, is divided into three equal rectangles, and that BEFG, the diameter, contains two of the three rectangles. Therefore the diameter equals two-thirds of the area, agreeably to the proposition.

PROPOSITION LVI.

In any triangle, whose circumference is seven, the diameter equals four-sevenths of the area.



Let the base AB be one, and AC and BC each three. Then the circumference of the triangle will be seven, and the radius of the inscribed circle is one-seventh of the perpendicular CE [Prop. 52.] Divide the perpendicular into seven equal parts, and it is manifest that BECD, which equals the area of the triangle, is divided into seven equal rectangles, and that the diameter, BAFG, is also divided into four such rectangles. Therefore the diameter is equal to four-sevenths of the area, agreeably to the proposition.

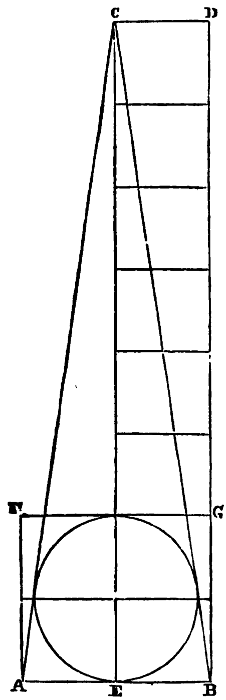


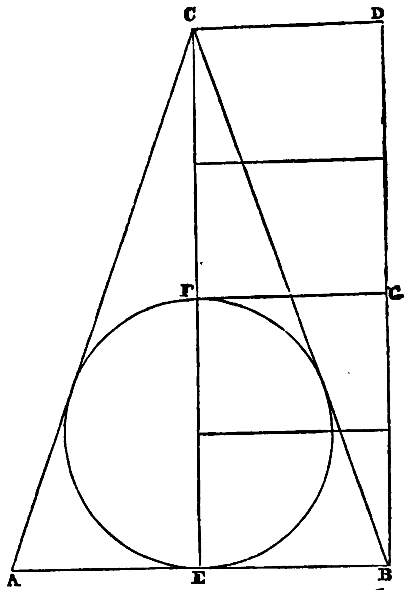
Again.—Let the base AB be two, and AC and BC each two and a-half. Then the circumference of the triangle is seven, and the radius of the inscribed circle is two-sevenths of the perpendicular CE. Divide the perpendicular into seven equal parts, and it is manifest that BECD, which is equal to the area, is divided into seven equal rectangles, and that the diameter, BEFG, contains four of them. Therefore the diameter equals four-sevenths of the area, agreeably to the proposition.

PROPOSITION LVII.

In any triangle, whose circumference is eight, the diameter equals half the area.

Let the base AB be one, and AC and BC each three and a-half. Then the circumference of the triangle is eight, and the radius of the inscribed circle is one-eighth of the perpendicular CE, [Prop. 52.] Therefore if the perpendicular be divided into eight equal parts, it is manifest that BECD, which equals the area of the triangle, is divided into eight equal rectangles, and that the diameter, BAFG, is divided into four such equal rectangles. Therefore the diameter equals half the area, agreeably to the proposition.



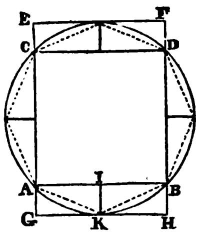


dia—ain.—Let the base AB be two, and AC and BC each three. e circumference of the triangle is eight, and the radius of ibed circle is two-eighths, or one-fourth of the perpendicu- Prop. 52.] Therefore divide the perpendicular into four , and it is manifest that BECD, which equals the area of the triangle, is divided into four equal rectangles, and that the diameter, BEFG, contains two of these rectangles. Therefore the diameter of any triangle, whose circumference is eight, equals half the area, agreeably to the proposition.

Corollary.—In all plane figures whatsoever, which can receive an inscribed circle, four times the area, divided by the diameter, equals the circumference.

PROPOSITION LVIII.

In a circle of *one* diameter, one side of the inscribed square equals the area of the inscribed octagon.

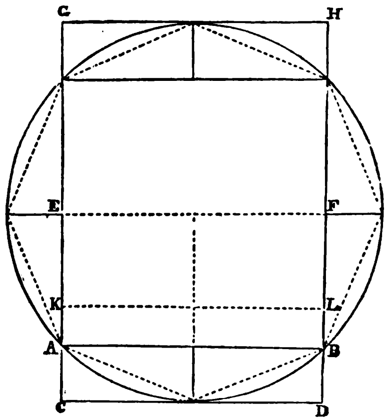


Let the diameter of the circle be one, [it is drawn at one inch,] and let ABCD be an inscribed square, and the dotted lines a regular inscribed octagon. One side of the square, as AB, has the same value in area as the octagon. The octagon is seen to be composed of the square, ABCD, and eight similar and equal right-angled triangles lying outside of the square and within the dotted lines. The value in area of any line is its length, whatever that is, and a breadth of one. Therefore the value of the line AB is the rectangle EFGH, for GH [equal to AB] is the *length*, and GE is the *breadth*, because it is equal to the diameter of the circle, that is, equal to *one*, and all lines have a breadth of one. Now the rectangle EFGH, which is the value of the line AB in area, is seen to be composed of the square ABCD and eight right-angled triangles similar and equal to the eight contained in the octagon. They are manifestly equal, because the rectangle ABGH is divided into two equal rectangles by IK, and each of these rectangles is divided into two equal right-angled triangles by the dotted diagonals KA and KB. And the rectangle CDEF is seen to be divided in the same manner into four equal triangles. Therefore the whole rectangle EFGH is equal to the octagon, [Axiom 4.] Therefore in the circle whose diameter is *one*, one side of the inscribed square equals the area of the inscribed octagon, agreeably to the proposition.

In the arithmetical calculation of the same square and octagon, the side of the square will be found to be half the square root of 2, viz., the decimal expression .7071+. And the area of the octagon will also be found to be half the square root of 2, viz., .7071+. Therefore the side of the square equals the area of the octagon in numbers, as well as in geometrical quantity.

PROPOSITION LIX.

In the circle whose diameter is *two,* one side of the inscribed square equals half the area of the inscribed octagon.

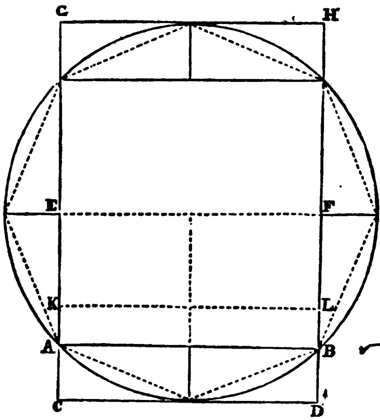


Let the diameter of the circle be *two* inches, and let a square and a regular octagon be inscribed, as in the last proposition. Then the octagon will equal the rectangle CDGH, [Prop. 58.] The value of one side of the inscribed square, as AB, is the rectangle CDEF; for the length of AB equals CD, and the breadth, being one, is equal to CE, that is, half of CG, which is two, being equal to the diameter of the circle. Therefore the rectangle CDEF, which is the value of the line AB, equals half the rectangle CDGH, and consequently equals half the octagon. Therefore in a circle whose diameter is two, one side of the inscribed square equals half the area of the inscribed octagon, agreeably to the proposition.

In the arithmetical calculation of the same square and octagon, the side of the square will be found to be the square root of 2, that is, 1.4142+, and the area of the octagon will be found to be the square root of 8, that is, 2.8284+. This last number is double the former; therefore the side of the square equals half the area of the octagon, in numbers as well as in geometrical quantity.

PROPOSITION LX.

In the circle whose diameter is *four,* one side of the inscribed square equals one-fourth of the area of the inscribed octagon.

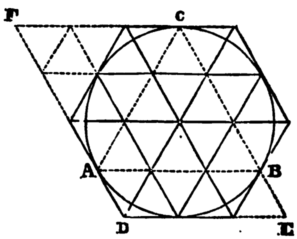


Let the diagram be the same as in the last proposition, and make diameter four. Take CK equal to one-fourth of CG; then CK will be one, and the rectangle CDKL will be the value in area of AB, one side of the square; for the length of AB is CD, and its breadth is CK, equal to one. But the rectangle CDKL is one-fourth of the rectangle CDGH, and therefore equal to one-fourth of the octagon. Therefore in the circle whose diameter is four, one side of the inscribed square equals one-fourth of the inscribed octagon, agreeably to the proposition.

In the arithmetical calculation of the same square and octagon, the side of the square will be found to be the square root of 8, viz., 2.828427+, and the area of the octagon will be found to be the square root of 128, viz., 11.3137+; and this last number divided by 4 produces 2.82842+, equal to the side of the square. Therefore in the circle whose diameter is 4, one side of the inscribed square equals one-fourth the area of the inscribed octagon, in numbers as well as in geometrical quantity.

PROPOSITION LXI.

In the circle, whose diameter is one, one side of the inscribed equilateral triangle is equal to the area of the circumscribed hexagon.

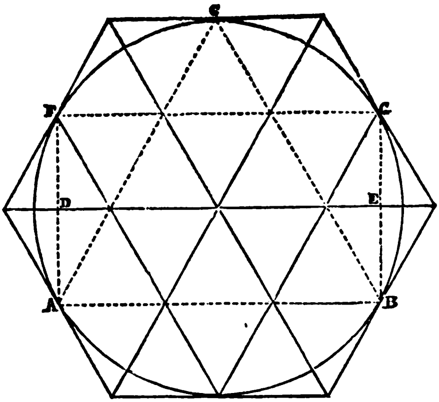


Let the diameter of the circle be one, and ABC an inscribed equilateral triangle; and let a regular hexagon be circumscribed about the circle. Then if each side of the triangle ABC be divided into three equal parts, and two straight lines drawn through each point of division parallel to the sides of the hexagon, the hexagon will be divided into twenty-four equal and equilateral triangles; for the triangle ABC being equilateral, and its sides equally divided, the lines drawn through the points of division must necessarily be equally distant from each other. AB, the base of the inscribed triangle, is seen to equal the bases of three of the small triangles. From D let the side or base of the hexagon be produced to E, and the side of the triangle CB produced to meet DE in E. Then DE will equal the bases of three of the small triangles, and will therefore equal AB. Again, produce the side of the hexagon from C to F, making FC equal to AB or DE. Then FC will also equal the bases of three of the small equal triangles. And D and F being joined, the parallelogram DEFC will be seen to contain twenty-four of the small equal triangles, and is therefore equal to the hexagon. But the parallelogram DEFC is equal to the line AB; for the line AB is equal to a rectangle whose length is AB, and whose breadth is one. The base of the parallelogram, DE, is equal to AB, and the perpendicular height of the parallelogram is equal to the diameter of the circle, and therefore is one; and a rectangle and a parallelogram of the same base and equal heights are equal, [Prop. 50.] Therefore the parallelogram DEFC is equal to AB, and the line AB is equal to the area of the hexagon. Therefore in a circle of one diameter, one side of the inscribed equilateral triangle equals the area of the circumscribed hexagon, agreeably to the proposition.

In the arithmetical calculation of the same triangle and hexagon, the side of the triangle will be found to be half the square root of 3—viz., .866+, and the area of the hexagon will also be found to be half the square root of 3—that is, .866+. Therefore the side of the triangle is equal to the area of the hexagon in numbers as well as in geometrical quantity, agreeably to the proposition.

PROPOSITION LXII.

In the circle, whose diameter is two, one side of the inscribed equilateral triangle equals half the area of the circumscribed hexagon.

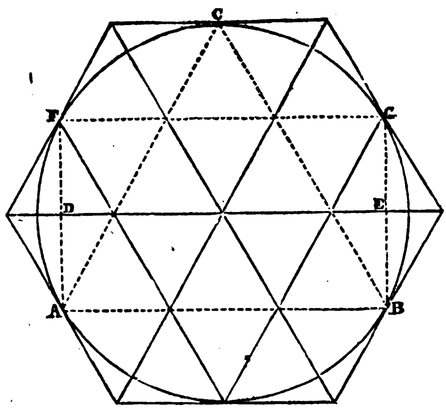


Let the diameter of the circle be two, and let a regular hexagon be circumscribed around it. Then if each side of the hexagon be divided into two equal parts, and two lines be drawn from each point of division and parallel to the two adjacent sides of the hexagon, so as to meet two opposite points of division, and if lines be drawn through the center of the circle to each angle of the hexagon, the whole hexagon will be divided into twenty-four equal and equilateral triangles, and ABC will be an equilateral triangle inscribed in the circle. The value or quantity of the line AB, one side of the triangle, is the rectangle ABFG, for the length is AB, and the breadth, AF, is one, being equal to half the diameter of the circle. AF is the length of the perpendiculars of two of the small equal triangles, and the diameter of the circle is manifestly the length of four such perpendiculars, therefore AF equals half the diameter. The rectangle ABFG is seen to contain twelve, or the value of twelve, of the small triangles, that is ten whole triangles and four halves. But the hexagon contains twenty-four triangles; therefore the rectangle ABFG equals half the hexagon. But the rectangle ABFG is the value of the line AB, one side of the triangle ABC, therefore the line AB equals half the hexagon; and therefore in a circle, whose diameter is two, one side of the inscribed equilateral triangle equals half the area of the circumscribed hexagon, agreeably to the proposition.

