Plato's Mathematical Imagination

by ROBERT S. BRUMBAUGH

The Mathematical Passages in the Dialogues and Their Interpretation

INDIANA UNIVERSITY PRESS
Bloomington

Preface

PROFESSORS Richard P. McKeon, Raymond Klibansky, Newton P. Stallknecht, and Guido Calogero have read this study in manuscript, and I am most grateful to them for their suggestions and comments. I hope that readers acquainted with Professor McKeon's approach to Greek philosophy will find in this study by one of his students and admirers a spirit and method with which they are familiar. Over a period of years, Professor Stallknecht has been unfailingly generous of his time and assistance. His insights into the aesthetic dimension of my problems, and his distinctive analysis of the nature and history of imagination, have added immeasurably to my own appreciation of the subject treated here. Professor Klibansky has contributed a number of incisive and illuminating comments on the Pythagorean and Platonic traditions and on some of the textual problems crucial in mathematical passages in Plato. Professor Calogero's general evaluation and specific comments were most helpful. The responsibility for any shortcomings of the text remains my own; but because this study departs at many points from any previous work in the Platonic tradition, it is a responsibility I should have hesitated to assume without the encouragement and help of these scholars.

I wish to thank Professor Edward Seeber, of Indiana University, for his help in the preliminary editing of copy. I am particularly grateful to Mr. Walter Albee, of the Indiana University Press, for copy-editing the manuscript and seeing it through the various stages of design and production and for suggesting many improvements in the design of the figures and in the coherence of the text of this study. Mr. Robert Williams and the Yale Cartographic Laboratory have been most helpful in preparing drawings for the figures. Professor Ronald B. Levinson, of the University of Maine, has helped me to clarify and improve Section 6 of Chapter III; I am sorry that his *In Defense of Plato* appeared too late to be cited where it is clearly relevant. The possibility of constructing a mirror which would give a non-reversed image as the one in Figure 88 does, a possibility that has been seriously doubted, was proven by Dr. Harrison Pemberton and Mr. George Starr, of Yale University. Mr. Thomas B. Rosenmeyer's criticism of my views on the *Critias* has also been helpful, not only in Chapter II, where his article is explicitly cited, but in the whole study.

My wife has been a model of patience, help, and industry; she has copied figures, turned thickets of words into English sentences, and listened intelligently to sporadic speeches on Platonic mathematics. I also want to thank Meda Z. Steele for her help in copying figures, and my son Robert, starting his editorial career at the age of five, for help with various operations and problems.

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Preparation of typescript copies of this manuscript was assisted by a Research Grant from Indiana University, and Yale University has provided funds and facilities for preparation of the figure drawings.

Acknowledgments

ACKNOWLEDGMENT of permission to include notes originally published in journals is made on page viii. Permission to quote from the following works has also been granted. The abbreviated title used in the text and notes is given first in each case.

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Introduction

Plato's "Mathematical" Passages

In this study, the theory has been advanced that the "mathematical" passages in Plato which have seemed nonsense or riddles to previous students in fact describe diagrams which Plato had designed, and were intended to accompany and clarify his text. The result has been, if my evaluation is correct, the recovery of new primary source material for the study of Plato.

Recovered diagrams meant to illustrate the *Timaeus* and *Republic* are evidently of importance and interest to anyone who has occasion to read or teach Plato. One of the facts that have made me think this study correct has been that these figures actually do clarify the text, even for the beginning student.

Consequently, the presentation has been made intelligible to a reader with no special command of Platonic Greek, of ancient mathematics, or of Plato scholarship. The text is in English, or accompanied by English translation, throughout; collateral information is supplied where needed; and so far as possible the discussions have been compressed, so that one gets to the figures at a reasonable pace. Nevertheless, it is important to omit or evade no problem affecting the correctness of the attribution of these illustrations to Plato, and some sections, in spite of all attempted shortening, have had to include many technical details of figure reconstruction.

For two thousand years certain "mathematical" passages have puzzled Plato's readers. Humanistic scholars have generally chosen to ignore them in favor of passages of more literary interest. Neo-Platonic enthusiasts have tended to center attention on them, with the apparent expectation that some mysterious, occult key to Plato's philosophy and to the nature of reality had been enciphered there. Modern treatments tend to fall into one or the other of these traditions. On the one hand, the humanistic scholar who reads his Plato as belles-lettres uses the term "literary effect" to bypass explanation of obscure mathematical passages. In literature, it may be allowable to introduce meaningless passages to create an effect of difficulty; in philosophy, it is not. Using this explanatory device, we should have to decide that Immanuel Kant devoted whole volumes, with a skill rare in literary history, to the creation of intrinsically nonsensical prose which admirably achieves the "literary effect" of obscurity. A corollary of this approach is that such passages in Plato need not be taken seriously, and might be deleted without any effect on the rest of the work. At the opposite extreme, adherents of the Neo-Platonic canons of interpretation invest these passages with tremendous significance; they believe them to contain esoteric keys to hidden doctrines which the author did not want to express more clearly. Sometimes they expect that such passages will be capable of interpretation for all subjects and at all levels of generality simultaneously; sometimes they assume that there is a hidden, specifically empirical reference to a single phenomenon or number, which an "interpretation" should discover; and sometimes, that there is in these passages a fixed symbolism with a key, in which various quantitative elements represent concepts, and images thus symbolize encoded propositions.

These two traditions illustrate perfectly Plato's remark, in *Epistle VII,* about the various ways in which a man's written works will be misread. Some readers, he says, not understanding the writing, will feel contempt for it and ignore it; others will expect too much from it, and study it in the belief that mastery of written formulae will make them experts in philosophy. The humanistic tendency to delete or bypass mathematical passages, on the ground that they are nonfunctional or not serious, is a misreading of the former sort, whereas the Neo-Platonic conviction that there is some sort of key to the universe hidden in this symbolism is a clear case of the latter kind of misreading.

It is easy for the modern reader to forget that Plato was heir to a pedagogical tradition in which the use of mathematical illustration seems to have been a standard device for clarifying discussion. It is equally easy to forget that behind this tradition lay the new discovery of mathematics as a promising instrument of science, and that successes with it in music and astronomy must have led one to expect equally effective applications to politics and medicine. We may even more easily forget that the simple technological notion of including illustrations with a book—e.g., drawing a diagram on a manuscript beside the text it was to illustrate—had not occurred to anyone when Plato wrote.

But if we forget these things, whole stretches of Platonic text in which (it will be suggested in this study) the constructions of diagrams to aid the reader are explained, become unintelligible, and what were meant as pedagogical aids become special textual problems, serving as obstacles instead.

Contemporary references, later tradition, mathematical vocabulary, and collateral material from the history of science all indicate something about the notion of "mathematics" that the Pythagoreans held. What they indicate appears quite clearly to be that mathematics was not sharply separated from other branches of natural science. The statements made about the relations of things and numbers by the Pythagoreans make sense only on the assumption (which their vocabulary confirms) that the language of this mathematics was intended to be metaphorical. By this I mean that the genera and differentiae into and by which numbers were classified were not regarded as peculiar to numbers, but were classifications derived from and shared by nonmathematical subject matters; thus, when a surface was called a "skin," or a line not amenable to measurement was called "irrational," that naming reflected a conviction that there is some common class including both skins and geometric surfaces, both moral stubbornness and incommensurability. Insofar as members of the same class have common properties, theorems about surfaces are also theorems about skins, and theorems about incommensurables are also theorems about moral behavior. Conversely, problems involving skins or morality may have analogues in pure mathematics, and it is these analogues that are crucial in the use of mathematics as a technique of research and of pedagogy.

If we can find numbers falling in the same classes as the entities we want to study, the relations of these entities can be illustrated and investigated by observing the correlated relations of the analogous classes of numbers. The interpretation of such illustrations involves (1) recognition that the properties involved are common to a broader class than quantity, in which class this given quantitative illustration is included, and (2) from this recognition, transference (by metaphor) of the insight gained from the quantitative case to the analogous nonmathematical cases.

Our modern view of mathematics has completely dissociated "number" from such attributes as sex or justice, and assimilated mathematics to the purely formal study of logic, rather than to ethics, physics, or aesthetics. This dissociation of mathematics began with Aristotle's separation of his "theoretic sciences," and was strengthened by later reaction against the Neo-Pythagoreans. The effect of this location of "mathematics" in our contemporary canon of sciences is to make the contemporary historian and reader find surprisingly little material in Platonic and pre-Platonic texts which he can recognize as distinctively "mathematical." Much of the work which the early Greek philosophers and scientists thought of as mathematics is not "mathematics" in its twentieth-century form at all. If, however, the contemporary reader becomes sensitive to the relation of this early mathematical work to metaphor (and our contemporary terms, such as "square" numbers, "roots," etc., still bear clear traces of such metaphorical thinking at their time of origin), he will find it possible to approach these earlier discussions of quantity and mathematics much more sympathetically and to interpret them more accurately. Naturally such an orientation would place great emphasis in teaching practice on the diagram and schematism, in social as well as natural science.

In addition to the practice of Plato in this matter, there is clear evidence of the existence and pervasive effect of such a pedagogical tradition in the writing of Aristotle. Although he is consistently critical of the tendency, which he detects in Platonists and Pythagoreans, to assimilate philosophy and mathematics, Aristotle's own lecture notes are interspersed throughout with mathematical illustration. Pedagogical diagrams involving mathematization are referred to throughout his discussions of logic and ethics and throughout whole sections of the physics and metaphysics, and on occasion also play a major role in the elucidation of biology. This seems to indicate that such diagram technique was accepted as a matter of course even by the arch-opponent of the philosophical orientation from which the technique originated.

Types of Mathematical Metaphor

What has been called "mathematical metaphor" may evidently vary in respect to the branch of mathematics from which it is drawn and, consequently, in the aspect of a subject matter to which it directs attention. For example, a proof taken from "pure" mathematics may illustrate a point of logic relevant to some discussion at hand. A set of arithmetical ("statistical" in a modern sense) descriptive details may call attention to the integration, or lack of it, in the institutions of a society. Again, a geometrical figure may present spatial analogues of the key relationships of concepts presented in a lecture or discussion. A metaphor from algebra may give a generalized formulation of the structure proper to a certain class of formulae. This range of source and function prevents any examination of the purely mathematical properties of such a set of illustrations from yielding any agreement as to what they illustrate; their common function of illustrating a context may be realized through many forms of functional quantitative structure.

Orientation and Limits of the Present Study

This study might be characterized as a "dialectical" interpretation. Two features seem to justify that characterization. The first is the consistent objective of establishing interpretations which make philosophic sense in their contexts (which does not, of course, relieve them of the need to make historical, mathematical, and philological sense as well). This is basically, however, a philosophic rather than a philological or historical approach, since the "contexts" of these passages are conceived as functional parts in the exposition of a philosophy. The second feature of the study is its recognition throughout of the responsiveness of the mathematical illustrations chosen to the contextual points they are intended to elucidate. A functional mathematical image must reflect these contextual methods, principles, and distinctions adequately, even if some deviation from the techniques of "pure" mathematics is required to secure such reflection. Past interpretations seem often to have postulated the detachability of all mathematical passages from their contexts, as though they enjoyed an independent meaning, like theorems in a system of geometry; or to have postulated that the passages could be understood by setting them all indiscriminately into some single eclectic context.

Since the present approach is basically philosophic, the proper criterion of its adequacy is whether or not it does, without intruding historic or philological implausibilities, illuminate the passages discussed, considered as parts of Plato's exposition of a philosophy. If an alternative approach had been chosen, the criterion might have been historical: Will the postulated interpretation explain and reconcile the extant statements of all interpreters, if we correct those statements in the light of our knowledge of the convictions and style intruded into the original by each? This is a separate and extensive enterprise, not here undertaken; in the present study antecedent interpretations are treated schematically, by criticizing or displaying the assumptions operative in their construction, not individually, except in those cases in which they have suggested some relevant insight. If a third approach had been chosen, the passages might have been studied as specimens of (presumably meaningful) classical Greek in which the meanings were expressed by means of syntactical and etymological devices analogous to those used by other Greek authors. Such an approach would be much more direct than either of the others, if it were only possible; but in every case where it has been tried (and it has been tried often) it has been necessary to resort to maxims of higher criticism which have generally taken the form of intruded, un-Platonic philosophical assumptions.