In the arithmetical calculation of the same triangle and hexagon, the side of the triangle will be found to be the square root of 3—viz., 1.732+, and the area of the hexagon will be found to be the square root of 12—that is, 3.464+, and half of this last number is 1.732+. Therefore one side of the triangle equals half the area of the hexagon, in numbers as well as in geometrical quantity, agreeably to the proposition.

PROPOSITION LXIII.

In the circle, whose diameter is four, one side of the inscribed equilateral triangle equals one-fourth of the area of the circumscribed hexagon.

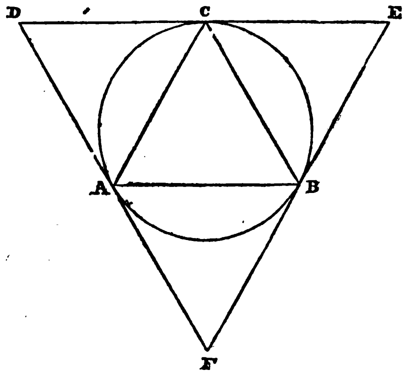


On the same diagram as in the last proposition, let the diameter of the circle be four, and let ABC be the inscribed equilateral triangle. The diameter of the circle being four, the perpendicular of each of the small triangles is one, for four of these perpendiculars is seen to equal the diameter of the circle. Therefore the line AB, one side of the triangle ABC, is equal to the rectangle ABDE, for the length is AB, and the breadth, AD, is one. The rectangle ABDE is seen to contain six of the small equal triangles—that is, five whole triangles and two halves. But the hexagon contains twenty-four such triangles, and six is one-fourth of twenty-four. Therefore the rectangle ABDE equals one-fourth of the hexagon; therefore the line AB equals one-fourth of the hexagon. Therefore in the circle, whose diameter is four, one side of the inscribed equilateral triangle equals one-fourth of the area of the circumscribed hexagon, agreeably to the proposition.

In the arithmetical calculation of the same triangle and hexagon, one side of the triangle will be found to be the square root of twelve—viz., 3.464+, and the area of the hexagon will be found to be 13.856+, which is equal to four times the former number, and is the square root of 192. Therefore the side of the inscribed triangle equals one-fourth of the circumscribed hexagon, in numbers as well as in geometrical quantity, agreeably to the proposition.

PROPOSITION LXIV.

The area of an equilateral triangle inscribed in a circle is one-fourth of the area of the equilateral triangle circumscribed about the same circle.

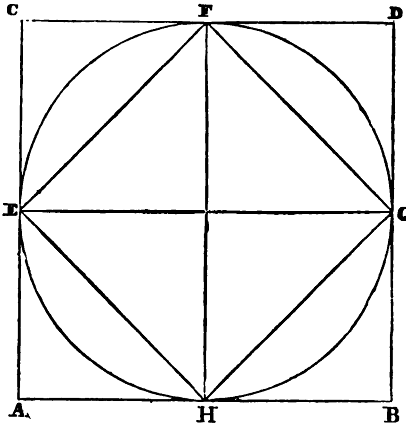


Let ABC be an equilateral triangle, each side equal to one, and let the circle be described about the triangle. Upon each side of the triangle ABC construct another equilateral triangle, then will DEF be an equilateral triangle circumscribed about the circle. The whole diagram contains four equilateral triangles, which are equal to each other; for the triangle ABG is equilateral, and one side of this triangle forms a side of each of the other triangles, which are also equilateral by construction, therefore all the sides of the four triangles are equal to each other; and the angles also being equal, the triangles would coincide if placed one upon another, therefore their areas must be equal. But the circumscribed triangle DEF contains the area of the whole four triangles, and the inscribed triangle, ABC contains the area of one triangle. Therefore the area of an equilateral triangle inscribed in a circle is one-fourth of the area of the equilateral triangle circumscribed about the circle, agreeably to the proposition.

In the arithmetical calculation of the same triangles, the area of the circumscribed triangle, DEF, will be found to be the square root of 3, viz., 1.732+, and the area of the inscribed triangle, ABC, will be found to be .433+, which is one-fourth of 1.732+. Therefore the area of the inscribed triangle is one-fourth the area of the circumscribed, in numbers as well as in geometrical quantity.

PROPOSITION LXV.

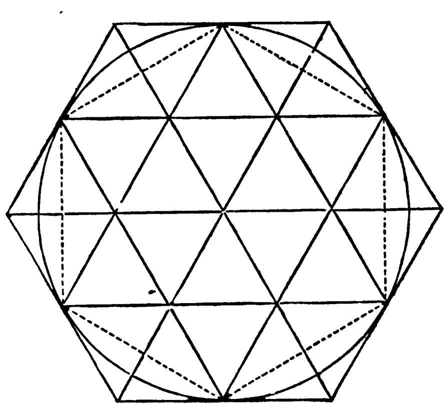
The area of a square inscribed in a circle is one-half the area of the square circumscribed about the same circle.



Let ABCD be a square, each side equal to two, and let a circle be inscribed in the square. Let each side of the square be divided into two equal parts in the points E, F, G, and H, and draw the straight lines EG and HF. Then the whole square will manifestly be divided into four smaller squares, equal to each other, and each equal to one, for each side of all the four squares is one. Let each of the four small squares be divided into two triangles by the diagonals EF, FG, GH, and HE. Then the whole figure will be divided into eight right-angled triangles, equal to each other, [Prop. 5.] But the four diagonals constitute a square, EFGH, inscribed in the circle, and this inscribed square, EFGH, is seen to contain four of the right-angled triangles. The circumscribed square ABCD contains eight such triangles. Therefore the area of a square inscribed in a circle is one-half the area of the square circumscribed about the same circle, agreeably to the proposition.

PROPOSITION LXVI.

The area of a regular hexagon inscribed in a circle is three-fourths of the area of the hexagon circumscribed about the same circle.



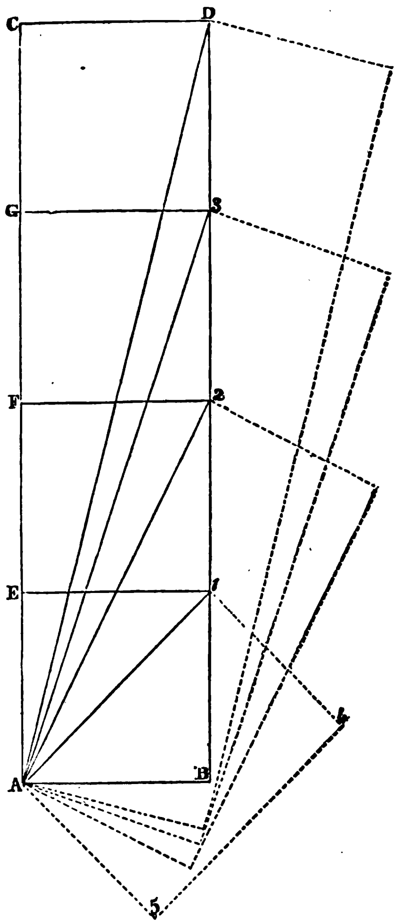
With a radius equal to one, describe a circle, whose diameter will then be two. Let a regular hexagon be circumscribed about the circle. Then if each side of the hexagon be divided into two equal parts, and two lines be drawn from each point of division and parallel to two sides of the hexagon, so as to meet two opposite points of division, and if lines be also drawn through the center of the circle to each angle of the hexagon, the whole circumscribed hexagon will be divided into twenty-four equal and equilateral triangles. That they must be equal and equilateral is manifest, because the hexagon being regular, its sides are equal, and each side being equally divided, and the lines from the points of division being drawn parallel to the sides of the hexagon, and therefore preserving everywhere respectively equal distances, the small triangles between them must all have equal perpendiculars and equal sides, and be equal to each other. The center of each side of the circumscribed hexagon touches the circle, and if these points of contact are joined by straight lines, as by the dotted lines in the diagram, these lines will constitute a regular inscribed hexagon. And the inscribed hexagon is seen to contain eighteen triangles, that is, twelve whole and twelve half triangles. But the circumscribed hexagon contains twenty-four such triangles, and eighteen is three-fourths of twenty-four. Therefore the area of a regular hexagon inscribed in a circle is three-fourths of the area of the hexagon circumscribed about the same circle, agreeably to the proposition.

In the arithmetical calculation of these hexagons, the diameter of the circle being two, the area of the circumscribed hexagon will be found to be the square root of 12, viz., 3.464+, and the area of the inscribed hexagon will be found to be 2.598+, which is three-fourths of 3.464+. Therefore the area of the inscribed hexagon is three-fourths of the area of the circumscribed hexagon, in numbers as well as in geometrical quantity.

REMARK.—From the principles established in these demonstrations it appears that every mathematical right line is a *rectangle,* whose breadth is one. And as every rectangle has a diagonal, it follows that every mathematical right line has a diagonal. Hence we deduce the following proposition.

PROPOSITION LXVII.

The diagonal of every right line is the square root of a quantity exceeding the square of the line by one.



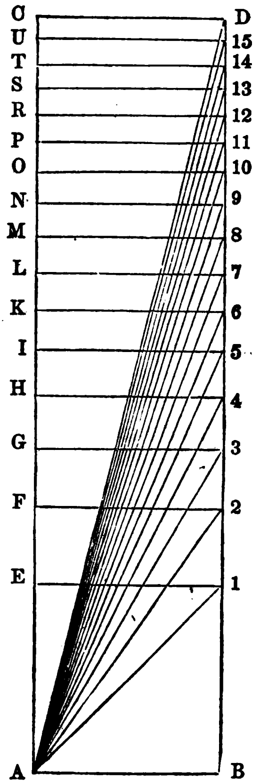
In the diagram, taking an inch for the unit, AB is 1, and AC is 4. Then the rectangle ABCD is a mathematical line of four inches. The rectangle ABE1 is one inch. Its diagonal, A1, is the square root of 2. The value of A1 in area is the rectangle A145. And the quantity of this last rectangle is such, that if multiplied into itself it would just equal two square inches. That is, it would exceed the square of one inch by one. Again, the rectangle ABF2 is a mathematical line of two inches. This line squared is 4 inches; and its diagonal, A2, squared, equals 5 square inches. Therefore the square of the diagonal exceeds the square of the line by one.

The rectangle ABG3 is a line of three inches, whose square is 9, and its diagonal A3, is the square root of an area, or the side of a square, equal to 10 square inches. And ABCD being a line of 4 inches, its square is 16, and its diagonal AD, squared, would equal 17 square inches. Thus the square of the diagonal of every mathematical line exceeds the square of the line by one.

The dotted lines in the diagram show the value in area of each diagonal. The general truth contained in this proposition, it will readily be seen, results from the celebrated theorem of Pythagoras, that "the square of the hypothenuse of a right-angled triangle equals the sum of the squares of the other two sides." For the diagonal of a line divides it into two right-angled triangles, the diagonal being the hypothenuse, as AD. The side of the triangle representing the breadth of the line, as CD, is always 1, and its square is 1. And since the square of the diagonal AD equals the square of AC plus the square of CD, it will always exceed the square of AC, whatever its length may be, by 1.

In the annexed diagram the rectangle ABCD is a mathematical line, the unit being one inch. AB is 1, and AC is 4. The whole line is divided into sections, which are the roots of quantities successively increasing by unity. And the diagonals also are the roots of quantities successively increasing by unity. The first section, ABE1, is the unit. It is one, and its root, AB, is also one. Its diagonal, A1, is the square root of 2. Take AF, equal to A1, and the rectangle ABF2 will also be the square root of 2. Then the diagonal of this rectangle, viz., A2, will be the square root of 3. Take AG, equal to A2, and the rectangle ABG3 will also be the square root of 3. In like manner A3 is the square root of 4, and the rectangle ABH4 is also the square root of 4. Each succeeding section of the line is the square root of a quantity larger by one square inch than the square of the preceding section. The diagonal A15 is the square root of 16, and the rectangle ABCD, being four square inches, is also the square root of 16.

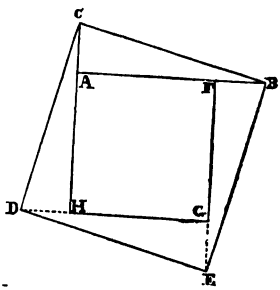
These quantities, as represented upon the diagrams, may all be determined with sufficient accuracy to test their truth simply by applying the rule and compasses.



PROPOSITION LXVIII.

In every right-angled triangle, the square of the hypothenuse equals four times the area of the triangle, plus the square of the difference of the other two sides.

REMARK.—I hardly know of a more beautiful demonstration in all geometry, than that which establishes the truth of this proposition. I discovered it in the month of June, 1850, when the greater part of this work was ready for the press. I was endeavoring to discover some more simple method than that given by Euclid and other geometers, of demonstrating the celebrated theorem of Pythagoras—that the square of the hypothenuse equals the sum of the squares of the other two sides. I was not, at the time, even aware of the existence of the principle stated in the above proposition, till it presented itself to me on the diagrams which I had drawn for another purpose. On stating my discovery to a mathematical friend, he informed me that this truth was known to mathematicians in the forms of arithmetic and algebra, and that a geometrical demonstration of it was considered a great desideratum, but was not supposed to be possible.

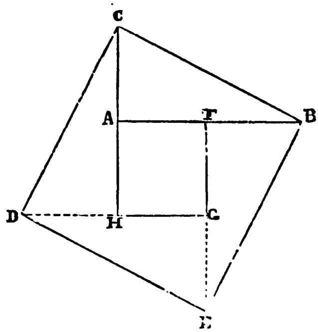


Let ABC be a right-angled triangle, and BC the hypothenuse. Square BC on the diagram, and it gives the square BCDE. From the base of the triangle, AB, cut off BF, equal to the perpendicular AC. Then the remainder of the base, AF, will be the difference of the two sides AB and AC. Join FE, and take EG equal to AC. Join GD, and take DH equal to AC, and join HA. Then AFGH will be a square, and it is the square of the difference of AB and AC. Besides this central square, the diagram is seen to contain four right-angled triangles, one of which is the original triangle ABC; and each of the other three is equal to this, because the hypothenuse of each triangle is one side of the same square, BCDE; these sides are therefore equal. The shortest side of each triangle is equal to AC by construction; and the remaining side of each triangle is equal to AB, that is, equal to AC plus one side of the central square. The four triangles are therefore equal to each other. And the square of the hypothenuse, BCDE, equals four times the area of the triangle ABC, plus the square of the difference of the other two sides, agreeably to the proposition.

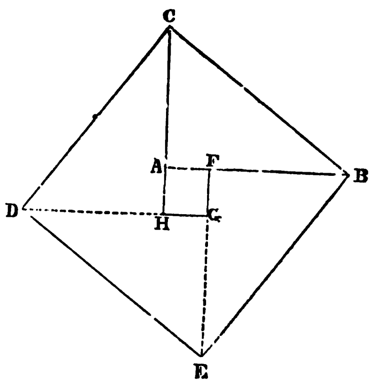
Remark.—However the sides AB and AC may be varied, the construction of the diagram will always be the same, and will always afford the same demonstration, until the two sides become equal to each other, and then the central square vanishes and the square of the hypothenuse equals four times the area of the triangle, as the following diagrams will show.

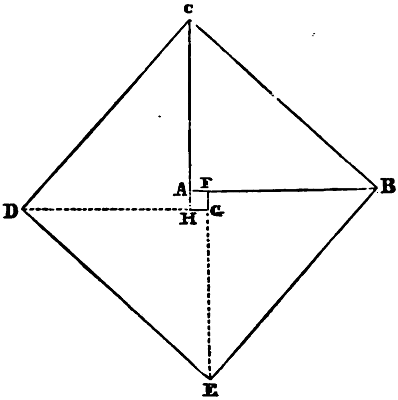
In the preceding diagram, the base AB is one inch, and the perpendicular AC a quarter of an inch.

In the present diagram AB is one inch, and AC is half-an inch; and yet the construction of the diagram proceeds in the same manner as in the last, and the demonstration is precisely the same.

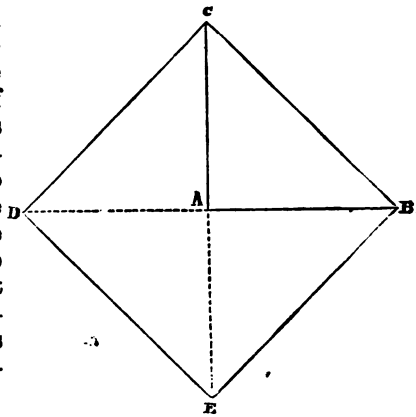


In the annexed diagram, AB is one inch, and AC three-quarters of an inch. BF is taken equal to AC; FE joined, and EG taken equal to AC; GD joined, and DH taken equal to AC, and HA joined. The results are seen to be the same as in the preceding demonstration, except that the central square becomes smaller.





In this diagram AB is one inch, and AC nine-tenths of an inch, making the difference of the two sides, AF, one-tenth of an inch, and the square of AF one-hundredth part of a square inch.



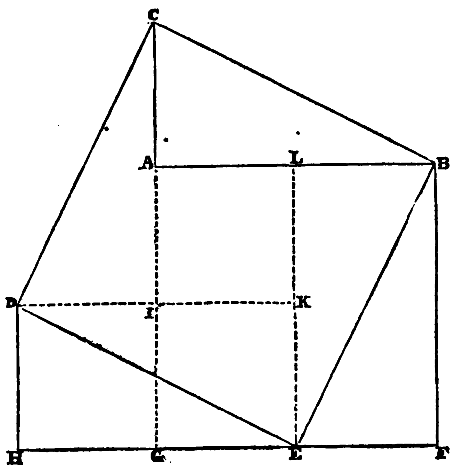
In this diagram AB and AC are equal, each being one inch, and the central square has vanished. The square of the hypothenuse, BCDE, is seen to contain four right-angled triangles, equal to ABC, for the hypothenuse of each triangle is one side of the same square, and the diagonals of this square bisect each other equally at A, making all the sides and angles of the four triangles respectively equal.

PROPOSITION LXIX.

The grand theorem of Pythagoras.—In every right-angled triangle, the square of the hypothenuse equals the sum of the squares of the other two sides.

Remark.—More than two thousand years ago this beautiful and important truth was discovered and demonstrated by the renowned philosopher of Samos. So delighted and impressed was he by the discovery, that he is said to have sacrificed a hundred oxen to the gods, in testimony of his joy and gratitude. And certainly it was no small cause of joy; for besides the great beauty and harmony of the changing forms which it presents, in point of practical utility, geometry can scarcely boast a more important demonstration. But the demonstration of this proposition, as given by Euclid and followed by most other geometers, is rather intricate and laborious, rendering it somewhat difficult for the unpracticed student to retain in his mind the different parts of the diagram, and see clearly their connection from the beginning to the end of the chain of reasoning. I was induced, therefore, to seek for a more simple and direct method of demonstrating this most valuable theorem; and it was while engaged in this attempt, as already intimated, that I discovered the truth demonstrated in Proposition 68. My original purpose, however, was also accomplished; for the same diagram, with a slight modification, gives a clear, simple, and beautiful demonstration of the great theorem of Pythagoras.

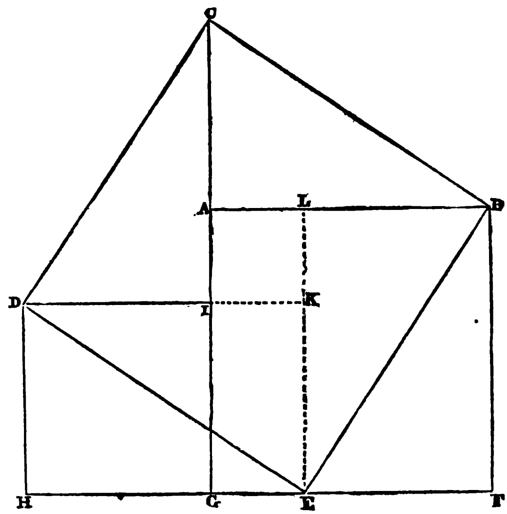
Let ABC be a right-angled triangle. Square the hypothenuse, BC, on the diagram, and it gives the square BCDE. Square the base, AB, and it gives the square ABFG. Take GI, equal to the perpendicular AC, and square it, and it gives the square GHDI. Take BL, equal to AC, and join LE. AL is the difference of AB and AC. Square AL, and it gives the square ALKI.



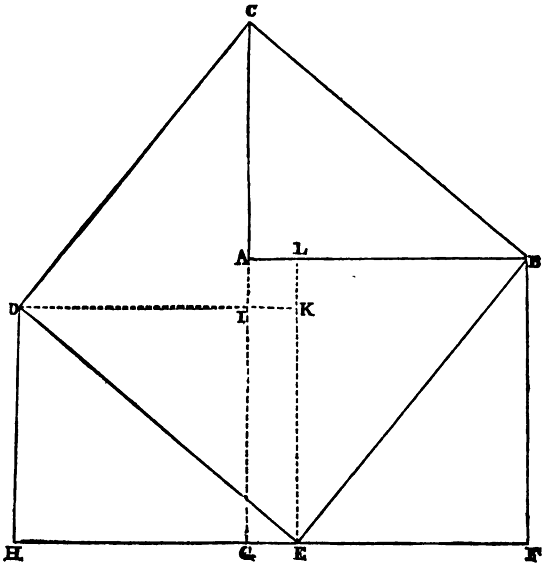
The square of the hypothenuse, BCDE, as

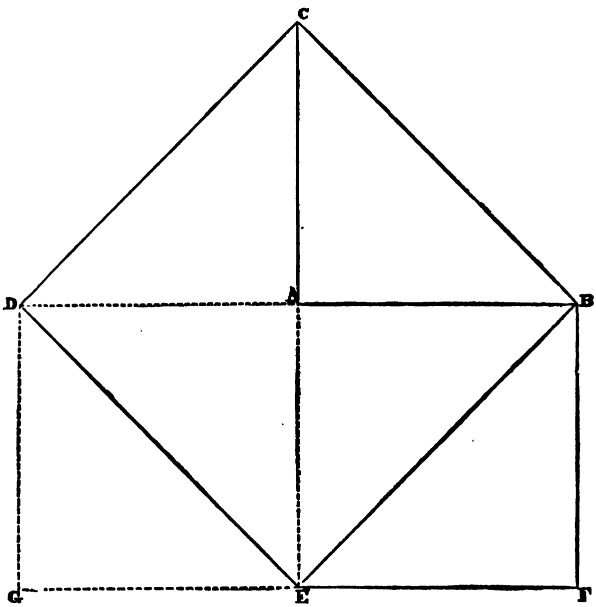
was shown in the last proposition, contains four triangles, plus the central square. The squares of the other two sides are, first, ABFG, and second, GHDI. Let these two squares be considered together as forming one figure, and take from it the central square. There will then remain two parallelograms, LBFE, and EKDH. And each of these parallelograms is seen to be divided into two right-angled triangles, each equal to ABC, for each has a side of the same square for its hypothenuse. Therefore the sum of the squares of AB and AC, equals four times the area of the triangle, plus the square of the difference of AB and AC. The square of BC has been shown to be equal to the same quantity; and things which are equal to the same are equal to each other. Therefore the square of the hypothenuse of a right-angled triangle equals the sum of the squares of the other two sides, agreeably to the proposition.

To show that the construction of the diagram and the demonstration will remain the same, however the sides AB and AC may be varied, three more diagrams are here given. In the preceding diagram AB is an inch and a half, and AC is three-quarters of an inch. In the present diagram, AB is an inch and a half, and AC an inch.



In this diagram AB is an inch and a half, and AC an inch and a quarter. It will be observed that in all these diagrams, the sides of the central square, produced, meet the angles of the large square, which is the square of the hypothenuse.





In this diagram AB and AC are equal, each being an inch and a half, and the central square has vanished. The square of the hypothenuse, BCDE, is seen to contain four right-angled triangles, each equal to ABC. The square of the base AB, which is ABFE, contains two such triangles, and AC, or its equal, AE, being squared, gives the square AEGD, which also contains two such triangles. The squares of AB and AC together equal four times the area of the triangle ABC; and the square of BC also equals four times the same triangle. Therefore the square of the hypothenuse of a right angle equals the sum of the squares of the other two sides, agreeably to the proposition.

PROPOSITION LXX.

If from any circle there be cut a segment of one diameter, the chord of half the arc of that segment is the square root of the diameter of the circle.