The history of Platonic scholarship goes far toward underscoring the need for the present mode of interpretation. The many studies of Plato of the historical and philological sort have uniformly, so far as the passages discussed in this study are concerned, contributed little or even negatively to our understanding of Platonic philosophy. Any reader of the *Protagoras* must agree that Plato himself believed that the proper method of textual interpretation was not that of Prodicus but of Socrates, the philosopher.

The main object of the present inquiry is Platonic mathematics as it is revealed in mathematical *imagery*. The concept of image is used here in its strict Platonic sense, in contrast to things, names, or formulae. Passages in which mathematics appears not as *illustrating* but as *being* what is talked about, as in the *Philebus, Statesman,* and *Parmenides* (though they have influenced the introductory sections, treating general technique and kinds of imagery) have been excluded. The Introduction to my doctoral thesis (*The Rôle of Mathematics in Plato's Dialectic,* [University of Chicago, 1942], pp. 1–7) still seems to me an exact statement of the distinction intended here between names, images, and formulae, though the present study is entirely independent of the dissertation.

Because of this limitation, the reader will feel that supplementary treatments are required to deal completely with the Platonic mathematical vocabulary (the *names* used), and with mathematical *formulae,* through study of which one should recover Plato's philosophy of mathematics.

However, until some agreement about the *images* has been reached, "any man who wants to may upset any argument" put forward to explain the *formulae.* An application to the study of the *Parmenides* of the conclusions and techniques of the present investigation has already convinced me that it is worth while even for a philosopher less interested in images than in realities to have undertaken this scrutiny of those images as a preliminary inquiry.

Final Comment on Tactics

The tactics of presentation used in this study have been to present translations of the passages concerned, with translations of relevant scholia, then discussion of alternative interpretations and problems, ending so far as possible with final interpretations resolving these problems.

It would have been impossible to construct my own translations of these passages which did not read into them my own notions of interpretation; and even if it had been possible to do so, any normally cautious reader would be rightly suspicious. It is easy to conjure a rabbit from a hat if you have already hidden it there, and the conflicting past assertions of philologists as to "the only meaning the Greek can bear" certainly indicate that translations of these texts leave plenty of room for conflicting interpretations. I have therefore developed interpretations from other English translations, chosen for their neutrality. In almost every case, these were made by translators who actually had in mind other interpretations than those developed in this study.

Translations of scholia are included both to show what sort of footnotes ancient scholars annexed to the mathematical passages in their manuscripts of Plato, and to prove that no set of diagrams derived from Plato's original manuscripts was known to Hellenistic scholars. On occasion, the text and the scholiast's marginal diagram are sufficiently at cross-purposes to rule out the possibility of any direct tradition.

In various introductory sections, an attempt is made to define sharply such terms as "matrix" and "analogy," which are used in the text to describe Platonic mathematical techniques and devices. Some concepts and notation of contemporary mathematical logic have been used to give the desired generality and precision to these statements. This use of concepts from twentieth-century formal logic certainly appears anachronistic, and in the present study is not explicitly related to the imagery that it describes. For the present, it is offered simply as a useful device for talking about the properties of classes of mathematical constructions and illustrations. In a later discussion of Plato's philosophy of mathematics, however, I expect to show that the modern concepts intruded here are essentially Platonic, and do not really involve what seems to be an anachronism.

One final fact requires emphatic statement: To analyze or explain an illustration is different from appreciating it directly. An author and his intended readers have in common certain habits of imagination and notions of pedagogy, and these readers need not analyze out as separate abstract relations the various properties of examples; they have an intuitive response to these without analysis. The more a reader recognizes what this direct awareness is like, the more keenly he will feel that an abstract analysis of imagery is missing something important. Since concrete aptness dissolves on analysis into many abstract relations of relevance, no one of which has the concrete vividness that we expect a good illustration to show, a commentator who uses such analytic technique is often accused of reading a great deal more into the text than its author intended, or of devising analyses with a neat or "slick" finish which most illustrations lack. Of course abstractions are neater than concrete cases; and of course no author sets about creating illustrative examples by separate consideration of a tremendous number of abstract connections. But the intuitive appreciation of the materials analyzed here presupposes an ancient Greek intuition to which we no longer have the key; so the choice is between this analytic approach to interpretation and no interpretation worth mentioning.

Part One

MATHEMATICAL IMAGES RELATIVELY INDEPENDENT OF THEIR DIALECTICAL CONTEXTS

CHAPTER I

Examples from Pure Mathematics of Methods and Class-Relations

Introductory Comment: Some Simple Illustrations

One of the most frequent and typical intrusions of mathematical imagery into the dialogues is as illustration of class-relations, particularly in connection with problems of method or definition. The *Euthyphro,*[1] *Meno,*[2] *Phaedo,*[3] and *Theaetetus*[4] all contain such illustrations. Another use is the citing of mathematics as an illustration of a specific art with its own definite subject matter; in the *Charmides,*[5] *Statesman,*[6] *Euthydemus,*[7] *Gorgias,*[8] and *Protagoras*[9] there are such illustrations. The only problems of interpretation which illustrations of this kind present are peripheral: the terminology, intended construction, or assumed differentiation of the branches of mathematics are sometimes not clear. In such cases, the function of the interpreter is to revive the sharpness and cogency of the intended illustration in its original setting; but since the function of these examples is, in context, perfectly clear, their elucidation will not throw much light on Plato's philosophy, and therefore very little is gained from more intensive interpretations.

The case is far otherwise with images and diagrams intended to illustrate or summarize crucial relations of terms in the contextual dialectic, or the contextual method. Often in these cases Plato's use of mathematical imagery to clarify discussions, where we no longer feel the relevance of such imagery to do so, poses a series of complex problems; and since the reconstruction of these diagrams provides a statement of how their context is to be interpreted, we should spare no effort to reconstruct them correctly.

The illustrations dealt with in this first chapter are all of the first sort; therefore, apart from supplying relevant diagrams and noting certain controversies over mathematical details of interpretation, the treatments of these passages have been kept brief. Even in this section, however, each of the images selected has some relevance to the study of Plato. The sacrifice of accuracy to dramatic consistency in the *Euthyphro* example, the geometry of class-inclusion in the *Meno,* the construction of the *Theaetetus,* all make some such contribution.

In following the suggested method of beginning with simple images which present no problem requiring extensive interpretation, and in postulating that the later, more complex images are similar in function to these (a postulate which can be tested by seeing how satisfactory the interpretations are that it suggests), we may profitably begin with the mention of a group of "trivial" illustrations; cases where neither subtle analysis of the context nor any knowledge of mathematics beyond its rudiments is required for interpretation. In *Republic* 428, for example, Socrates says that if we are looking for four things, and have found three, the fourth must be the one that is left. This is a simple way of illustrating the contextual point that in the state, if courage, wisdom, and temperance have been defined, the cardinal virtue that remains will be justice. In *Republic* 458, Glaucon contrasts "geometrical" and "erotic" necessity, the one proceeding from reason, the other from the passions. In *Republic* 337, Socrates asks Thrasymachus how a man could define 12 if he were forbidden to name any of its factors in his definition. (Thrasymachus has just challenged Socrates to define justice, without saying that it is good, expedient, beneficial, or "any such nonsense.") In the *Theaetetus,*[10] as a case of error, the example of the man who believes that $7 + 5 = 11$ is considered. In the *Symposium,*[11] Aristophanes represents Zeus as calculating that if he divides each man in half he will get twice as many sacrifices. In *Euthyphro* 7, Socrates points out that metric techniques settle disagreements over number, weight, and measure. In the *Meno,*[12] the difference between a class and its members is illustrated by reference to the difference between defining "figure" and enumerating specific figures.

These six passages are offered as instances of uncomplicated mathematical allusion, functional in illustrating problems of methods in a quantitative subject matter where their pattern will show most clearly. There is nothing about them that is obscure or mysterious, or nonfunctional in context. Since their pedagogical intention, contextual relevance, and mathematical content are very clear, these examples have been chosen because they show most typically characteristics which, as this study is intended to establish, all Platonic mathematical images possess. It must be admitted, however, that an attempt to extend this list much further—for example, by checking citations of ἀριθμός or πλῆθος in a Platonic lexicon—would quickly arouse controversy. The present passages are atypical in the degree of their apparent independence of relatively remote context, which gives them perhaps less interest and artistic merit than, say, the opening of the *Timaeus* with a simple enumeration of the persons present for the conversation,[13] or the discussion of odd and even number in the *Phaedo;* [14] but this very fact of apparent relative isolation gives them particular pedagogical suitability as starting-points for a program of analysis designed to lead from simpler to more complex mathematical imagery.

I. Definition of Odd and Even Number

Euthyphro 12 *

SOC.: Then if piety is a part of justice, I suppose that we should enquire what part? If you had pursued the enquiry in the previous cases; for instance, if you had asked me what is an even number, and what part of number the even is, I should have had no difficulty in replying, a number which [is *isoskeles,* not *skalene* †]. Do you not agree?

* Trans. Jowett, *Dialogues,* I, 394.

† The transliteration in brackets is substituted for Jowett's rendering ("represents a figure having two equal sides") because the substitution seems to give the translation greater neutrality as to interpretation.

Then we are wrong in saying that where there is fear, there is reverence; and we should say, where there is reverence there is also fear. But there is not always reverence where there is fear; for fear is a more extended notion, and reverence is a part of fear, just as the odd is a part of number, and number is a more extended notion than the odd. I suppose that you follow me now?

Scholia *

(*Euthyphro* 12D)

The scalene triangle has three unequal sides. The isosceles, however, has two sides equal to one another, and a third unequal. Since an even number is divisible into two equal numbers, as for example 8 (into two 4's), but the odd into unequal, for example 5, the one is called "isosceles," and the other "scalene."

• • •

Scalene: crooked and irregular. There are three kinds of triangles —equilateral, isosceles, and scalene.

These peculiar definitions of odd and even numbers, as "scalene" and "isosceles," respectively, have occasioned some puzzled comment, because they seem either capricious or inexact. The basis of the peculiarity is dramatic, not mathematical.

The Platonic dialogue is an artistic representation of a conversation in which the illustrations offered by the speaker are appropriate to the character of his listener; when they are not, the listener says so. The demands of artistic consistency require that the speaker use examples dramatically intelligible to his audience.

If, in this connection, we consider the character of Euthyphro, we find a prototype of the "ageometrical" man, denied entrance by the inscription over the door of the Academy. He consistently substitutes piety and religious precedent for independent intellectual inquiry; he is neither interested nor skilled in matters of dialectic or mathematics. Consequently, when Socrates wants

* Greene, *Scholia Platonica,* 419, 3.

to suggest, early in their discussion, that there should be some techniques for resolving political disagreements, he cites measuring and weighing as examples of demonstrative technique.[15] The practical empirical aspect of mathematics, represented by everyday use of ruler and balance, is the only sort of "mathematical" reference which Euthyphro finds familiar. Consequently, when odd and even numbers are cited as examples of definable classes, we should expect any definition that he can understand to have the same reference to everyday experience.

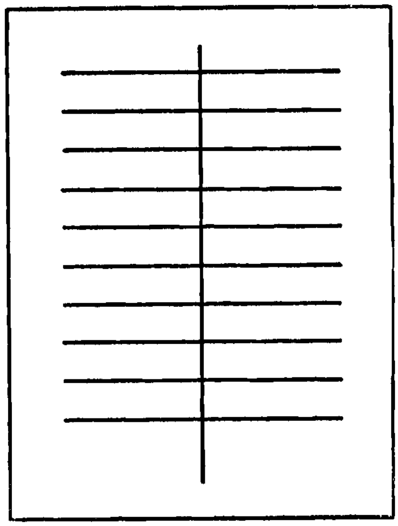
The obvious analogue to the ruler and scale is the abacus or counting-board.[16] A set of pebbles on the board can be placed in pairs on either side of the central line; the two lines of pebbles are either equal, or one is longer. In the former case, the pebbles outline an "equal-sided" rectangular figure; in the latter case, an irregular quadrilateral ("scalene").

This peculiar mode of abacus-centered operational definition does not reflect Plato's own theory of number; it is simply Socrates' concession to the mental backwardness of his companion; the images of familiar abacus pebble-configurations seem the only counterparts Socrates can find to more accurate definitions which would be artistically inadmissible, because they would be beyond Euthyphro's dramatically established powers of mathematical and abstract imagination. (See figures 1 and 2.)

II. Proof of the Recollection Theory of Knowledge

The following discussion with the slave boy in the *Meno* is one of the clearest cases of a passage giving step-by-step directions to the reader for drawing the diagram referred to. The directions are so clear-cut that from the time the passage was written there seems to have been no controversy over its interpretation. Some unusually geometrically-minded scholiast has carefully constructed all the figures referred to, representing each by a separate figure, rather than as subdivisions of a single, more complex figure, as do later diagrams illustrating the passage.[17] This use of separate figures makes even clearer the exact

Figure 1



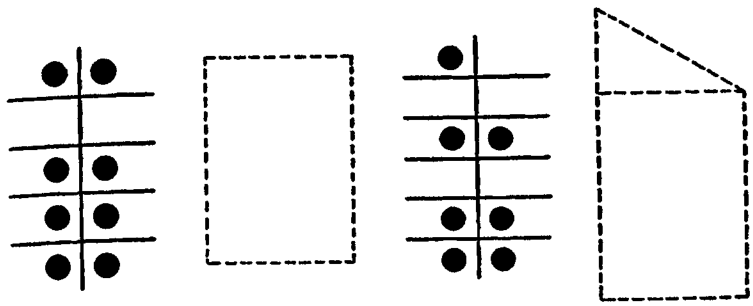
GREEK ABACUS

(Schematic Representation)

Figure 1 is a schematization of the only Greek abacus which has been discovered (the "Salaminian table"). The table is discussed by J. Gow, *Short History of Greek Mathematics* (Cambridge University Press, 1884), pp. 33–36. Although the reported lettering on the table indicates that at least one of its uses was for monetary reckoning,[18] it has also been suggested that the table was used as a gameboard, and its design partly dictated by this function. In any case, the existence of such computers' aids in Plato's time is certain; and Plato's coupling of references to the game of πεττεία with computation (in the Myth of Theuth and Thamus in the *Phaedrus*[19]) is one which is immediately suggested by the similarity of manipulating counters on an abacus and pieces on a checkerboard. The important point for the present study is that an abacus is a device taking advantage of spatial orientation to differentiate kinds of relation, a general technique discussed in Chapter III, Introduction, in connection with the concept of "verbal matrices."

Figure 2

ABACUS PEBBLE-CONFIGURATIONS



"ISOSCELES" SET "SCALENE" SET

The analogy of sets to figures was basic in Pythagorean mathematics and is still retained in the terms "square" and "cube" numbers in our own mathematical vocabulary. The use of a dot notation, representing each number as a spatially arranged set of units, facilitated the formulation of such analogies, which gave the Pythagoreans a kind of reversed analytic geometry in that they could represent continuous figures by analogous discrete sets. See John Burnet, *Early Greek Philosophy,* 4th edition (London, 1930), pp. 99–112, 276–309; Aristotle, *Metaphysics* 1092b 8 (Eurytos). See also note 38, Chapter IV, following.

aspect of the diagram intended. These scholia figures are reproduced on pages 27–29. They have been cross-referenced in the text,[20] so that each of Socrates' statements is referred to the figure showing what part of the construction he is pointing to.

Meno 82 ff.*

MENO: . . . If you can prove to me that what you say [that knowledge is recollection] is true, I wish that you would.

SOC.: It will be no easy matter, but I will try to please you to the utmost of my power. Suppose that you call one of your numerous attendants, that I may demonstrate on him.

MENO: Certainly. Come hither, boy.

* Trans. Jowett, *Dialogues,* I, 361–65. References to figures are in brackets.

SOC: He is Greek, and speaks Greek, does he not?

MENO: Yes, indeed. He was born in the house.

SOC.: Attend now to the questions which I ask him, and observe whether he learns of me or only remembers.

MENO: I will.

SOC.: Tell me, boy, do you know that a figure like this is a square? [Fig. 3]

BOY: I do.

SOC.: And do you know that a square figure has these four lines equal? [*AB, AC, CD, BD,* Fig. 3]

BOY: Certainly.

SOC.: And these lines which I have drawn through the middle of the square are also equal? [*EF, GH,* Fig. 3]

BOY: Yes.

SOC.: A square may be of any size?

BOY: Certainly.

SOC.: And if one side of the figure be of two feet, and the other side be of two feet, how much will the whole be? [*ABDC,* Fig. 3] Let me explain: if in one direction the space was of two feet, and in the other direction of one foot, the whole would be of two feet taken once? [Fig. 4]

BOY: Yes.

SOC.: But since this side is also of two feet, there are twice two feet?

BOY: There are.

SOC.: Then the square is of twice two feet?

BOY: Yes.

SOC.: And how many are twice two feet? Count and tell me.

BOY: Four, Socrates.

SOC.: And might there not be another square twice as large as this, and having like this the lines equal?

BOY: Yes.

SOC.: And of how many feet will that be?

BOY: Of eight feet.

SOC.: And now try and tell me the length of the line which forms the side of that double square: This is two feet—what will that be?

BOY: Clearly, Socrates, it will be double.

SOC.: Do you observe, Meno, that I am not teaching the boy anything, but only asking him questions, and now he fancies that he knows how long a line is necessary in order to produce a figure of eight square feet; does he not?

MENO: Yes.

SOC.: And does he really know?

MENO: Certainly not.

SOC.: He only guesses that because the square is double, the line is double.

MENO: True.

SOC.: Observe him while he recalls the steps in regular order. (To the boy) Tell me, boy, do you assert that a double space comes from a double line? Remember that I am not speaking of an oblong [such as *ABDC*, Fig. 5] but of a figure equal every way, and twice the size of this—that is to say, of eight feet; and I want to know whether you still say that a double square comes from a double line?

BOY: Yes.

SOC.: But does not this line become doubled if we add another such line here? [$AC + CG$, Fig. 6]

BOY: Certainly.

SOC.: And four such lines will make a space containing eight feet?

BOY: Yes.

SOC.: Let us describe such a figure: Would you not say that this is the figure of eight feet? [Fig. 6]

BOY: Yes.

SOC.: And are there not these four divisions in the figure, each of which is equal to the figure of four feet? [*ABDC, CDHG, BDFE, FDHI,* Fig. 6]

BOY: True.

SOC.: And is not that four times four?

BOY: Certainly.

SOC.: And four times is not double?

BOY: No, indeed.

SOC.: But how much?

BOY: Four times as much.

SOC.: Therefore the double line, boy, has given a space, not twice, but four times as much.

BOY: True.

SOC.: Four times four are sixteen—are they not?

BOY: Yes.

SOC.: What line would give you a space of eight feet, as this gives one of sixteen feet;—do you see?

BOY: Yes.

SOC.: And the space of four feet is made from this half line? [$AB = \frac{1}{2}\, AE$, Fig. 6]

BOY: Yes.

SOC.: Good; and is not a space of eight feet twice the size of this, and half the size of the other? [i.e., $2 \cdot ABDC$, $\frac{1}{2}\, AGIE$, Fig. 6]

BOY: Certainly.

SOC.: Such a space then, will be made out of a line greater than this one, and less than that one? [greater than *AB,* less than *AE,* Fig. 6]

BOY: Yes, I think so.

SOC.: Very good; I like to hear you say what you think. And now tell me, is not this a line of two feet and that of four?

BOY: Yes.

SOC.: Then the line which forms the side of eight feet ought to be more than this line of two feet, and less than the other of four feet?

BOY: It ought.

SOC.: Try and see if you can tell me how much it will be.

BOY: Three feet.

SOC.: Then if we add a half to this line of two, that will be the line of three. [$AB + BC$, Fig. 7] Here are two and there is one; and on the other side, here are two also and there is one [*AB, BC,* and *AD, DE,* Fig. 7], and that makes the figure of which you speak? [*AFGE,* Fig. 8 = *AEFC,* Fig. 7]

BOY: Yes.

SOC.: But if there are three feet this way and three feet that way, the whole space [*AEFC,* Fig. 7] will be three times three feet?

BOY: That is evident.

SOC.: And how much are three times three feet?

BOY: Nine.

SOC.: And how much is the double of four?

BOY: Eight.

SOC.: Then the figure of eight is not made out of a line of three?

BOY: No.

SOC.: But from what line? Tell me exactly, and if you would rather not reckon, try and show me the line.

BOY: Indeed, Socrates, I do not know.

SOC.: Do you see, Meno, what advances he has made in his power of recollection? He did not know at first, and he does not know now, what is the side of a figure of eight feet; but then he thought that he knew, and answered confidently as if he knew, and had no difficulty; now he has a difficulty, and neither knows nor fancies that he knows.

MENO: True.

SOC.: Is he not better off in knowing his ignorance?

MENO: I think that he is.

SOC.: If we have made him doubt, and given him the 'torpedo's shock,' have we done him any harm?

MENO: I think not.

SOC.: We have certainly, as would seem, assisted him in some degree to the discovery of the truth; and now he will wish to remedy his ignorance, but then he would have been ready to tell all the world again and again that the double space should have a double side.

MENO: True.

SOC.: But do you suppose that he would ever have enquired into or learned what he fancied that he knew, though he was really ignorant of it, until he had fallen into perplexity under the idea that he did not know, and had desired to know?

MENO: I think not, Socrates.

SOC.: Then he was the better for the torpedo's touch?

MENO: I think so.

SOC.: Mark now the farther development. I shall only ask him, and not teach him, and he shall share the enquiry with me: and do you watch and see if you find me telling or explaining anything to him, instead of eliciting his opinion. Tell me, boy, is not this a square of four feet which I have drawn? [*ABED,* Fig. 9]

BOY: Yes.

SOC.: And now I add another square equal to the former one? [*BEFC,* Fig. 9]

BOY: Yes.

SOC.: And a third, which is equal to either of them? [*DEHG,* Fig. 9]

BOY: Yes.

SOC.: Suppose that we fill up the vacant corner? [*EFIH*, Fig. 9]

BOY: Very good.

SOC.: Here, then, there are four equal spaces?

BOY: Yes.

SOC.: And how many times larger is this space than this other?

BOY: Four times.

SOC.: But it ought to have been twice only, as you will remember.

BOY: True.

SOC.: And does not this line, reaching from corner to corner, bisect each of these spaces? [*DB, BF, FH, HD*, Fig. 9]

BOY: Yes.

SOC.: And are there not here four equal lines which contain this space? [*DBFH*, Fig. 9]

BOY: There are.

SOC.: Look and see how much this space is.

BOY: I do not understand.

SOC.: Has not each interior line cut off half of the four spaces?

BOY: Yes.

SOC.: And how many spaces [i.e., half-squares] are there in this section? [I.e., in *DBFH*, Fig. 9]

BOY: Four.

SOC.: And how many in this? [*ABDC*, Fig. 10 = *ABED*, Fig. 9]

BOY: Two.