The diagram presents eight circles—three perfect and five broken—there not being room on the diagram to complete them. The diameter of the first circle, AB, is *one*, [one inch.] The diameter of the second, AC, is 2. The diameter of the third, AD, is 3, and so on, the diameters increasing by unity till the last circle, whose diameter is 8. These circles all touch a common point at A, and their centers are all in the same straight line, AD, produced. From all these circles, except the smallest, the chord GH cuts a segment, each segment having the same diameter, AB, which is 1.

From the second circle, whose diameter is 2, the segment cut off by the chord GH is seen to be half the circle; and the chord of half the arc of that segment is seen to be A1, and A1 is the square root of 2—viz., it is 1.4142+.

A2 is the chord of half the arc of the segment cut from the third circle, whose diameter is 3; and A2 is the square root of 3—viz., it is 1.732+.

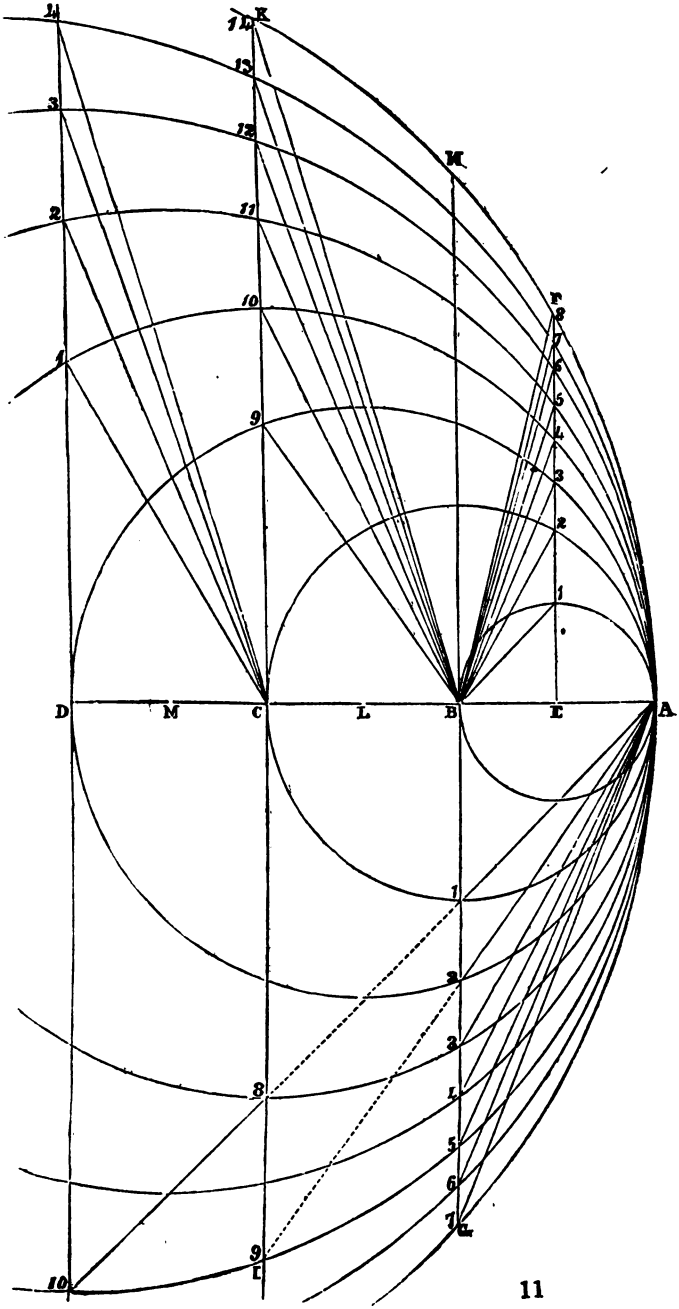
A3 is the chord of half the arc of the segment of the fourth circle, whose diameter is 4, and A3 is the square root of 4—that is, the chord A3, with perfect drawing and perfect measurement, would be just two inches.

A4 is the chord of the fifth circle, whose diameter is 5, and A4 is the square root of 5.

A5 is the chord of the sixth circle, whose diameter is 6, and A5 is the square root of 6.

A6 is the chord of the seventh circle, whose diameter is 7, and A6 is the square root of 7.

A7 is the chord of half the arc of the segment cut off by GH from the largest circle, whose diameter is 8, and A7 is the square root of 8—viz., it is 2.8284+.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PROPOSITION LXXI.

If from any circle there be cut any segment whatever, the chord of half the arc of that segment is the square root of the diameter of the circle multiplied by the diameter of the segment; or the chord is a mean proportional between the diameter of the circle and the diameter of the segment.

AC, the diameter of the second circle, is 2. Therefore the chord IK cuts segments from all the larger circles, each segment having a diameter of 2.

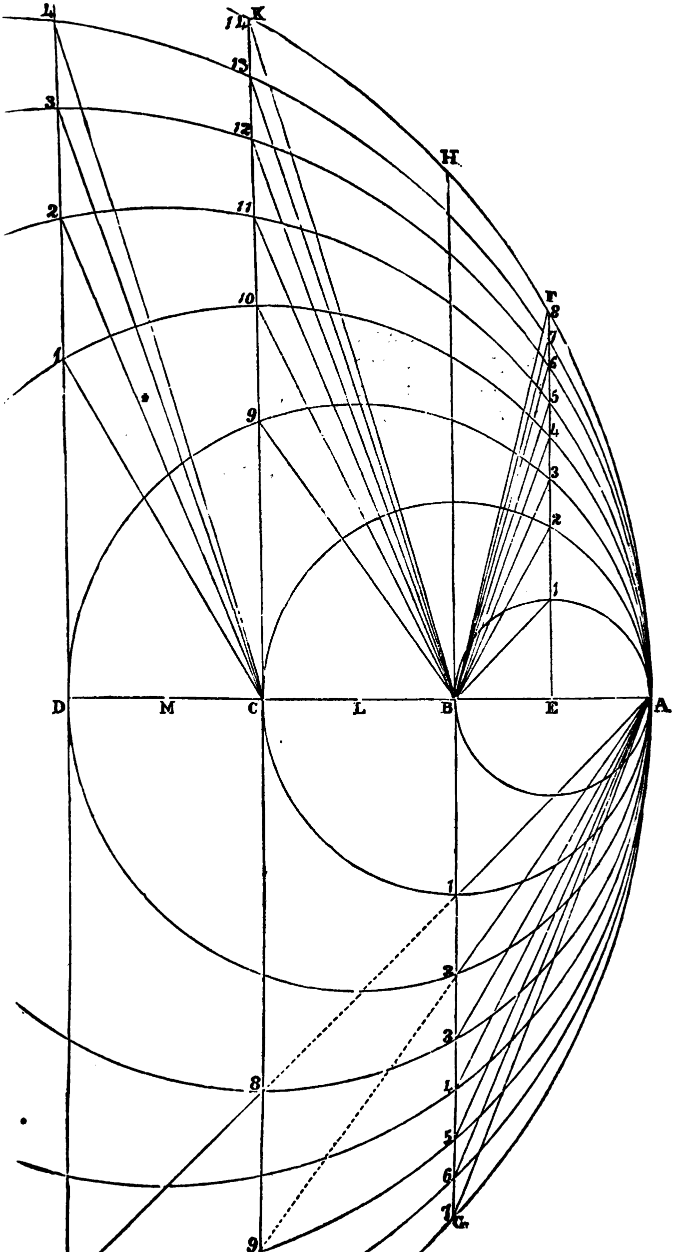
A8 is seen to be the chord of half the arc of the segment cut from the fourth circle. This circle has a diameter of four, which multiplied by 2, the diameter of the segment, makes 8; and the chord A8 is the square root of 8—viz., it is 2.8284+.

A9 is seen to be the chord of half the arc of the segment cut from the sixth circle. This circle has a diameter of 6, which multiplied by 2, the diameter of the segment, makes 12; and the chord A9 is the square root of 12—viz., it is 3.464+.

AD, which equals 3, is the diameter of the segments cut off by the chord passing through D; and A10 is seen to be the chord of half the arc of the segment cut from the sixth circle. This circle has a diameter of 6, which multiplied by 3, the diameter of the segment, makes 18; and A10 is the square root of 18.

Again: let the segments cut from all the circles each have a diameter of half of one, that is, equal to AE. If lines were drawn from A, terminating on the line EF at the points 1, 2, 3, 4, &c., they would be the chords of half the arcs of the segments thus cut off. To make the lines more distinct they are drawn from the point B, and are manifestly of the same length as they would be if drawn from A. Now, these lines are respectively the square roots of *half* the diameters of the circles on which they terminate:—that is, B1 is the square root of half of 1, B2 is the square root of half of 2, B3 is the square root of half of 3, B4, which terminates on the circle whose diameter is 4, is the square root of half the diameter, and B8, terminating on the circle whose diameter is 8, is the square root of 4. So that, if the diagram were perfectly drawn and perfectly measured, B8 would be just 2 inches.

In like manner, if the segments cut off had a diameter equal to one-fourth of the unit, the chords would be the square roots respectively of one-fourth of the diameters of the circles.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PROPOSITION LXXII.

The curve line is the measure of extension in every possible direction. [Def. 38.]

The truth of the proposition, and the propriety of the definition on which it is based, may be illustrated on the annexed diagram, upon which may be drawn lines representing the square roots of *any quantities whatever*, and series of lines representing the square roots of quantities regularly increasing by *any ratio whatever*.

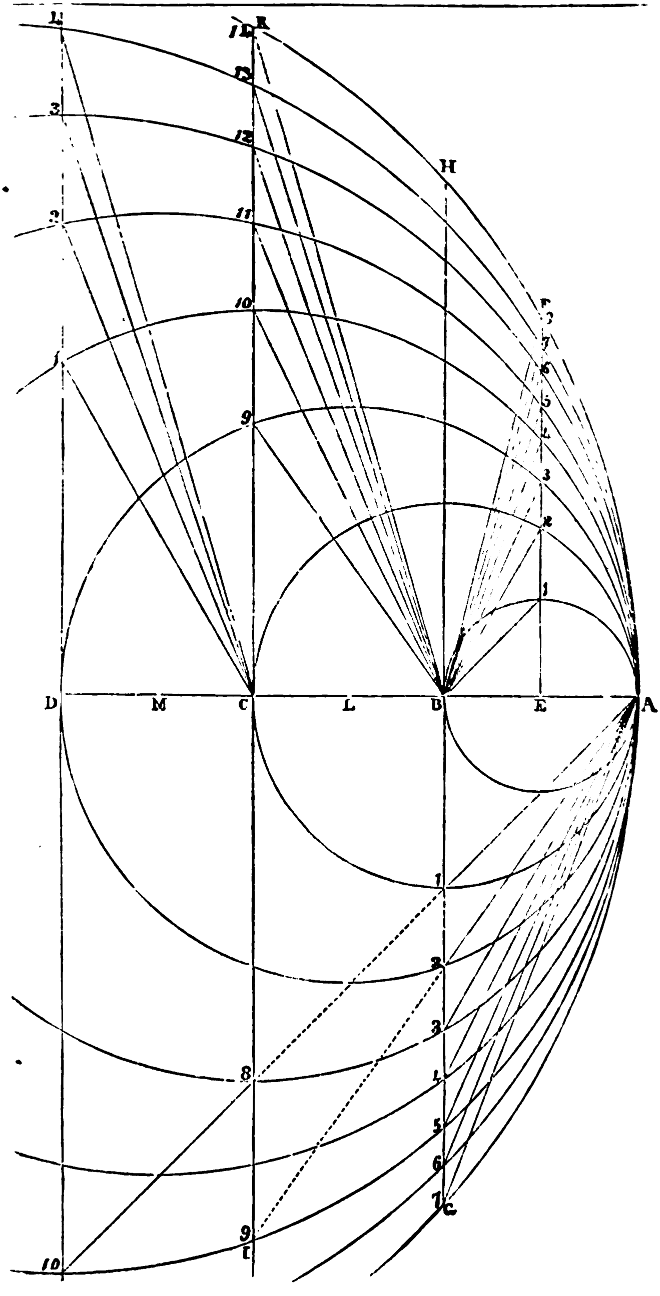
The series of chords drawn from A, and terminating on the line BG, are severally the square roots of quantities which increase regularly by a ratio of one; that is, AB is the square root of 1, A1 is the square root of 2, A2 is the square root of 3, A3 is the square root of 4, and so on, as far as the series may be carried.