SOC.: And four is how many times two?

BOY: Twice.

SOC.: And this space [*DBFH*, Fig. 9] is of how many feet?

BOY: Of eight feet.

SOC.: And from what line do you get this figure?

BOY: From this.

SOC.: That is from the line which extends from corner to corner of the figure of four feet? [*DB*, Fig. 9]

BOY: Yes.

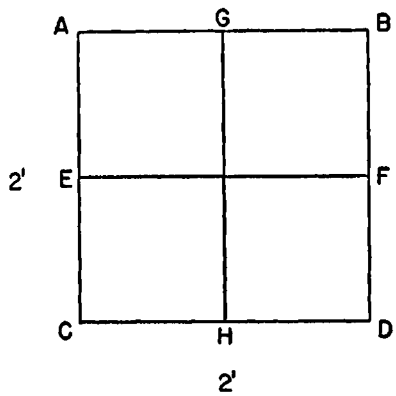
SOC.: And that is the line which the learned call the diagonal. And if this is the proper name, then you, Meno's slave, are prepared to affirm that the double space is the square of the diagonal? [An alternative illustration of this is given in Fig. 11.]

BOY: Certainly, Socrates.

The "Meno" Discussion: Figures from the Scholia

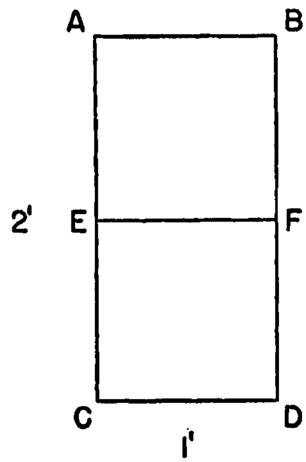
(Figures 3 through 11 from Greene, *Scholia Platonica*, pp. 171–73. Figure 8 and Roman lettering are my additions. Nonfunctional numbers 2′, 6′ are omitted in Figure 5 and nonfunctional 1′ marks in Figure 6.)

Figure 3



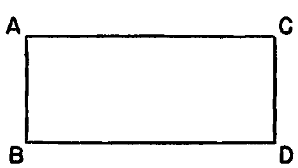
INITIAL FOUR-FOOT SQUARE
Area = 2′ x 2′ = 4 sq. ft.

Figure 4



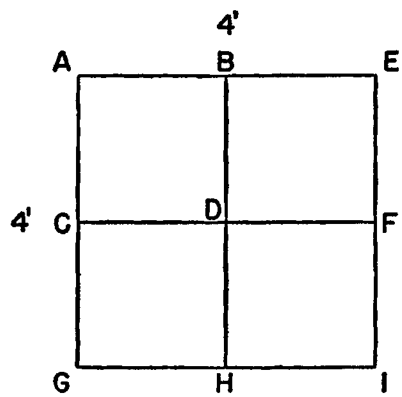
CALCULATING SQUARE AREA
Area = 2′ x 1′ = 2 sq. ft.

Figure 5



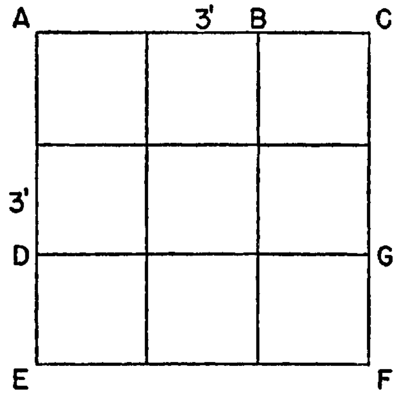
SHAPE OF AN "OBLONG"

Figure 6



FIRST CONSTRUCTION: SQUARE WITH AREA OF SIXTEEN SQUARE FEET
Whole = 16 sq. ft.

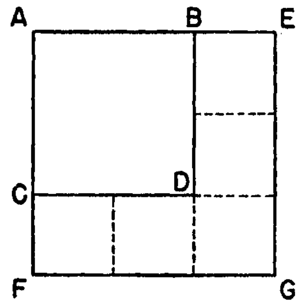
Figure 7



SECOND CONSTRUCTION: SQUARE WITH AREA OF NINE SQUARE FEET

Area = 9 sq. ft.

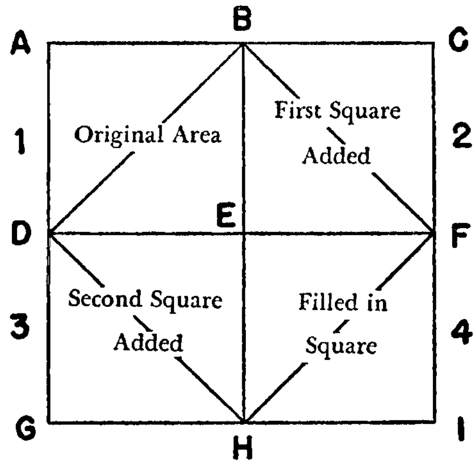
Figure 8



DEMONSTRATION OF AREA OF NINE-FOOT SQUARE

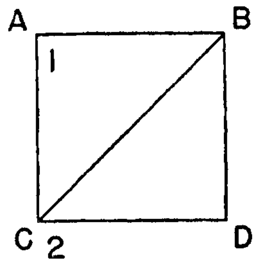
The area of the 3 x 3 square is computed by summing the unit squares. *ABCD* has already been shown to equal four of these, and the added gnomon, *CBDEFG,* contains five more.

Figure 9



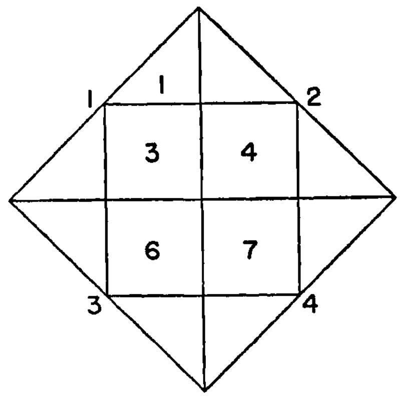
FINAL CONSTRUCTION: FOUR FOUR-FOOT SQUARES JOINED, WITH DIAGONALS DRAWN

Figure 10



DEMONSTRATION THAT DIAGONAL DIVIDES UNIT SQUARE INTO TWO HALF-UNIT TRIANGLES

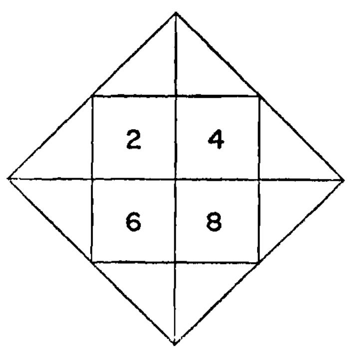
Figure 11



FINAL SCHOLION FIGURE

The numbers as they stand do not make sense. Figure 12, following, represents a suggested emendation of this figure which makes it interpretable as an alternative intuitive demonstration of the relative area of squares on the diagonal and the side.

Figure 12

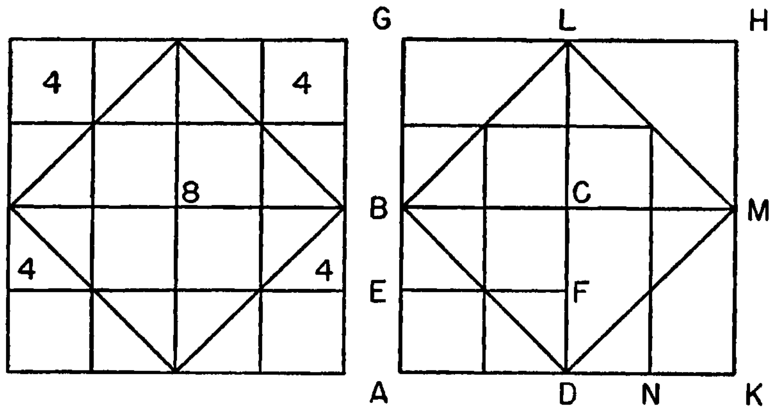


PROPOSED EMENDATION OF FINAL SCHOLION FIGURE

The final diagram represents an original excursion of the scholiast's own; he is advancing a figure for an alternative proof of the general theorem that "the square on the diagonal is double the square on the side." [21] At some point, this final figure has been corrupted in transcription, so that its numbering no longer reproduces this original intention. The intended proof proceeds, as does Plato's, by computing areas through summation of equal triangles; eight unit triangles are combined to form a square, about which, by adjoining eight more unit triangles, a second square with sides equal to the diagonal of the first is circumscribed. A second figure is subjoined, showing that each triangle equals $\frac{1}{2}$ the unit of square area, if its side $= 1$ (Fig. 10, preceding). The area of the inscribed square is indicated by numbers summing up the number of the half-unit triangles through its four quadrants: These numbers should therefore be 2–4–6–8, rather than the 3–4–6–7 which are in the manuscripts.[22] The areas of the two squares are thus $\frac{1}{2} \times 8 = 4$, and $\frac{1}{2} \times 16 = 8$, as in the preceding figure, which corresponds to Plato's example. The diagram does not correspond to any figure in Euclid, but resembles figures associated with a suggested early method of proof of Euclid I. 47. It would be of some possible interest to inquire further where the scholiast found this particular diagram.

Figure 13

LATER REPRESENTATIONS OF THE *MENO* FIGURE



JOWETT'S FIGURE
(*Dialogues*, I, 362)

HEATH'S FIGURE
(*History*, I, 298)

In Jowett's figure, stages are not labelled; Heath's line *AN* is the side of the three-foot square which represents the slave boy's second attempted solution. Either figure gives a clear picture of the diagram as it develops during the discussion, by stages which the scholia represent separately.

The purpose of this demonstration in context is to convince Meno that knowledge is recollection. Meno's association with Gorgias has lead him to accept a Sophistic notion of teaching and learning which identifies knowledge with literal recall of past experiences and precepts. A. E. Taylor has shown that the Sophistic notion of virtue as a τέχνη would imply that it could be learned and taught in this dogmatic way. It is this conviction that "teaching" is dogmatic statement that causes Meno to agree that Socrates did not "teach" anything to the boy, though later educators not in the Sophistic tradition have repeatedly cited this passage as an example of teaching procedure. Socrates, convinced that learning is a matter of insight, confronts Meno with a proof that if, as Meno maintains, all knowledge is past experience recollected, he must postulate both a very peculiar "experience" and a very extended "past." The argument need not be taken as proof of a literal belief on Plato's part, but makes equal sense read as his development of the implications of the Sophistic position along a line which shows that even that position requires a theory of ideas if all cases of knowledge are to be explained. (The converse form of the same point is developed in the *Theaetetus,* where no identification of knowledge with temporal psychological process can be found that will apply to mathematical error and insight.)

For this demonstration, Socrates needs an example that will be something the slave boy clearly has never learned, but it must also be capable of being presented in a form intuitively evident enough for Meno and the slave to understand.

To avoid any possibility that the construction is one which the boy may have learned, Socrates chooses an example from the field of "higher mathematics" to convince Meno. The properties of the incommensurable were one of the fields of advanced mathematical research in Plato's time. The theorem chosen has the added advantage that when the correct construction is completed, its truth is so evident from the diagram, that even a novice like Meno can see the demonstrated relationship. An example from quantum mechanics would be analogous in difficulty in a contemporary dialogue, as one from calculus would have been in Leibniz' time. This theorem, however, can be proved intuitively by a construction which is clear without any special mathematical training. One cannot infer that since the problem Socrates chooses is "advanced," he assumes Meno to have some technical mathematical education. This fact is highly relevant to the mathematical passage in the *Meno* discussed in the following section, since many of the interpretations proposed seem to require that Meno be a competent mathematician.

III. The Geometer's Method of Hypothesis

(*Meno* 89 *)

". . . as regards a given area, whether it is possible for this area to be inscribed in the form of a triangle in a given circle. The answer might be "I do not yet know whether this area is such as can be so inscribed, but I think I can suggest a hypothesis which will be useful for the purpose; I mean the following. If the given area is such as, when one has applied it (as a rectangle) † to the given straight line in the circle (the diameter), it is deficient by a figure (rectangle) similar to the very figure which is applied, then one alternative seems to me to result, while again another results if it is impossible for what I said to be done to it. Accordingly, by using a hypothesis, I am ready to tell you what results with regard to the inscribing of the figure in the circle, namely, whether the problem is possible or impossible."

* Trans. Heath, *History*, I, 299.

† If the reader will accept Heath's (undoubtedly correct) identification of "the line in the circle" as the "diameter," and tentatively disregard the parenthetical references to rectangles, which represent Heath's scholion, derived from the later technical sense of the terms "applying" and "falling short," the translation has suitable neutral accuracy. The interpretative slant represented by the references to "rectangles" is discussed in the comment following Figure 15.

Scholia *

MENO 86E
[Socrates is speaking of] the inscription of the triangle in the circle.
MENO 87A
concerning the inscription of the triangle in the circle.

At present, apparently from the conviction that Plato was expert in pure mathematics and therefore must have introduced many allusions to specific mathematical problems in his writings, scholars seem to be tending to make the objects of these allusions more and more complex and mathematically advanced. Although this shows a commendably high opinion of Plato, it is also a little reminiscent of the Pythagoreans' attributing all new discoveries to Pythagoras himself.

Three main interpretations of this passage have been advanced and defended as though they were mutually contradictory.

Benecke and Gow [23] suggest that the problem is illustrated with the same two-foot square that was used in the earlier discussion with the slave boy; in which case the inscription is possible if the radius of the given circle equals the side of the given area, represented as a square. They urge that a more technical construction would be out of place, since it would involve an inconsistency in the character of Meno, who is not a mathematician, and to whom the problem should therefore be presented in intuitively evident form. One could well cite Socrates' tempering of the rigor of his definition of odd and even number to the capacities of Euthyphro as an analogous concession to a nonmathematical companion. The geometer's condition then is: "If, when you erect on the diameter a square equal to the area, an identical square can be constructed on the remaining segment, the inscription is possible."

Heath and others prefer a more general theorem as the intended illustration; if this method is illustrated by a contemporary geometric problem, they feel it should be less trivial than

* Greene, *Scholia Platonica,* pp. 480, 173.

the theorem involved in Benecke's interpretation.[24] In fact, the problem of inscription in Plato's time could have been presented in just this way, as possible under certain determinate conditions, impossible under others. The inscription is possible if the area can be applied to the diameter as a rectangle in such a way that its corner point lies on the circumference of the circle; and, by the converse of Euclid iii. 35,[25] this will happen if the rectangle on the remaining segment of the diameter is similar to the one applied. This gives the theorem more generality and more mathematical interest than Benecke's interpretation, but at the expense of (*a*) presupposing more background in geometry than Meno can consistently have and (*b*) disconnecting this figure from the ones Socrates has already drawn, whereas his references to "this triangle," "this area," etc., seem clearer if we allow some such connections.[26]

Demme and, by implication, Taylor[27] believe that the identity of the theorem is irrelevant; the needs of the dialogue are met if this statement is "the sort of thing that a geometer might say." While there is no doubt that this is essentially true, it seems likely that some specific figure was intended to illustrate the abstract statement of method; and, in a context in which Socrates has been illustrating his points by constructing figures, it is reasonable to expect that he is constructing or referring to a figure of some sort here also.

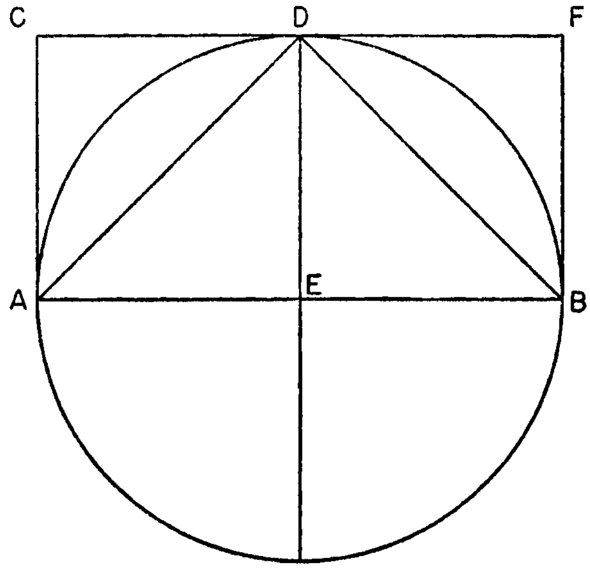
In the first place, one may certainly allow the correctness of the third set of interpretations insofar as the mathematical detail of the illustration falls outside of its dialectical relevance (except for a point discussed in Appendix B), in a way typical of such Platonic illustrations of method. It is less plausible to assume that there was no intended technical mathematical reference at all.

The figure of Benecke's choice represents a figure for the simplest case of the more general theorem. In Euclid iii. 35, the case of two intersecting lines in a circle which coincide with perpendicular diameters is treated separately as the initial case in the proof of the theorem.[28] There is no reason why Socrates, if he had the more general problem in mind, should not have presented to Meno the most intuitively evident special case from a complete demonstration which Meno could not have understood. This suggestion combines the merits of Benecke's insistence on consistency of character and economy of imagery, and the suggestion of the second group of interpreters, that some more general theorem is required to make the example mathematically interesting or meaningful. This case is thus analogous to the *Euthyphro* definition of odd and even number by associated intuitive configurations; only so much of a technically valid mathematical illustration is introduced by Socrates as his audience can consistently be shown to grasp.

In summary it seems that at least three conditions are met by this illustration, of which each interpreter has emphasized one. The passage must be "the sort of thing a geometer would say," i.e., a statement illustrating a general method, quite apart from the exact figure intended. It must, or at least should, be something that would hold some intrinsic mathematical interest, deriving from the relation of the method to the illustration chosen (an interest that certainly would attach to the inscription problem, where the conditions but not the solution were known). It must also be dramatically appropriate, in not exceeding the posited ability of the audience to whom it is addressed and, if possible, in developing from antecedent figures used for methodological illustration in preceding context. If we take Benecke's figure, treated as a special case of the general theorem of inscribability, the mathematical interest of the general problem is still suggested by this special case, though considered as a theorem in its own right, apart from that context, it is not very significant or mathematically interesting. The general theorem, to gain its relevance to the contextual discussion, must itself be taken as a special case of a method used by the geometer at work, and one may be able to appreciate that technique without laboring over the detail of a postulated, specific example.

Still, there are many diagrams which would satisfy all three of these conditions. In Appendix B, following, a further consideration of the intuitive connection of pure mathematical examples to the conceptual problems in their contexts will be presented, with an analysis to show why, out of the possible set of instances possessing interest, appropriateness, and methodological relevance, the inscription theorem was the specific example here chosen.

Figure 14

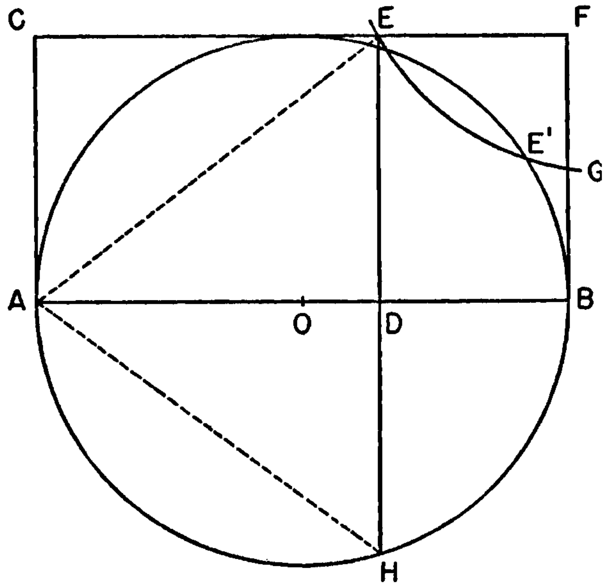


THE "MENO" HYPOTHESIS: BENECKE'S INTERPRETATION
(A. Benecke, *Ueber die geometrische Hypothesis in Platons "Menon"* [Elbing, 1867])

This interpretation identifies the area to be inscribed with the square used in the previous demonstration (so that *ACDE* in the present figure is identical with *ABED* in Figure 9). As Socrates refers to inscribing "this area," this interpretation visualizes him as turning again to the other diagram and pointing to *ABED.* If he drew a circle at *Meno* 74E, while discussing "roundness," he might also have pointed to the circle in stating the present problem.

In this figure, if *AE* is less than the diameter of the circle, *AB,* by an amount equal to itself, then a square *BEDF* equal to the square *ACDE* can be constructed, and the area contained in the triangle *ADB,* formed by drawing the diagonals *AD, DB,* equals *ACDE,* inscribing the area as a triangle. The inference which shows *ADB* equal to *ACED* is the same summing of half-unit triangles used in the earlier demonstration with the slave boy.

Figure 15

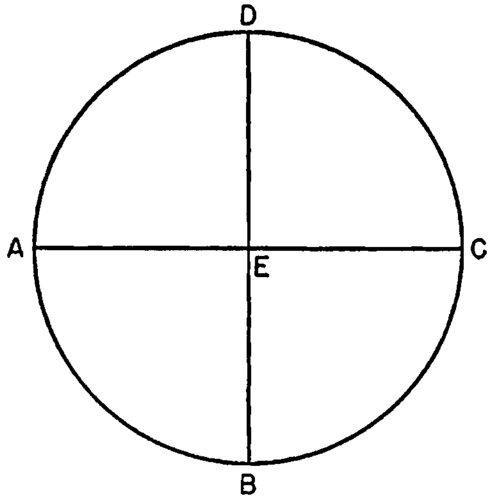


THE "MENO" HYPOTHESIS
(Heath, *History,* I, p. 300)

"In order, therefore, to inscribe in the circle an isosceles triangle equal to the given area (*X*), we have to find a point *E* on the circle such that, if *ED* be drawn perpendicular to *AB,* the rectangle *AD·DE* is equal to the given area (*X*) ('applying' to *AB* a rectangle equal to *X* and falling short by a figure similar to the 'applied' figure is only another way of expressing it). . . . This is an equation of the fourth degree which can be solved by means of conics, but not by means of the straight line and circle."

If *ACED* is similar to *DEFB, DE* is a mean proportional between *AD* and *DB,* and *ACED* can be inscribed as the triangle *AEH.* Since *AB* is the diameter, *DE* equals *DH,* and $DE \cdot DH = DE^2 = AD \cdot DB$. Conversely, if this condition is satisfied, *E* represents a point on the circle of which *AB* is the diameter.

Figure 16



FIRST CASE OF EUCLID III. 35

The theorem is that if two straight lines intersect within a circle, the area of the rectangle bounded by the segments of one is equal to that of the rectangle bounded by the segments of the other. The converse of this would establish that if, in the figure, *ED* is a mean proportional between *AE* and *EC* and if *ED* is perpendicular to *AC*, points *A,C,* and *D* will lie on the same circle, regardless of whether *E* lies at the center of the circle.

The first case: If both lines pass through the center of the circle, the proof is evident; for, all four radii being equal, the rectangles bounded by them are equal as well.

Compare with Benecke's diagram, Figure 14, preceding.

IV. Definition of Roots and Surds

Theaetetus 147 *

THEAET.: Yes, Socrates, there is no difficulty as you put the question. You mean, if I am not mistaken, something like what occurred to me and to my friend here, your namesake Socrates, in a recent discussion.

* Trans. Jowett, *Dialogues,* IV, 199–200.

SOC.: What was that, Theaetetus?