The series of chords drawn from B, and terminating on the line EF, [being the lengths of chords supposed to be drawn from A and terminating at the same points,] are the square roots of quantities increasing regularly by a ratio of *half of one*. B1 is the square root of half of one, B2 is the square root of one, B3 is the square root of one and a-half, B4 is the square root of two, B5 the square root of two and a-half, &c. Expressed in decimal numbers, B1 is the square root of .5; B2 is the square root of 1; B3 is the root of 1.5; B4 is the root of 2; B5 is the root of 2.5, and so on, the square of each succeeding line containing more area by half a square inch than the square of the preceding line.

A series of lines drawn from A, and terminating on the second chord, IK, at the intersections of the circles, would be the square roots of quantities increasing regularly by a ratio of 2. Thus, AC is the square root of 4; the chord of the next circle, if drawn, would be the square root of 6; the next, A8, is the square root of 8; the next would be the square root of 10; and the next, A9, is the square root of 12.

The series of lines drawn from B, and terminating on the line CK at the intersections of the circles, are also the square roots of quantities increasing regularly by a ratio of 2; that is, they are the square roots of all the *odd numbers*, as far as the series may be carried. BC is the square root of 1; B9 is the square root of 3; B10 is the square of 5; B11 is the square root of 7; B12 the square root of 9, and so on.

The series of lines drawn from C, and terminating on the next chord at the intersections of the circles, are the square roots of quantities increasing regularly by a ratio of 3. Thus CD is the square root of 1; C1 is the square root of 4; C2 is the square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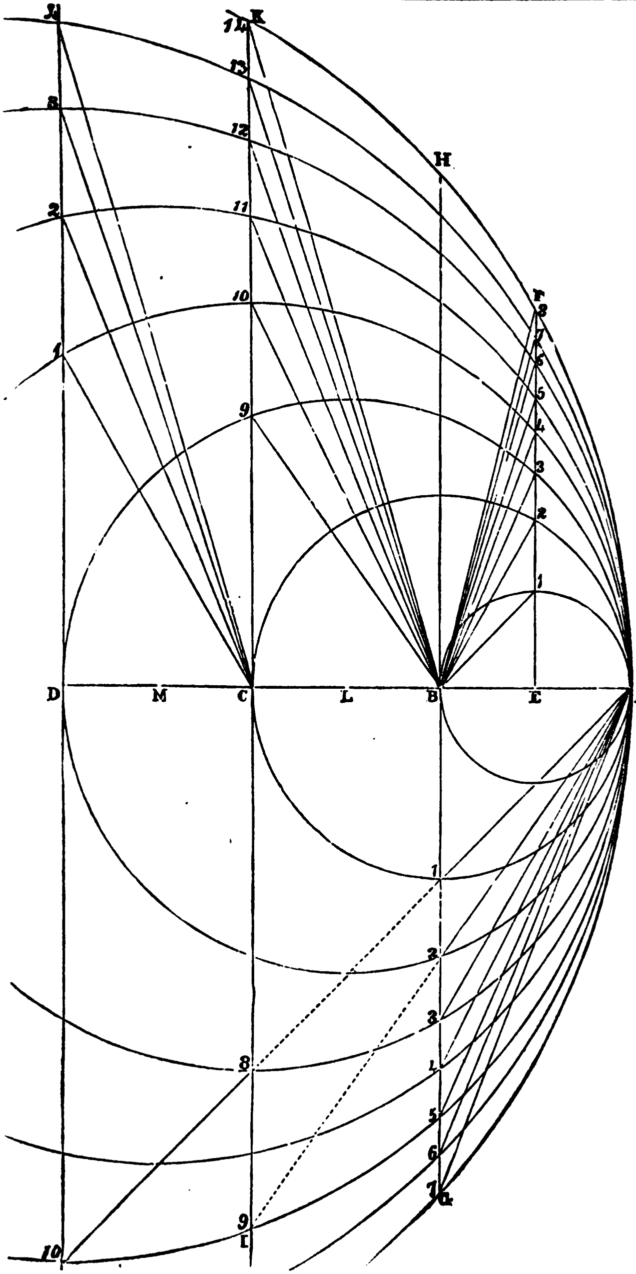
root of 7; C3 is the square root of 10; and C4 is the square root of 13.

REMARK.—The examples thus far given on the diagrams under the last three propositions seem to be sufficient to show that the circle is the measure of extension in every possible direction, and in every possible quantity. The diagrams afford the means of proving the truth of every one of the examples given under these three propositions, as well as innumerable others which might be given, simply by the application of the theorem of Pythagoras, that the square of the hypothenuse equals the sum of the squares of the other two sides. For instance, let it be required to obtain the length of B9, terminating at 9 on the third circle. This circle has a diameter of 3, and its center is L. A line drawn from L to 9 would be the radius, and therefore equal 1.5. This squared is 2.25. CL is .5, which squared is .25. Subtract the square of CL from the square of L9, and it leaves the square of C9 equal to 2. Now C9 square added to CB square equals B9 square. But C9 square is 2, and CB square is 1, making 3, therefore B9 square is 3, and the square root of 3, viz., 1.732+, is the length of B9.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PART THIRD.

HARMONIES OF GEOMETRY.

REMARK.—Of the following *harmonies*, numbering something over a hundred, more than three-quarters were original discoveries with me, and probably most of them will be new to mathematicians in general. These discoveries were reached entirely by the Greek method of rule and compasses, and calculations by arithmetical numbers; and probably very few of them would ever have been reached by anybody, by the methods of algebra. The *surd* quantities which we have to deal with in mathematics are very numerous, while those quantities which have perfect roots are comparatively few. Algebra is entirely blind to relations and agreements existing between surd quantities; whilst arithmetical numbers, by carrying out the roots to a few decimal places, can see and show these relations and agreements as clearly and satisfactorily as in quantities with perfect roots.

For instance, when arithmetic shows us that the square root of 8 is 2.8284+, and that the square root of 2 is 1.4142+, although it has not in either case shown us what the perfect root is, but only an approximation to it, it has nevertheless shown us by satisfactory proof the *perfect relation* between those roots when their quantities are made perfect, viz., that the quantity of one root is precisely half the quantity of the other. And so in a thousand cases the *perfect relations* of surd quantities are shown by arithmetic, as clearly as the relations of quantities with perfect roots.

HARMONIES OF PLANE FIGURES.

CIRCLES AND SQUARES.

1.

The circumference of any square whatever, divided by the circumference of its inscribed circle, produces the same quotient, viz., 1.273+, and this is *the square of the diameter* of another circle whose area equals one square. The square root is 1.1283+, and this is the diameter of a circle whose area equals one square.

2.

The circumference of any circle whatever, divided by the circumference of its circumscribed square, produces the same quotient, viz., .78539+, and this is the *area* of a circle whose diameter is one square.

3.

The area of any square divided by the area of its inscribed circle, also produces the square of the diameter of another circle whose area equals one square; viz., the quotient is always 1.273+.

4.

The area of any circle, divided by the area of its circumscribed square, always produces the area of a circle whose diameter is one square, viz., .78539+.

5.

The circumference of one square, divided by the circumference of a circle whose area equals one square, produces the *diameter* of a circle whose area equals one square, viz., 1.1283+.

6.

The circumference of a circle whose area equals one square, divided by the circumference of one square, produces the square root of the area of a circle whose diameter is one square, viz., .88622+.

7.

Twice the square root of the circumference of any given square, produces the circumference of another square, whose area equals the diameter of the given square.

8.

Twice the square root of the circumference of any given circle, produces the circumference of another circle whose area equals the diameter of the given circle.

9

Twice the square root of the diameter of any given square is the diameter of another square, whose area equals the circumference of the given square.

10

Twice the square root of the diameter of any given circle is the diameter of another circle, whose area equals the circumference of the given circle.

11

Four times the square root of the area of any given circle equals the circumference of another circle, whose area is equal to the circumscribing square of the given circle.

12

The area of a square inscribed in a circle is half the area of a square circumscribed about the same circle.

13

The area of a circle inscribed in a square is one-half the area of a circle circumscribed about the same square.

14

Half the circumference of any circle, multiplied by half its diameter, equals the area of the circle.

15

Half the circumference of any square, multiplied by half its diameter, equals the area of the square.

16

Half the circumference of any plane figure whatever, multiplied by half its diameter, equals the area of the figure. (Diameter always being the diameter of the inscribed circle.)

17

The difference of the circumferences of any two squares, divided by the difference of their diameters, produces the circumference of a square of one diameter, viz., 4.

18

The difference of the circumferences of any two circles, divided by the difference of their diameters, produces the circumference of a circle of one diameter, viz., 3.14159+.

19

The sum of the circumferences of any two squares, divided by the sum of their diameters, produces the circumference of a square of one diameter, 4.

20

The sum of the circumferences of any two circles, divided by the sum of their diameters, produces the circumference of a circle of one diameter, viz., 3.14159+.

21

The square root of the circumference of any given circle is the circumference of another circle, whose area equals one-fourth of the diameter of the given circle.

22

The square root of the circumference of any given square, is the circumference of another square, whose area equals one-fourth of the diameter of the given square.

23

To find a circle and a square whose areas shall be equal to each other. Take any square and its inscribed circle, that is, a square and a circle of the same diameter, and extract the square root of the circumference of each. Double the root from the square for the circumference of *a new square*, and double the root from the circle for the circumference of *a new circle;* then shall the areas of the new square and the new circle be equal to each other.

CIRCLES AND EQUILATERAL TRIANGLES.

24.

The area of a circle inscribed in an equilateral triangle is one-fourth of the area of a circle circumscribed about the same triangle.

25.

The area of an equilateral triangle inscribed in a circle is one-fourth of the area of an equilateral triangle circumscribed about the same circle.

26.

Twice the square root of the circumference of any given equilateral triangle is the circumference of another equilateral triangle whose area equals the diameter of the given triangle.

27.

Twice the square root of the diameter of any given equilateral triangle is the diameter of another equilateral triangle, whose area equals the circumference of the given triangle.

28.

In any equilateral triangle, the square of the perpendicular, divided by the square root of 3, equals the area of the triangle. And double the perpendicular, multiplied by the square root of 3, equals the circumference of the triangle.

29.

To find a circle and an equilateral triangle, whose areas shall be equal to each other. Take any equilateral triangle and its inscribed circle—that is, a triangle and circle of the same diameter, and extract the square root of the circumference of each. Double the root from the triangle, for the circumference of *a new triangle*, and double the root from the circle for the circumference of *a new circle;* then shall the areas of the new triangle and the new circle be equal to each other.

30.

In the equilateral triangle, if the perpendicular is one, the circumference is *twice the square root of three;* but if the area is one, the perpendicular is *the square root of three twice extracted*, or the biquadratic root of three.

REMARK.—It has long been noticed by mathematicians, as a particular and remarkable fact, that if the area of an equilateral triangle is one, the perpendicular happens to be exactly the square root of three *twice extracted.* But in examining this *particular fact,* and the relations between the different parts of equilateral triangles, I discovered that, like most other particular and remarkable facts in geometry and mathematics, it was but the expression of a *general principle,* which applies universally to all equilateral triangles. The general principle is this:

31.

In all equilateral triangles, the biquadratic root of three times the square of the area equals the perpendicular.

Thus, if the area of the triangle is 1, the square of the area is 1, and three times the square of the area is 3, and the square root of three, twice extracted, [the biquadratic root,]—viz., 1.316+, is the perpendicular.

Again, if the area of the triangle is 2, the square of it is 4, and three times the square is 12, and the square root of 12, twice extracted—viz., 1.8612+, is the perpendicular.

It follows, that in every equilateral triangle whose area is a whole number, the perpendicular twice squared will be a whole number, as the following ten examples will show:

If area is 1, the perpendicular is the square root twice extracted from 3.
If area is 2, the perpendicular is the square root twice extracted from 12.
If area is 3, the perpendicular is the square root twice extracted from 27.
If area is 4, the perpendicular is the square root twice extracted from 48.
If area is 5, the perpendicular is the square root twice extracted from 75.
If area is 6, the perpendicular is the square root twice extracted from 108.
If area is 7, the perpendicular is the square root twice extracted from 147.
If area is 8, the perpendicular is the square root twice extracted from 192.

If area is 9, the perpendicular is the square root twice extracted from 243.
If area is 10, the perpendicular is the square root twice extracted from 300.

CIRCLES AND ALL POLYGONS.

32.

Twice the square root of the circumference of any given circle is the circumference of another circle whose area equals the diameter of the given circle.

33.

Twice the square root of the circumference of any given polygon is the circumference of another similar polygon, whose area equals the diameter of the given polygon.

34.

Twice the square root of the diameter of any given circle is the diameter of another circle, whose area equals the circumference of the given circle.