THEAET.: Theodorus was writing out for us something about roots, such as the roots of three or five, showing that they are incommensurable by the unit; he selected other examples up to seventeen—there he stopped. Now as there are innumberable roots, the notion occurred to us of attempting to include them all under one name or class.

SOC.: And did you find such a class?

THEAET.: I think that we did; but I should like to have your opinion.

SOC.: Let me hear.

THEAET.: We divided all numbers into two classes: those which are made up of equal factors multiplying into one another, which we compared to square figures and called square or equilateral numbers;—that was one class.

SOC.: Very good.

THEAET.: The intermediate numbers, such as three and five, and every other number which is made up of unequal factors, either of a greater multiplied by a less, or of a less multiplied by a greater, and when regarded as a figure is contained in unequal sides;—all these we compared to oblong figures, and called them oblong numbers.

SOC.: Capital; and what followed?

THEAET.: The lines, or sides, which have for their squares the equilateral plane numbers, were called by us lengths or magnitudes; and the lines which are the roots of (or whose squares are equal to) the oblong numbers, were called powers or roots; the reason of this latter name being, that they are commensurable with the former (i.e., with the so-called lengths or magnitudes) not in linear measurement, but in the value of the superficial content of their squares; and the same about solids.

SOC.: Excellent, my boys; I think that you fully justify the praises of Theodorus, and that he will not be found guilty of false witness.

THEAET.: But I am unable, Socrates, to give you a similar answer about knowledge, which is what you appear to want; and therefore Theodorus is a deceiver after all.

The use of incommensurables to illustrate the proper method of definition in the *Theaetetus* is certainly intended as a memorial to what was one of Theaetetus' important mathematical achievements. The memorial, however, has been selected with a view to its appropriateness in the context of the reported conversation. The moral of the whole dialogue is one that might well be symbolized by a theorem of incommensurability; knowledge turns out, whatever unit of comparison we employ, to be incommensurable with opinion. Various items of knowledge, particularly the mathematical, cannot be identified with or explained by any process of perception or physical construction. The relation of knowledge to opinion cannot be described simply by identifying items of knowledge and opinion, but must be stated in some other way. What is needed is to find some common classification for all these items of knowledge, and to find a statement of their relation to our perceptions and habits of operation, as Theaetetus and Young Socrates have defined and stated the relations of magnitudes and roots.

This example has a mathematical reference which claimed great interest from Plato, and continues to attract the attention of historians of mathematics. Its relevance to the dialogue itself is only that of the other mathematical examples introduced as paradigms of definition in the *Meno* and *Euthyphro*.[29] A good definition must state a principle of classification, not simply enumerate various individual cases. The development of the treatment of irrationals, in which Plato and the Academy took a special interest,[30] provided an excellent geometrical example of this difference. Only so much of the detail of this history as is immediately relevant to the dialogue is included by Plato, but this much gives almost the only clue we have for filling in a series of technical developments not discussed fully.

The proof of the irrationality of the diagonal of a unit square cited by Aristotle,[31] based on the demonstration that if the side and diagonal are assumed to be commensurable, the same number must be both odd and even, is usually taken as a Pythagorean starting-point in the investigation of irrationals. With the work of Eudoxus, carried out in collaboration with Plato, a complete theory of irrationals, that contained in Euclid's *Elements,* enters the history of geometry. Somewhere in the intervening period, the investigations and proofs must have been extended and generalized. But the present passage in Plato, with its confirmation by a scholion on Euclid x.9, is the only source of insight into this intervening period of exploration and generalization.

Historians agree that the procedure of Theodorus seems to have had the peculiarity that each case needed to be demonstrated separately, and that what impressed Plato most in Theaetetus' procedure was his generalization of this episodic technique. From the theorems and methods of inquiry available to the mathematician in Theodorus' time, conjectural reconstructions of his proofs have been proposed, falling into three main groups, of which one seems to fit the evidence of Plato's account better than either of the others.[32]

(1) It has been suggested that since there was available a technique for successive approximations to the numerical values of irrationals (the use of side- and diagonal-numbers, explained by Theon, and attributed by him to pre-Platonic Pythagoreans),[33] Theodorus may have employed an extension of this method. The principal objection is that such a line of inquiry would establish only a *presumption,* not a *proof,* of the unending character of the approximation, hence would not really establish incommensurability. With the Pythagorean demonstration of the irrationality of the square root of two as a model, Theodorus would hardly have been satisfied with a proof which was so much less effective, nor could we explain Theaetetus' apparent satisfaction with the validity of the proofs of the separate cases which Plato's dialogue makes him express.

(2) The older Pythagorean proof lends itself to an extension to the cases investigated, showing that for each irrational, the assumption of its rationality (when it is represented as the ratio, m/n, of two integers) leads to the contradiction that m is and is not divisible by a given number.[34] This procedure probably did occur somewhere in the transition period from Pythagoras to Eudoxus, but its assignment to Theodorus has been challenged (and, it would seem, conclusively so) on the ground that this method would already have the generality which, according

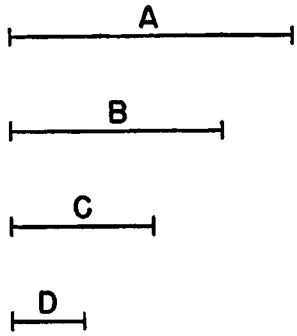
to Plato's report, is just what Theodorus' procedure lacked.

(3) Zeuthen and Heath [35] suggest a geometrical demonstration of a type perhaps discovered and certainly most simply illustrated by the attempt to find the greatest common measure of a line cut in extreme and mean ratio. When the shorter segment of such a line is laid off on the longer, the difference is itself in the same ratio to the shorter segment that the latter had to the longer. Since the two initial segments do not have an integral ratio, it follows that no common unit will be approached by the difference of the two segments, no matter how often the one is subtracted from the other. This satisfies both conditions attaching to Theodorus' proof: (*a*) that it is a valid proof of incommensurability, not merely an empirical illustration of a presumption; and (*b*) that it requires a separate geometric figure for each case, such that the same ratio can be shown (by similar figures) to repeat for every subtraction of the shorter segment (an integral length line of the figure) from the longer (representing the given constructed irrational).

The final form of Theaetetus' generalization of Theodorus' results was probably the source of Euclid's theorem x.9, which is attributed specifically to Theaetetus. This connects the ratios of the sides of similar figures with previously demonstrated arithmetical theorems about the ratios of square numbers.[36]

However, there were probably several intermediate stages in Theaetetus' work before he arrived at this final algebraic generalization, and Plato's passage may recall such an intermediate development, which supplied in a more typically geometric form the generality lacking in Theodorus' approach. Such an intermediate step may have consisted in the application of the extension of the Pythagorean proof, mentioned above as one of the conjectured reconstructions of Theodorus' method, to any figure in which some standard construction established the incommensurability of the relation of diagonal and sides. The proof would then have been developed as applying to the square root of any number, the geometric representation of which as an area would have been possible only under the given conditions of construction.

Figure 17



A PROOF ATTRIBUTED TO THEAETETUS: EUCLID X.9 (Heath, *Euclid, II,* pp. 29–31)

"The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number; and squares which do not have to one another the ratio which a square number has to a square number will not have their sides commensurable in length either.

"For let *A,B* be commensurable in length; I say that the square on *A* has to the square on *B* the ratio which a square number has to a square number.

"For, since *A* is commensurable in length with *B,* therefore *A* has to *B* the ratio which a number has to a number (X.5).

"Let it have the ratio which *C* has to *D.*

"Since then, *A*:*B*::*C*:*D,* while the ratio of the square on *A* to the square on *B* is duplicate of the ratio of *A* to *B,* for similar figures are in the duplicate ratio of their corresponding sides; (VI.20, Por.) and the ratio of the square on *C* to the square on *D* is duplicate of the ratio of *C* to *D,* for between two square numbers there is one mean proportional number, and the square number has to the square number the ratio duplicate of that which the side has to the side; (VIII.11) therefore also, as the square on *A* is to the square on *B*, so is the square on *C* to the square on *D*.

"Next, as the square on *A* is to the square on *B*, so let the square on *C* be to the square on *D*; I say that *A* is commensurable in length with *B*.

"For since, as the square on *A* is to the square on *B*, so is the square on *C* to the square on *D*, while the ratio of the square on *A* to the square on *B* is duplicate of the ratio of *A* to *B*, and the ratio of the square on *C* to the square on *D* is duplicate of the ratio of *C* to *D*, therefore also, as *A* is to *B*, so is *C* to *D*.

"Therefore *A* has to *B* the ratio which the number *C* has to the number *D*; therefore *A* is commensurable in length with *B*.

"Next, let *A* be incommensurable in length with *B*; I say that the square on *A* has not to the square on *B* the ratio which a square number has to a square number.

"For, if the square on *A* has to the square on *B* the ratio which a square number has to a square number, *A* will be commensurable with *B*;

"But it is not; therefore the square on *A* has not to the square on *B* the ratio which a square number has to a square number.

"Again, let the square on *A* not have to the square on *B* the ratio which a square number has to a square number; I say that *A* is incommensurable in length with *B*.

"For, if *A* is commensurable with *B*, the square on *A* will have to the square on *B* the ratio which a square number has to a square number.

"But it has not; therefore, *A* is not commensurable in length with *B*.

"Therefore, ...

. . . .

"A scholium to this proposition (Schol. x No. 62) says categorically that the theorem proved in it was the discovery of Theaetetus."

Part Two

MATHEMATICAL IMAGES CLOSELY DEPENDENT ON THEIR DIALECTICAL CONTEXTS

CHAPTER II

"Social Statistics": Arithmetic Detail

I. Atlantis and Its Institutions

In his description of Atlantis in the *Critias,* Plato gives the exact numbers and measures of almost every phase of its geography, public works, and political institutions. In the description of ancient Athens, in the same dialogue, there is only one numerical detail given (the total fighting strength of the state).[1] This suggests that the use of such specific figures is a device peculiarly appropriate to a description of the Atlantean state and that the specific figures which Plato invents have some characteristics intended to reflect peculiarly Atlantean principles of legislation and technology.

The institutions and customs of ancient Athens can be and are adequately specified by reference to a normative standard embodied in legislative principles; the exact measurements can be summed up by the statement that they are those which are best adapted to proper functioning. In a disordered and only loosely unified state such as Atlantis, on the other hand, institutional and technological details are not determined and coordinated by a rational unifying plan. The closest analogue to the structural statements made about ancient Athens (where the structure of the society was organized around rational legislation) in an account of Atlantis is, therefore, the separate description of the institutions and public works of which this social structure happens to be composed.

The substitution of some set of specific figures for considerations of proper function in a total plan is peculiarly appropriate to the description of the type of disunity and disorder which

Atlantis illustrates. That the specific figures given have in common an arithmetical characteristic which emphasizes Atlantean irrationality and confusion will be shown in the discussion which follows.

Poseidon seems to have been an ancestor not likely to produce philosophic and mathematically minded offspring; for, if we compare his ordering of circles of land and sea in Atlantis to the circles of the heavens described in the *Timaeus,* it becomes evident that, when this god geometrizes, he does it like a carpenter's apprentice. And the institutions preserved by the descendants of Poseidon who rule Atlantis show that, in fact, the offspring have made no improvement, philosophically or mathematically, on the insight of their ancestor. The key to the selection of all the numbers in the *Critias* is the statement that these rulers "met alternately every fifth and every sixth year, paying equal honor to the odd and to the even." [2] That this shows a total and fundamental lack of understanding of the nature of number is clear if this passage is compared with the careful distinction of kinds of sacrifices which should be made in odd and those which should be made in even numbers, in *Laws* 717A. Not only is the confusion of even and odd (which are the basic contrary principles of the most elementary mathematical science) a sign of total lack of theoretical ability, but the specific numbers cited here, which represent the even and the odd, reflect this same confusion, the one being the sum and the other the product of the first odd and the first even number.