35.

Twice the square root of the diameter of any given polygon is the diameter of another similar polygon, whose area equals the circumference of the given polygon.

36.

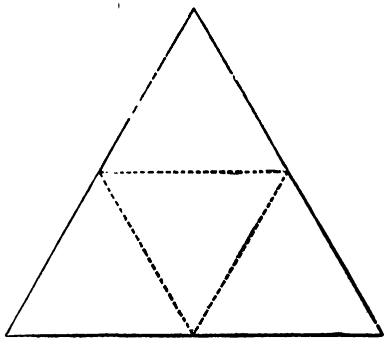
To find a circle, whose area shall equal the area of a polygon, which is similar to any given polygon that can receive an inscribed circle. Extract the square root of the circumference of the given polygon, and also the square root of the circumference of the inscribed circle. Double the root from the polygon, for the circumference of *a new similar polygon*, and double the root from the circle for the circumference of *a new circle;* then shall the area of the new circle equal the area of the new polygon, which is similar to the given polygon.

HARMONIES OF SPHERES AND THE PLATONIC BODIES.

Remarks.—The harmonies and remarkable agreements and coincidences of geometry, which are discovered in the examination of *solids*, are perhaps more beautiful and interesting, than even those which appear in the consideration of plane figures. And in the study of the general principles of the science, as Dr. Barrow remarks in the Preface to his Euclid, "the noble contemplation of the *five regular bodies* cannot, without great injustice, be pretermitted."

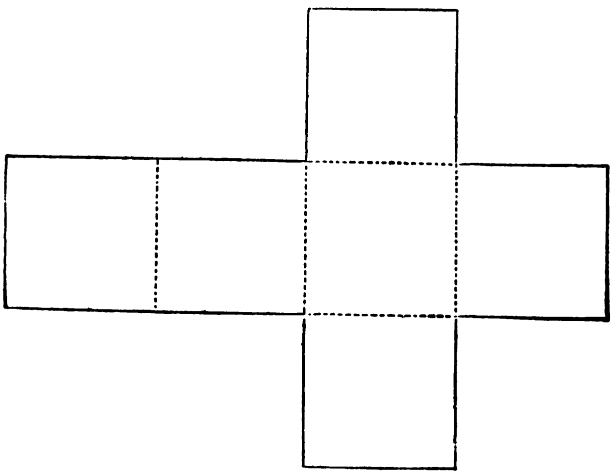
Attempts to represent solids and parts of solids by diagrams, in the page, or on a plane surface, generally afford but little aid to the student, and sometimes perplex and embarrass more than they enlighten; for it often costs the unpractised student more labor to learn and recollect the parts of the solid represented by such diagrams, than it would to demonstrate half a dozen propositions if the palpable solid in its proper form were before him. Therefore instead of attempting to illustrate the principles and proportions of solid figures by diagrams, I recommend to the student, in examining and demonstrating the following harmonies, by all means to have before him *proper models* of the solids and parts of solids which he is considering. They may readily be cut from soft wood, and even temporary ones from a vegetable, a turnip or potatoe. Perhaps the easiest and pleasantest mode however of constructing them, with nearly accurate proportions, is to cut them from paste-board, as represented in the annexed diagrams. These may be easily and pleasantly made by *ladies;* and I hope the day is not distant when this most perfect and beautiful science shall become a favorite study with ladies.

Let the plane figures, as represented in the diagrams, be cut out of common paste-board, and on the dotted lines cut half through with a sharp knife. Then by turning up the folds till the several edges meet, and fastening them with paste or some adhesive substance, you will have what are called "the five regular solids of Plato," viz., the tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron.

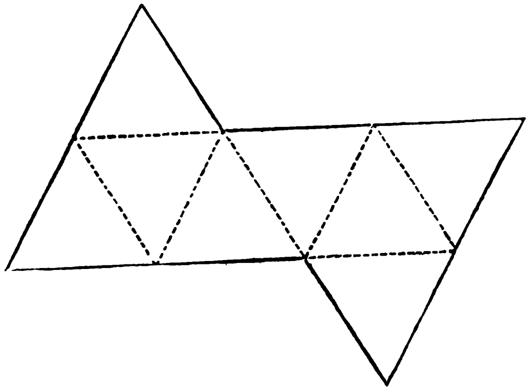


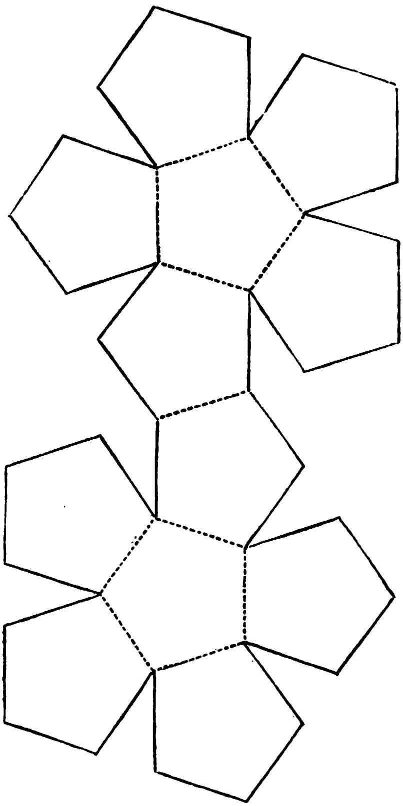
Tetrahedron.—The diagram presents an equilateral triangle, each side being two inches. Let each side be divided in the center, and the points of division joined by the dotted lines, and the diagram then presents four equilateral triangles, each side of which is one inch. If the whole diagram be cut from pasteboard, and the dotted lines cut half through, then by turning up three of the triangles till their edges meet we have a solid figure with four triangular faces and four solid angles.

Hexahedron, or Cube.—The surface of the hexahedron consists of six squares. The diagram cut half through on the dotted lines, and folded, makes the cube, having the form of the *geometrical unit.*

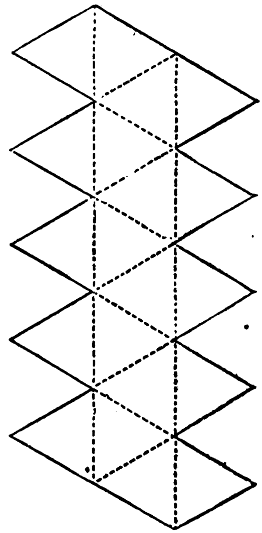


Octahedron.—The surface of the octahedron is composed of eight equilateral triangles.





Dodecahedron.—The surface of the dodecahedron is composed of twelve equilateral pentagons.



Icosahedron.—The surface of the icosahedron is composed of twenty equilateral triangles.

In each of these solids the planes, constituting the surface, are all similar in form, equal in extension, and meet each other at equal angles. They are therefore called "regular solids;" and besides these five, it is not possible for another regular solid to be formed; that is, a solid, the planes of whose surface are all similar in form, equal in extension, and meet each other at equal angles.

Legendre, the most distinguished French geometer, in proposing some changes in definitions and names applied to parts of these solids, remarks that, "as the theory of those solids has hitherto been *little investigated*, no great inconvenience could arise from introducing any new expressions which are called for by the nature of the objects."

From the following enumeration of *harmonies* in the Platonic bodies, and from the general principles laid down in the first part of this work, it will be seen that the "*theory of those solids*" is very simple, and that in the relations of diameter, solidity and surface, one simple and uniform law applies to them all. It will be seen also, that the principles developed in these new elements of geometry required some new definitions, such as the distinction between the *faces* and *surface* of a solid, and a new definition for diameter both in plane figures and solids. For the reader must not forget that the diameter of every solid is the diameter of its inscribed sphere. Instead of hexahedron, in what follows, we shall generally use the name cube, as being a more convenient word.

SPHERES AND CUBES.

37.

The surface of a cube of one diameter, divided by the surface of its inscribed sphere, produces the square of the diameter of another sphere, whose surface equals the surface of the given cube.

38.

The surface of *any* cube, divided by the surface of its inscribed sphere, produces the square of the diameter of another sphere, whose surface equals the surface of a cube of one diameter.

39.

The surface of a sphere of one diameter, divided by the surface of its circumscribed cube, produces the solidity of the given sphere.

40.

The surface of *any* sphere, divided by the surface of its circumscribed cube, produces the solidity of a sphere of one diameter.

41.

The surface of any cube, divided by the surface of its inscribed sphere, produces the cube or third power of the diameter of a sphere, whose solidity is one, or equal to the solidity of a cube of one diameter.

42.

The solidity of any cube, divided by the solidity of its inscribed sphere, produces the cube or third power of the diameter of a sphere, whose solidity is equal to a cube of one diameter.

43.

The solidity of any sphere, divided by the solidity of its circumscribed cube, produces the solidity of a sphere of one diameter.

44.

The solidity of any given sphere, divided by the solidity of a sphere of one diameter, produces the solidity of the cube circumscribing the given sphere.

45.

The surface of a cube inscribed in a sphere, equals one-third of the surface of the cube circumscribed about the same sphere.

46.

The surface of a sphere inscribed in a cube, equals one-third of the surface of the sphere circumscribed about the same cube.

47.

If a sphere be inscribed in a cube and another sphere circumscribed about the cube, the square of the diameter of the inscribed sphere equals one-third of the square of the diameter of the circumscribed.

48.

If a cube be inscribed in a sphere and another cube circumscribed about the sphere, the square of the diameter of the inscribed cube equals one-third of the square of the diameter of the circumscribed.

49.

The cube root of the surface of a cube, whose diameter is six, equals the surface of a cube whose solidity is one.

50.

The cube root of the surface of a sphere, whose diameter is six, equals the surface of a sphere whose solidity is one, or equal to one cube.

51.

In both the cube and the sphere, if diameter is six, solidity equals the surface; and if surface is six, solidity equals the diameter.

52.

In both the cube and the sphere, six times the solidity, divided by the diameter, equals the surface.

53.

If 1 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 3.

54.

If 2 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 12.

55.

If 3 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 27.

56.

If 4 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 48.

57.

If 5 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 75.

58.

If 6 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 108.

59.

If 7 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 147.

60.

If 8 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 192.

61.

If 9 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 243.

62.

If 10 is the diameter of a cube, the diameter of its circumscribed sphere is the square root of 300.

SPHERES AND TETRAHEDRONS.

63.

The surface of a tetrahedron of one diameter, divided by the surface of its inscribed sphere, produces the square of the diameter of another sphere, whose surface equals the surface of the given tetrahedron.

64.

The surface of *any* tetrahedron, divided by the surface of its inscribed sphere, produces the square of the diameter of another sphere, whose surface equals the surface of a tetrahedron of one diameter.

65.

The diameter of a sphere inscribed in a tetrahedron, equals half the perpendicular or height of the tetrahedron.

66.

The perpendicular of a tetrahedron inscribed in a sphere, equals two-thirds the diameter of the sphere.

67.

If a sphere be inscribed in a tetrahedron and another sphere circumscribed about the tetrahedron, the diameter of the inscribed sphere equals one-third the diameter of the circumscribed, the surface of the inscribed equals one-ninth of the surface of the circumscribed, and the solidity of the inscribed equals one twenty-seventh of the solidity of the circumscribed sphere.

68.

If a tetrahedron be inscribed in a sphere and another tetrahedron circumscribed about the sphere, the diameter of the inscribed tetrahedron equals one-third the diameter of the circumscribed, the surface of the inscribed equals one-ninth of the surface of the circumscribed, and the solidity of the inscribed equals one-twenty-seventh of the solidity of the circumscribed tetrahedron.

69.

If the linear edge of a tetrahedron is 1, the surface equals the square root of 3.

70.

If the diameter of a tetrahedron is 1, the solidity equals the square root of 3.

71.

The height or perpendicular of a tetrahedron equals the square root of two-thirds the square of its linear edge.

72.

The surface of a tetrahedron of one diameter, divided by the surface of its inscribed sphere, or sphere of one diameter, produces the cube or third power of the diameter of a sphere whose solidity equals the solidity of the given tetrahedron.

73.

The surface of *any* tetrahedron, divided by the surface of its inscribed sphere, produces the cube or third power of the diameter of a sphere whose solidity equals the solidity of a tetrahedron of *one* diameter.

74.

The solidity of a tetrahedron, divided by the solidity of its inscribed sphere, produces the cube or third power of the diameter of a sphere whose solidity equals the solidity of a tetrahedron of one diameter.

75.

In both the sphere and tetrahedron, if diameter is six, solidity equals the surface; and if the surface is six, solidity equals the diameter.

76.

In both the sphere and tetrahedron, six times the solidity, divided by the diameter, equals the surface.

77.

If the surface of a tetrahedron is 6, the linear edge is the square root of 12, twice extracted, or the biquadratic root of 12. The linear edge also equals the diagonal of an octahedron, whose surface is 6.

SPHERES AND OCTAHEDRONS.

78.

In any octahedron, the square of the diameter equals two-thirds the square of the linear edge.

79.

In any octahedron, the square of the linear edge equals one-half the square of the diagonal.

80.

In any octahedron, the square of the diameter equals one-third the square of the diagonal.

81.

In both the sphere and octahedron, if diameter is 6, the solidity equals the surface; and if surface is 6, the solidity equals the diameter.

82.

In both the sphere and octahedron, six times the solidity, divided by the diameter, equals the surface.

83.

The surface of an octahedron of one diameter, divided by the surface of its inscribed sphere, produces the square of the diameter of another sphere, whose surface equals the surface of the given octahedron.

84.

The surface of *any* octahedron, divided by the surface of its inscribed sphere, produces the square of the diameter of another sphere, whose surface equals the surface of an octahedron of one diameter.

85.

The solidity of an octahedron of one diameter, divided by the solidity of its inscribed sphere, produces the cube or third power of the diameter of another sphere, whose solidity equals the solidity of the given octahedron.

86.

The solidity of *any* octahedron, divided by the solidity of its inscribed sphere, produces the cube or third power of the diameter of another sphere, whose solidity equals the solidity of an octahedron of one diameter.

87.

If an octahedron be inscribed in a sphere, and another circumscribed about the sphere, the square of the diameter of the inscribed octahedron equals one-third of the square of the diameter of the circumscribed; the square of the surface of the inscribed equals one-ninth the square of the surface of the circumscribed; and the square of the solidity of the inscribed equals one twenty-seventh of the square of the solidity of the circumscribed octahedron.

88.

If a sphere be inscribed in an octahedron, and another sphere circumscribed about the octahedron, the square of the diameter of the inscribed sphere equals one-third the square of the diameter of the circumscribed; the square of the surface of the inscribed equals one-ninth the square of the surface of the circumscribed; and the square of the solidity of the inscribed equals one twenty-seventh of the square of the solidity of the circumscribed sphere.

89.

In the octahedron whose diameter is 1, the solidity equals half the square root of 3; the linear edge equals the square root of one and a-half, or 1.5; the diagonal equals the square root of 3; and the surface equals the square root of 27.

90.

In the octahedron whose diameter is 2, the linear edge equals the square root of 6; the diagonal equals the square root of 12; the solidity equals the square root of 48; and the surface equals the square root of 432. 48 is one-ninth of 432, and the square root of 48 is one-third the square root of 432.

91.

In the octahedron whose diameter is 3, the linear edge equals the square root of 13.5; the diagonal equals the square root of 27; the solidity equals the square root of 546.75; and the surface equals the square root of 2187. The square root of the last of these numbers is double the square root of the preceding number; therefore when the diameter of the octahedron is 3, the solidity equals half the surface.

92.

In the octahedron whose diameter is 4, the linear edge equals the square root of 24; the diagonal equals the square root of 48; and the solidity equals two-thirds the surface.

93.

In the octahedron whose diameter is 5, the linear edge equals the square root of 37.5; the diagonal equals the square root of 75; and the solidity equals five-sixths of the surface.

94.

In the octahedron whose diameter is 6, the linear edge equals the square root of 54; the diagonal equals the square root of 108; the solidity is the square root of 34992, and the surface is also the square root of 34992, viz., 187.06148+. Therefore when diameter is 6, the solidity equals the surface.

95.

In the octahedron whose *surface* is 6, the linear edge equals the square root of 3, *twice extracted;* the diagonal equals the square root of 12, *twice extracted;* the diameter equals the square root of 1.333333+, *twice extracted;* and the solidity also equals the square root of 1.333333+, *twice extracted.* Therefore when surface is 6, the solidity equals the diameter.

CUBES AND OCTAHEDRONS.

96.

In the cube whose diameter is 1, the diagonal equals the square root of 3.

97.

In the octahedron whose diameter is 1, the diagonal equals the square root of 3.

98.

In the cube whose diameter is 2, the diagonal is the square root of 12.

99.

In the octahedron whose diameter is 2, the diagonal is the square root of 12.

100.

And in all cubes and octahedrons of equal diameters, the diagonals are also equal.

TETRAHEDRONS AND OCTAHEDRONS.

101.

In the tetrahedron whose linear edge is 1, the surface equals the square root of 3.

102.

In the octahedron whose diagonal is 1, the surface equals the square root of 3.

103.

In the tetrahedron whose linear edge is 2, the surface equals the square root of 48.

104.

In the octahedron whose diagonal is 2, the surface equals the square root of 48.

105.

And universally, if the linear edge of a tetrahedron equals the diagonal of an octahedron, the surfaces of the two bodies are equal.

106.

If four tetrahedrons, whose faces are severally equal to the faces of an octahedron, be applied to four alternate faces of the octahedron, the whole will constitute a regular tetrahedron.

107.

If the linear edges of a tetrahedron be all equally bisected, and the four vertices or solid angles of the tetrahedron be taken away by planes cutting through the points of bisection, the part that is left will be a regular octahedron.

TETRAHEDRONS AND EQUILATERAL TRIANGLES.

108.

If the perpendicular of a tetrahedron be 1, the solidity equals one-sixteenth of the circumference of an equilateral triangle whose perpendicular is 1.

109.

If the perpendicular of a tetrahedron be 2, the solidity equals four-sixteenths of the circumference of an equilateral triangle whose perpendicular is 2.

110.

If the perpendicular of a tetrahedron be 3, the solidity equals nine-sixteenths of the circumference of an equilateral triangle whose perpendicular is 3.

111.

If the perpendicular of a tetrahedron be 4, the solidity equals sixteen-sixteenths, that is, it *equals* the circumference of an equilateral triangle whose perpendicular is 4.

112.

And universally, the circumference of any equilateral triangle, divided by 16, and multiplied by the square of its perpendicular, equals the solidity of a tetrahedron of the same perpendicular.

THE THREE ROUND SOLIDS OF ARCHIMEDES.

REMARKS.—The great geometer of Syracuse, who has been styled the "Newton of antiquity," discovered and demonstrated the proportions and relations to each other, of the cylinder, the sphere, and the cone; and at his decease the figures of these solids were carved upon his tomb in honor of his distinguished contributions to science. But the works of genius are not commemorated by monuments of marble or brass. A hundred and thirty-six years after the decease ot Archimedes, Cicero, on visiting the Island, sought for the monument of the great mathematician and philosopher, but there was no one to point it out to him. The light of science had almost become extinguished in Syracuse, and the name of Archimedes nearly forgotten. After considerable exertion, however, Cicero discovered the monument, overgrown with thorns and briars, and was still able to read the half-effaced inscriptions, and to behold the figures of the cylinder and the sphere. That monument has long, long since crumbled to dust; two thousand years have passed away; but the beautiful theorems of Archimedes still live and flourish, undying evergreens in the gardens of science.

Archimedes discovered and demonstrated, that the perpendiculars of a cylinder and a cone, and the diameters of their bases, and the diameter of a sphere, all being equal, the solidity of the cone equals one-half the solidity of the sphere, and the solidity of the sphere equals two-thirds the solidity of the cylinder. And consequently that the cone equals one-third the cylinder.

Had Archimedes discovered the simple law of solidity, diameter, and surface of all solids, he would have seen relations and harmonies existing in these round bodies, more remarkable and more beautiful than even those which he demonstrated. For the surfaces of cylinders and cones being composed partly of *plane* and partly of *curved* surfaces, it is indeed a beautiful and wonderful illustration of the truth and universality of the law of solidity, diameter, and surface, already so fully explained in this work, to find it governing the cylinder and the cone precisely as it governs the cube and the tetrahedron, or any other solid.

The geometrical diameter, or diameter of extension, of all solids, it must be remembered, is the diameter of an inscribed sphere. And as no cylinder can have an inscribed sphere, except those in which the altitude or axis equals the diameter of the base, it is manifest that no other cylinders, properly speaking, have a geometrical diameter. But every right cone, however the base or altitude may be varied, can contain an inscribed sphere. Therefore all right cones have a geometrical diameter.

A cylinder of *one* diameter, and a circle of *one* diameter, are in every respect identically the same figure; precisely as the unit of a line or the unit of a surface is identically the same figure with the unit of a solid. One inch of area, and one inch of solidity, in geometry, have precisely the same value, and there is no geometrical difference between a square inch and a cubic inch. When we consider the unit in area, or as a square inch, we consider only one face of the unit, disregarding entirely its thickness. But when we consider the unit in a solid, we look to all its dimensions, as contained under six faces.

Precisely in the same manner is explained the truth stated above—that a cylinder of *one* diameter and a circle of *one* diameter are in every respect the same figure. As a circle, it is a plane figure, and belongs to area; and in considering it we look at one end or one base of the cylinder only, disregarding its other dimensions. As a cylinder, it becomes a solid, and is considered in all its dimensions, as contained under a curve surface and two plane bases.

The area of a circle of 1 diameter is .78539+. And the base of a cylinder of 1 diameter being a circle of 1 diameter, the area of the base of the cylinder is also .78539+. To obtain the solidity of a cylinder we multiply the area of the base by the height. But the diameter of the cylinder being 1, the height is 1. And multiplying the area of the base by 1, still gives .78539+ for the solidity of the cylinder. Therefore the solidity of a cylinder of one diameter and the area of a circle of one diameter are precisely the same thing.

SOLIDITY, DIAMETER, AND SURFACE OF CYLINDERS AND CONES.

If the *diameter* of a cylinder or cone is *one*, solidity equals one-sixth of the whole surface.

If diameter is two, solidity equals one-third of the surface.

If diameter is three, solidity equals one-half the surface.

If diameter is four, solidity equals two-thirds of the surface.

If diameter is five, solidity equals five-sixths of the surface.

If diameter is six, solidity and surface are equal.

If the *surface* of a cylinder or cone equals *one*, solidity equals one-sixth of the diameter.

If the surface is two, solidity equals one-third of the diameter.

If surface is three, solidity equals one-half the diameter.

If surface is four, solidity equals two-thirds of the diameter.

If surface is five, solidity equals five-sixths of the diameter.

If surface is six, solidity and diameter are equal.

EXAMPLES.

First. **In the cylinder, whose diameter is *one*, the solidity equals one-sixth of the surface.**

It has already been shown that the solidity equals the area of a circle whose diameter is one—viz., .78539+. The curve surface of a cylinder is obtained by multiplying the perimeter or circumference of the base by the height of the cylinder. This circumference in a cylinder of one diameter, or a circle of one diameter, is 3.14159+ ; and this quantity is not varied by multiplying it by the height, for the height is 1.

Therefore the curve surface is - -	3.14159+
The surface of the base is - - -	.78539+
The surface of the opposite base is - -	.78539+
And the *whole* surface is - - -	4.71237+.
One-sixth of the last number is - -	.78539+.

Therefore the solidity of a cylinder, whose diameter is one, equals one-sixth of its surface.

Second. **In the cylinder, whose diameter is *six*, the solidity equals the surface.**

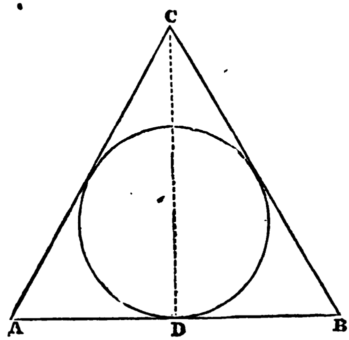
The base of the cylinder is a circle, whose diameter is 6. Therefore the area of the base is 28.2743+, and this area multiplied by 6, the height of the cylinder, gives for the *solidity* ha9.6458+. The perimeter of the base is 18.8495+; and this, A c plied by 6, the height of the cylinder, gives for the *curve*

. - - - - - - -	113.0972+
meter, ea of the base is - - -	28.2743+
figure ; pr of the opposite base is - -	28.2743+
le surface - - -	169.6458+.

This last sum equals the solidity. Therefore in the cylinder whose diameter is six, the solidity equals the surface.

Third. In the *cone,* whose diameter is *one,* the solidity equals one-sixth of the surface.