Since in Plato's mathematical images and formulae in dialectical and cosmological contexts the basic opposition of odd and even is observed and since in contexts dealing with legislative detail ease of manipulation or religious propriety is the determining factor (and in the latter case the basic distinction is again that of odd and even), while in mythical contexts periods and distances are poetically dismissed as "myriads" (perhaps composed of lesser, proportionately related periods, which are indicated by smaller powers of ten), the absence of anything remotely resembling "alternating fives and sixes" in other Platonic contexts is causal, not accidental. The choice of "five and six alternately" by the Atlantean kings is not only an example of mathematical ignorance on so grand a scale that they cannot distinguish the natures of the odd and the even but also a sign of a lack of rational statesmanship so great that no real principle of any sort is observed in the fixing of these meetings of the rulers, the state's most important political and religious festival.

Reflecting and leading up to this final detail, where an explicit statement is given of the underlying confusion which accounts for its selection, all the other numbers and measurements cited, however casually, are (except one) either (*a*) multiples of 6 or 5 or (*b*) parts of a sum, product, or ratio which in its entirety is a multiple of 6 or 5 or (*c*) 6 or 5.

Poseidon himself begat *five* pairs of twin sons; [3] his statue depicts him driving *six* horses; [4] his engineering consists in the construction of *five* circles (three of sea and two of land) [5] about a central island with a diameter of *five* stades.[6] Further, the total widths of the circles of sea are to the total widths of those of land in the ratio of 6:5.[7] (This consideration of total widths is relevant, since precisely this type of relation gives the adumbrated geometrical structure operative in Plato's assignment of relative sizes to the rims of the whorls described in the Myth of Er.) [8]

The divinity of Poseidon's nature reveals itself only in the ease with which he performs his mechanical operations: he established the circles of sea and land "with ease, as a god might." [9] His descendants, like him in this respect, created public works of a magnitude that appeared incredible,[10] but they were also like their ancestor in their partiality for 6's and 5's. Working with a plain (which was oblong and crooked, not perfectly square or straight) 6,000,000 square stades in area,[11] they constructed an intersecting network of canals.[12] The outermost canal of this network, encircling the plain, had its breadth related to its depth in the ratio of 6:1. The total length of this ditch was 10,000 stades (since its sides were, respectively, 2,000, 3,000, 2,000, and 3,000 stades long).[13] In these dimensions and details a 6:1 ratio is used; and 6 is represented as the product of 2 and 3, 10 as the sum of 2, 3, 2, and 3. Confusion of the even and the odd in these numbers is reflected, in the first two cases, by the alternatives of multiplying and adding the first even and first odd number. In this context, therefore, the representation of 10 as a sum of 3's and 2's is not really an exception to the rule of the prominence of 6's and 5's.[14]

The military arrangements of Atlantis [15] afford a remarkable array of 6's.

As we survey these two sets of figures, a second principle of selection is also seen to be operative: the vastness of the numbers and distances involved is reflected by the prominence of myriads as units of description. The ratios which give a qualitative aspect and dialectical point to these various precise statements of distances and numbers, however, remain, no matter what the scale, 6's and 5's.

The way in which this principle is carried out in the principal religious-political festival of the state has already been shown. It is further represented, however, in the law that no king may be sentenced to death without concurrence in the sentence of at least six of the members of the council.[16] In the state religion, the six steeds in the statue of Poseidon carry on this principle; and the dimensions of Poseidon's temple are stated in such a way that, while their basic ratio is 2:1, the immediate reduction of stades to πλέθρα, which the form of statement suggests, would yield a 6:3 ratio instead.[17]

The number of Nereids (100) [18] emphasizes the continuing influence of the cult of Poseidon, the Atlantean familiarity with the sea, and perhaps also their tendency to make everything bigger than they should. These are the factors emphasized by Plato's parenthetical remark in 116E, which underscores the deviation from tradition. But it is a deviation which merely doubles a set of 5 times 10 and which, in conjunction with the contextual mention of the number of steeds, shows in the state religion the same basic confusion that is reflected in the alternation of 6's and 5's.

In summary, the apparently random numbers so liberally interspersed in Plato's account of the Atlantean state are not inserted simply to give an impression of great size or simply to create an effect of artistic verisimilitude (though, in fact, they do perform both these functions). These "random" numbers are constructed on a dialectical and artistic principle in such a way that each reflects some aspect of the rulers' basic and traditional confusion in mathematics and philosophy. Plato's selection of these objective metric statements of structural details of Atlantean politics, public works, and geography illustrates, and adds further insight and precision to, his eloquent disapproval and condemnation of Atlantis as a whole. Plato reveals his philosophic and artistic precision and his sensitivity to the significance of detail in inventing the history of a bad state as well as in describing the archetype of a good one.

One of the most peculiar of the Atlantean mathematical details is the 2:1 ratio of the ground-plan of Poseidon's temple. Although, as was pointed out above, these dimensions are stated as 6:3 plethra, it seems odd that Plato would even by such tenuous implication criticize what is elsewhere his own favorite ratio of 2:1. However, the actual dimensions of Greek temples suggest a possible explanation.[19] Though we have asserted that Plato was not himself a skilled artisan, the fact remains that he was an interested observer who found the raw material for many of his analogies in the arts and crafts. Since the range of these illustrations includes fish-trapping, shipbuilding, spinning and weaving, dyeing, potting, fulling, the minting of money, statue painting, theatrical scene-design, and other equally diverse activities, there is no reason to believe that this interested observation did not also extend to architecture. It must have seemed to Plato a striking oddity of architectural design that the typical ground plan of the most beautiful temples had sides that were not in simple ratio, and that may, indeed, have been planned by a construction which made them incommensurable. The aesthetic effect of buildings planned out by simple geometrical constructions which involved incommensurables may well be one of the sources inspiring Plato's remarks in the *Protagoras* that the same proportions are not always characteristic of pure functionality and of aesthetic effect.[20] But the aesthetic value of the effect was not lost on Plato; for the temple, like the religious myth, has a function to which the pleasure of its appearance is relevant.

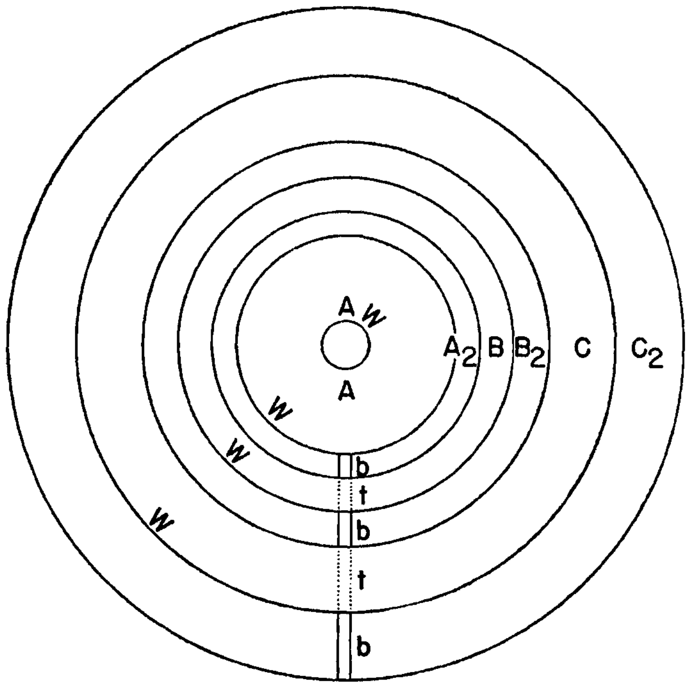
A temple based on the simple 2:1 ground plan would have seemed to an architect of Plato's time (as it does to one today) a dull, boxlike structure, not suited to its function of presenting an object of beauty and apparent harmony to the worshipper. In his judgment of this temple, that "there was something barbaric in its appearance," Plato may well be thinking not only of the garish external application of orichalcum and gold, but also of the lack of aesthetic subtlety revealed in the architectural ground plan and elevation. It is the sort of unsubtle product one might expect of a Milesian army engineer, temporarily turned temple-architect; and the same insensitivity to the proper social integration of function that characterizes the public works of the Atlantean engineers is reflected in their religious architecture also.

It is interesting to speculate further on the possible relation of architectural procedure to Plato's "diagonal of five" in the nuptial number passage.[21] Hambidge's measurements suggest that lines in ratio $1{:}\sqrt{5}$ may have been a sufficiently familiar builder's line to have had some special designation.[22] Such technical familiarity with the quantity, however, though it could have helped assure Plato that the line he referred to would be recognized, does not seem in any way to explain the specific nomenclature that Plato has chosen.

The details of Figure 18 should be compared with Plato's preferred city planning in *Laws* 778C, where it is prescribed that the city shall have no wall, in contrast to the 4 inner walls and to the outer wall (50 stades in radius; not shown in Figure 18) of the greater city of Atlantis. The tunnels permit shipping to come to the very center of the city, aggravating all the disorders and disadvantages which the Athenian Stranger in the *Laws* describes as typical of a seaport town. There is no separation of residence for soldiers, as the Acropolis in the *Republic* or the dispersed garrisons in the *Laws* would be.

If this city plan, with its central island containing the king's residence, is compared with Ortygia and its role in the history of

Figure 18



PLAN OF THE INNER CITY OF ATLANTIS

(R. G. Bury, *Timaeus, etc.*, map facing p. 286)

"A = central island (acropolis), 5 stades in diameter, with sacred pillar, temple, altar, etc. (116A, C ff.)

A_2 = inner belt of water, 1 stade wide

B = inner belt of land, 2 stades wide, with temples, gardens, barracks, etc. (117C)

B_2 = middle water-belt (2 stades)

C = outer land belt (3 stades) with hippodrome (117C), barracks, etc.

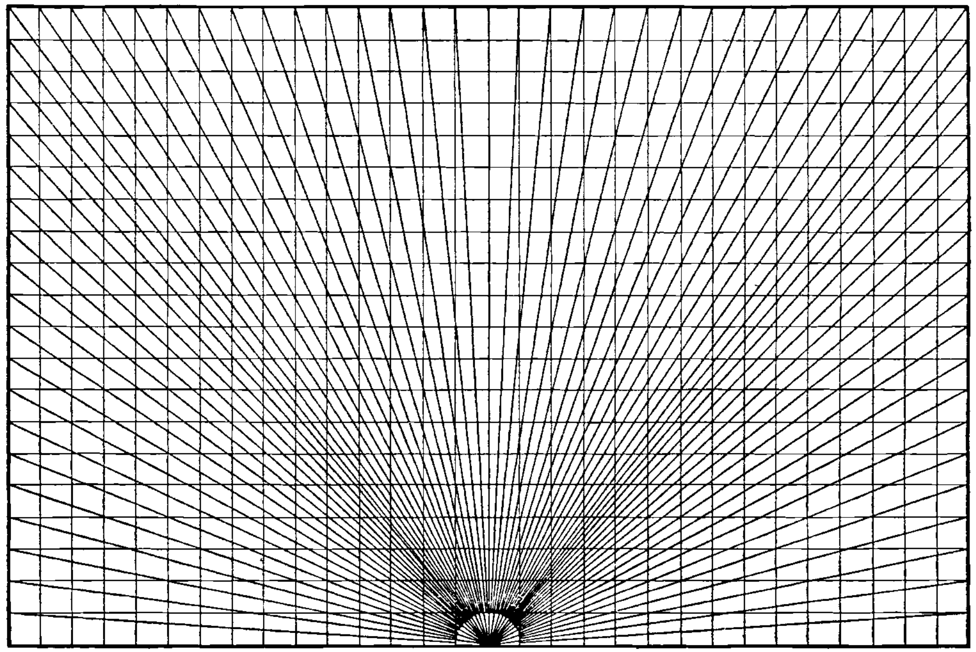
C_2 = outer water-belt (3 stades)

bb = bridges, with gates and turrets at each end, joining AB, BC, CD (116A)

tt = tunnels for ships under B and C (115D, E)

ww = ring-walls (4), round A,B,C, and *sacrarium* in A (116A)."