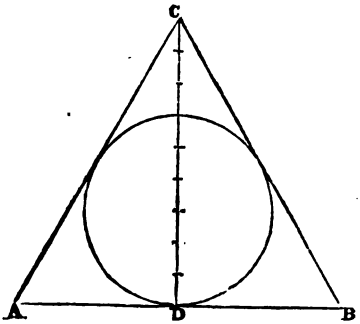
If any right cone containing an inscribed sphere be bisected by a plane passing through the vertex and the center of the base, the section will present a triangle with an inscribed circle, like the annexed diagram. And if the slant height of the cone is equal to the diameter of the base, the triangle thus presented will be equilateral. From this diagram of the vertical section we can obtain the dimensions of the cone. The triangle being equilateral, the diameter of the circle equals two-thirds the perpendicular of the triangle, [Prop. 7.] Let the diameter therefore be *one.* [In the diagram the unit is one inch.]



Then the perpendicular, CD, is one and a half—viz., 1.5; and this squared is 2.25. One-third of 2.25 is .75, which added to 2.25 makes 3. Then the square root of 3 equals one side of the triangle, for the side of an equilateral triangle is the square root of one-third added to the square of the perpendicular. Therefore the square root of 3—viz., 1.732+, equals the slant height of the cone, AC or BC, and also the diameter of the base, AB.

The diameter of the base being 1.732+, the area of the base is 2.356+, and this multiplied by one-third the perpendicular height —viz., .5, gives 1.178+ for the *solidity* of the cone. The perimeter of the base is 5.4412+, and this multiplied by half the slant height—viz., .866+, gives 4.712+ for the *curve* surface. Add to the curve surface the area of the base, 2.356, and we have 7.0681+ for the whole surface of the cone. Divide this last number by 6, and it gives 1.178+, equal to the solidity. Therefore in the cone whose diameter is *one,* the solidity equals one-sixth of the surface.

Fourth. In the cone, whose diameter is *six,* the solidity equals the surface.



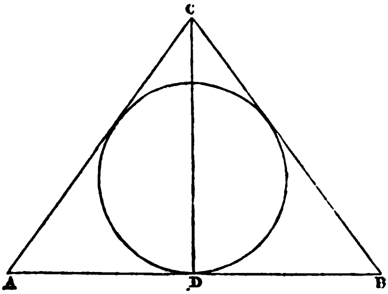
Make diameter 6, and divide the perpendicular CD into nine equal parts. Diameter being 6, the perpendicular CD equals 9, [Prop. 7.] The square of 9 is 81, and one-third added makes 108. Therefore the square root of 108—viz., 10.3923+, equals the slant height of the cone AC or BC, and also the diameter of the base, AB. The diameter of the base being 10.3923+, the area of the base is 84.82285+, and this area multiplied by 3, one-third of the perpendicular, gives 254.468+ for the *solidity* of the cone. The perimeter of the base is 32.64834+, and this multiplied by half the slant height—viz., 5.19615+, gives 169.64567+ for the *curve* surface. Add to the curve surface the area of the base, and we have 254.468+ for the *whole* surface of the cone, which thus equals the solidity. Therefore in the cone whose diameter is six, the solidity equals the surface.

REMARK.—In tracing out these problems of the cone I discovered another remarkable general principle. It is this:

In every right cone, whose slant height equals the diameter of its base, if the square of the perpendicular be made the diameter of a circle, *the circumference of that circle shall equal the surface of the cone.*

For instance, in the last example the perpendicular of the cone is 9, and the square of 9 is 81. If 81 be made the diameter of a circle, and multiplied by the circumference of one—viz., 3.14159+, it gives for its circumference 254.468+, which equals the surface of the cone.

And in the preceding example, the perpendicular is 1.5, the square of which is 2.25. If 2.25 be made the diameter of a circle, its circumference will be 7.068+, which equals the surface of the cone.



Fifth. Let the diameter of a right cone be *one*, and the diameter of the base two. Then the diagram will present a vertical section through the center. The diameter of the base, AB, is two inches, and the diameter of the inscribed circle or inscribed sphere one inch. The slant height of the cone, AC or BC, is one and two-thirds, 1.66666+, and the perpendicular, CD, is one and one-third, 1.33333+. Diameter of the base being 2, the area of the base is 3.14159+, which being multiplied by one-third of the perpendicular, gives 1.3962+ for the *solidity* of the cone. The perimeter of the base is 6.28318+, which being multiplied by half the slant height gives 5.23598+ for the *curve* surface. Add to the curve surface the area of the base, 3.14159+, and it gives 8.37757+ for the *whole* surface of the cone. And this surface divided by 6 gives 1.3962+, equal to the solidity. Therefore in any cone whose diameter is one, the solidity equals one-sixth of the surface.

That the sides AC and BC, in the last example, are each 1.6666+, and the perpendicular, CD, 1.3333+, may be proved by applying the principle demonstrated in Proposition 52, viz., that the whole circumference of a triangle is to the base as the perpendicular is to the radius of the inscribed circle. Thus, AB is 2, AC is 1.6666+, and BC is 1.6666+; and these being added together, the whole circumference of the triangle is 5.3333+. Then we have the proportion 5.3333+ : 2 : : 1.33333+ : .5

$$\begin{array}{r} 2 \\ \hline 5.3333+ \left\{ \begin{array}{l} 2.66666+ \\ 2.66666+ \end{array} \right\} .5 \end{array}$$

which gives .5 for the radius of the inscribed circle. But the diameter was made 1 in the proposition; therefore the radius is .5, and agrees with the demonstration.

NOTE.

THE foregoing Treatise was commenced with a remark of Lord BACON, that "the invention [or discovery] of *forms* is of all other parts of knowledge the worthiest to be sought, if it be possible to be found;" and I feel constrained now to add, that in every stage of progress through this work, and the researches and reflections to which it has led me, the justice and importance of that remark have been more and more deeply impressed on my mind. I am led strongly to conjecture that geometrical forms underlie all the works of nature. The wonderful and endless harmonies discoverable in those forms, show them to be capable of infinite adaptations. We already know that every note of music which strikes the ear, is governed by geometrical laws; and the time may yet come when every shade of color which delights the eye, and every odor and every taste which regales the sense, may be referred to different geometrical forms of matter. The time may yet come when geometrical forms shall be found to be the mainspring of all the motions and all the forces of nature—a mainspring receiving an eternal impress from the finger of the *Almighty*, and forever and unceasingly doing his will.

Sir ISAAC NEWTON, in the preface to his *Principia*, makes this remark: "I am induced by many reasons to suspect, that they [the phenomena of nature] may all depend upon certain forces, by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards each other, and cohere in *regular forms*, or are repelled and recede from each other; which forces being unknown, philosophers have hitherto attempted the search of nature in vain."

Let us fix our attention for a moment on one simple law of Geometry, viz., that in any given quantity of matter or space the relation of solidity, diameter and surface is an infinitely varying relation, changing with every change of *form* in the given quantity. As the equilateral triangle and the circle are the two extremes of this infinitely varying scale in plane figures, so the tetrahedron and sphere are the two extremes of the infinitely varying scale in solid figures. If we take a given surface equal to six square inches, and measure the solidity or bulk of matter or space inclosed by that surface under different forms, we shall find that in the form of the tetrahedron it will inclose the least possible bulk that it can inclose in any form whatever, and if brought into the form of a perfect sphere, it will inclose the greatest possible bulk that a surface of six inches can inclose under any form whatever.

For example: let the six square inches of surface be brought into the form of the tetrahedron, and it will contain a little more than three-fourths of a cubic inch, viz., .759+.

Let the same surface be brought into the form of a hexaedron or cube, and it will contain just *one* cubic inch.

Let the same surface be brought into the form of an octahedron, and it will contain a little more than one cubic inch, viz., 1.074+.

Let the same surface be brought into the form of a sphere, and it will contain more than one inch and one-third, viz., 1.381+.

And between the tetrahedron and the sphere, the same amount of surface may contain an infinite series of magnitudes, all differing in quantity.

Even to our limited powers of comprehension it does not seem difficult to suppose, that this simple law of Geometry, in the hands of the Almighty, may be made the basis of an infinite variety of attractive or repulsive forces in matter.

Among the valuable publications of the English Society for the Diffusion of Useful Knowledge, is a paper on the "objects, advantages, and pleasures of science," to which is attached the following note:—"The application of mathematics to *chemistry* has already produced a great change in that science, and is calculated to produce still greater improvements. It may be almost certainly reckoned upon as the source of new discoveries, made by induction after the mathematical reasoning has given the suggestion. The learned reader will perceive that we allude to the beautiful doctrine of *Definite* or *Multiple Proportions*. Take an example: the probability of an oxide of arsenic being discovered, is impressed upon us, by the composition of arsénious and arsenic acids, in which the oxygen is as 2 to 3; and therefore we may expect to find a compound of the same base, with the oxygen as *unity*." This is an interesting hint to the chemist and the natural philosopher. And we may suggest further: If chemistry presents its products in mathematical proportions, represented by numbers, when we recollect that all mathematical numbers are but representatives of magnitudes, and that all magnitudes have *forms*, what other conclusion can we come to, but that all chemical changes are simply changes in the forms of matter? Changes perhaps, in which diameters, solidities, and surfaces find new relations, and become subject to new impulses.

It is supposed that all substances are susceptible of crystallization by nature or art. And all crystals, if produced in situations which allow perfect freedom of motion in the particles during the formation of the crystal, are known to assume perfect geometrical forms. Thus, sea-salt generally crystallizes in the form of cubes; sometimes in the form of octahedrons. Nitre takes the form of a hexaedral prism. Sugar appears in four or six-sided prisms, with trihedral terminations. Alum in pure water crystallizes in octahedrons, &c., &c. Some substances assume different forms, according to the temperature in which the crystallization occurs. Thus the carbonate of lime, for instance, takes a great variety of forms. Heat and light both have a remarkable agency in the formation of crystals. It is stated in the London Magazine of Science, that "prismatic crystals of sulphate of nickel, exposed to a summer's sun in a close vessel, had their internal structure so completely altered, without any exterior change, that when broken open they were composed internally of *octahedrons*, with square bases." And in the same way prismatic crystals of zinc are said to be changed in a few seconds by the heat of the sun to *octahedrons*.

The distinguished Hauy developed the theory of crystals so far as to show "that in every crystallized substance, whatever may be the difference of figure which may arise from modifying circumstances, there is in all its crystals *a primitive form*, the nucleus, as it were, of the crystal, invariable in each substance." The primitive forms he reduced to six, viz., the parallelopipedon, which includes the cube; the rhomb, including all the solids terminated by six faces, parallel two and two; the tetrahedron; the octahedron; the regular hexaedral prism; and the dodecahedron. But Hauy analyzed these primitive forms still farther, cleaving off their parallel layers, till he discovered their germs, so to speak, and found them to consist of but three still more simple forms, which he called integrant particles. These are the tetrahedron, the simplest of the pyramids; the triangular prism, the simplest of the prisms; and the parallelopipedon, including the cube, the simplest of solids, which have their faces parallel two and two.

Thus it would seem that the *Almighty* has been pleased to subject all matter to the control of perfect geometrical laws. And while he has endowed man with reason, and left him to search out these laws and study their beauties and perfections, he has manifestly given to the lower orders of animals, in many instances, and for aught we know, in all instances, an instinctive faculty or power of being guided by these laws in the pursuits adapted to their natures. The eagle, which frequents high places, in descending from height to height, or from the mountain top to the plain, is believed by curious and philosophical observers to descend in that peculiar curve line called a *cycloid*. [A nail, or any point in the rim of a wheel, moves in this curve through the air as the wheel rolls along the plain.] Now, mathematical demonstrations prove, that if the eagle would descend from the mountain top to the plain in *the least possible time*, it must go, not in a straight line, nor in any other curve line, but exactly in the cycloid.

Beavers are observed to build their dams uniformly on mathematical principles. The side up stream is built at a certain slope or angle; and mathematicians demonstrate that this angle is the very one which gives to the dam the greatest possible power of resistance to the stream.

The wonderful honey-bee is a perfect geometer in all his works. He fills up his hive or his hollow tree with perfect geometrical figures, constructed on the truest principles. The walls of the little cells to contain his honey are perfect hexagons; and they are closed at top and bottom by triangular planes meeting in a point, and always at a certain angle. Now the form of these cells is mathematically proved to be precisely that which combines the greatest possible capacity in the given space for storing the honey, the greatest possible strength in the cells, and the least possible quantity of material for their construction.

And the spider too, even "the villain spider"—

> "Who taught the spider parallels design,
> "Sure as De Moivre, without rule or line?"

But I must close, lest what was intended only as a brief Note should run into an Essay. I can but add my conviction that there is yet a wide field for the future progress of human knowledge in the investigation of geometrical forms and their relations to matter. When we see all nature, both animate and inanimate, everywhere around us, wearing the impress of perfect geometrical laws, well may we exclaim with the pious Dr. Barrow—"O Lord! how great a geometer art thou!"