Figure 19



CANALS IN THE ATLANTEAN PLAIN

Syracuse, the similarity suggests that some of the elements of Plato's picture of a royal residence on an island within a city derive from contemporary affairs in Sicily. The legendary extra-Mediterranean location is probably derived from sailors' stories, of the sort which had earlier inspired Homer's invention of Ogygia.

The fact that both this and the ideal cities share a circular plan underscores Plato's point that technological accuracy is no substitute for functionality.

The image in Figure 19 is a maze rather than a regular grid; since the purpose of some of the "transverse" canals is to communicate with the city, I have drawn them as they would be planned to serve that function, rather than in the regular grid which would be reasonable if their principal function were irrigation rather than water communication.

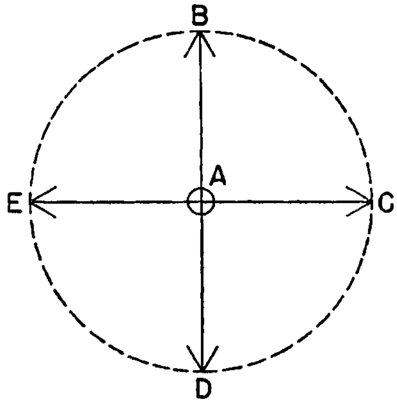
Compare Bury's map (*Timaeus*, etc., facing p. 285). The use of two large primes (29 and 31) helps to reinforce the maze effect; here the technology has completely escaped any rationale. Note that the city, if we interpret the canal net in the present way, is a focus of disorder; the lot plans become more and more irregular as one gets closer to it. A good brief discussion of the rectangular and radial theories of city planning in Plato's time is given in H. Diels, *Antike Technik* (3d ed.; Leipzig and Berlin, 1924), pp. 15–16.

Two specific observations are suggested by this canal net: the first is the Atlantean's characteristic waste of effort through lack of planning; the second is the similarity of the irrigation canal net and the rectangular grid used in city planning. Hippodamas, a Pythagorean engineer, had used such a rectangular plan in his redesign of the Piraeus. Plato himself, as Figure 21 shows, favored a radial plan. As the *Laws* describes the planning of the state, radiating highways were to be built establishing direct communication between the city and the frontier. One was to pass directly from center to periphery without (as an Atlantean dependent on his irrigation canals would) traveling along both legs instead of the hypotenuse of a right triangle. By describing the Atlanteans as first digging canals establishing communication by such legs of right triangles, then adding the net representing their hypotenuses separately, Plato seems to bring out a criticism of the Hippodamian plan. The radial plan seems to have been thought the more "radical"; as usual with such new ideas, Plato adopts it, and Aristophanes enthusiastically opposes himself to any such unconventionality.

This resemblance of the canal net to a local street plan (reinforced by the resemblance of the fortifications planned by the Atlanteans and by Hippodamas) suggests that the inspiration of the Atlantis story is not taken as a unit from any single source, but is a combination of details selected from the contemporary scene. In that case, most of the proposed identifications of the sources of the story are probably correct in their identification of some of its elements, but wrong in assuming that all the elements have the same source. As one recalls the criticisms that Plato offers of Pericles' public-works program, the exaggerated emphasis on public works in Atlantis sounds like another criticism of the Athenian projects. In particular, the wall and tower fortifications and the rectangular, irrigation canal grid are replicas of Hippodamas' city planning for the Piraeus and Thurium. The inclusion of the "voracious elephant" among the Atlantean fauna is probably an invitation to the reader to compare the Atlanteans and the Carthaginians. The elephant was peculiar to Carthage in Plato's time, and he seems to introduce it here as a "type" (a local animal chosen by a city for the design of its coinage). The end of the story, the overnight disappearance of a great sea power, is so close to the actual fate of the Minoan civilization that there probably were traditions preserved which suggested it. Certainly the details of Atlantean bull-worship were taken from legends associating such a cult with a hostile Mediterranean power. And so throughout Plato's description: legend and contemporary fact have been drawn on when either suggested detail for the composite photograph of a state dominated by imperialism and monomania.

In specifying the landmarks which represent the extent of the Acropolis in ancient times, Plato picks equidistant points, bringing out the basically circular geometrical form of the old city. In contrast with Atlantis, the regions are connected and functionally allocated. In describing the borders of the country, however, the landmarks mentioned do not seem intended to suggest a definite geometric figure. (The shape of Attica could not easily be derived by normal geologic processes from a prehistoric territory of postulated geometric regularity.) Instead, these early borders of the country underscore the continuously mountainous frontier, constituting a natural defense against land attack, where Plato pre-

Figure 20



TOPOGRAPHY OF ANCIENT ATHENS

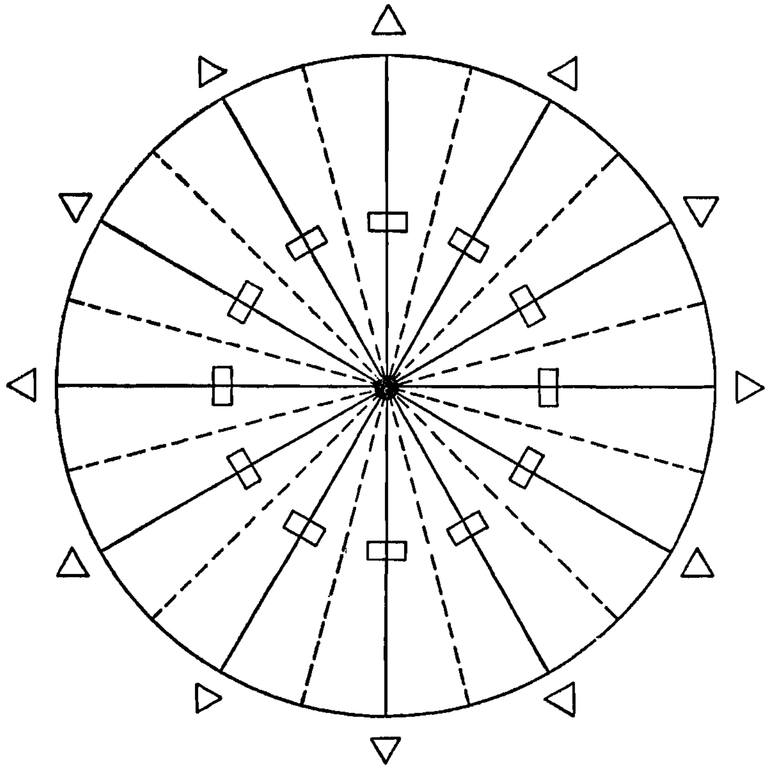
A = Acropolis
B = Eridanus
C = Lycabettus
D = Ilissus
E = Pnyx

sumably intended to locate the defeat of the invading Atlanteans.

The credibility of the story is heightened throughout by explaining present topography as the result of natural forces acting over a period of time on the ancient features "recorded" in the Egyptian inscription. Whether some ancient tradition suggested an original connection of Pnyx, Acropolis, and Lycabettus in a single plateau, or whether the same sort of inference is used here as in defending the historicity of Atlantis by the mud shallows in the Atlantic, later archaeological findings support Plato's reconstruction of the prehistoric connection of the three hills.

In Figure 21 the city is located in the most efficient way, in respect to its territory, for purposes of administration, defense, and trade. Unlike Atlantis, which lies at one side of its territory, and on a visualized map appears as a focus of disorder, the central city in this plan is a center of symmetry. (The suitability of the rectangular plan for purposes of location, which makes it functional in our own culture, would not be a great advantage for a

Figure 21



A FUNCTIONAL CIVIC PLAN (*Laws*)

● Centrally located capital city
—— Radial highways
▭ Regional markets and temples
- - - - District boundaries
△ Frontier garrisons

community of the size of a small Greek city state, nor would its superiority to the radial plan in this respect seem so evident to a culture which was not pervaded by the use of Cartesian co-ordinates for specifying location.)

Figure 22

MILITARY STRENGTH OF ATLANTIS

(Conscription is based on 60,000 military districts.)

I. PERSONNEL	
Archers	120,000
Hoplites	120,000
Slingers	120,000
Javelin-throwers	180,000
Light-armed slingers	180,000
Horsemen and charioteers	240,000
Sailors	240,000
Total *	1,200,000
II. EQUIPMENT	
Ships	1,200
Chariots (1/6th supplied by each allotment)	10,000

* This is exactly 60 times the force that the ancient Athenian state maintained (*Critias* 112D.; this is the only statistical detail given in connection with the description of ancient Athens).

II. The Social Institutions of the "Laws"

a. Mathematics and the Law

From the frequent specifications of the political need for the right order in society, one would expect the statistical details of the *Laws* to present a marked contrast to the Atlantean pattern. As in the *Republic,* legislation is described as a way of introducing harmony of classes and institutions into a state.[23] Unlike the *Republic,* however, the practicable legal statute must combine two types of such harmony. The order of Zeus, which apportions wealth and excellence to merit, is an order by geometrical proportion, the only type of structure employed in the *Republic.* The arithmetical ratio, on the other hand, gives equally to all citizens.[24] In a practicable legal code, the two orders must be blended. Also, unlike a discussion of political principles, a statute must make specific quantitative evaluations: it is not enough to agree that "wedding feasts must not be excessively expensive"; the law must state how much "excessive" expenses are.[25]

The mathematical aspect of the *Laws* is prominent in three connections. In the first place the population and administrative organization of the state are determined by logistical considerations which facilitate the mechanics of civil administration, representation, military conscription, tax assessment, etc. (The careful detail with which numbers suitable for this function are chosen contrasts effectively with the indefinite bigness of the comparable figures for Atlantis.) The state laws governing weights and measures are also determined by logistical considerations designed to make all transactions easy and readily calculable. These considerations are based purely on facility of calculation, an aspect of social organization not admitted to importance in the *Republic,* nor present in the state of Ancient Athens described in the *Critias.* (The population figures of these two states fall rather in the group of "powers of ten" typical of Plato's "mythical and historical" numbers.)

In the second place, the compulsory public education, designed to train students for socially effective participation in the state, puts great emphasis on the mathematical study of measure as a technique needed to evaluate social effectiveness and the efficacy of various social implementations of the state's political ideal.[26]

In the third place, the criminal code, which must be provided by the legislator, since these laws are designed for an actual rather than an ideal community, involves a very intricate application of mathematics in its proportioning of status and penalty.[27] Since there is a blending of oligarchy in the make-up of the state, possession of property is treated as involving greater civic importance, hence greater social responsibility. Probably one reason for including the study of mathematics in the schools was to give students the background requisite for understanding this aspect of the state laws in the required civics course devoted to the statutes and their explication.

The pattern which these figures represent shows a conscious, over-all plan at work, in complete contrast to the fives and sixes of Atlantis. It is a plan which adopts a different rationale for each of the three aspects of social planning: logistic for population and representation and for property distribution laws; commensurability for courses aimed at civic planning, which are based on geometry; and a mixture of harmonics and calculation embodied in the fines and sumptuary laws of the statute code.

b. Administrative Logistic

In the *Laws,* the scorn for calculation as an inelegant applied form of number theory, which Socrates expresses in the *Republic,*[28] is replaced by an intense interest in computation. The mechanisms of coinage, fines, taxation, calendar, standards of weight and measure, as well as of election, military conscription, irrigation and agriculture, must catch the dialectical proportions of the political philosopher in a lawgiver's specific metric net, a net capable of ready logistical adjustment and manipulation to meet the demands of new social contingencies.

For purposes of representation, defense, etc., it is necessary to subdivide the citizens into varying numbers of equal groups. Consequently, in establishing the optimum population of his state, Plato seeks, with all the resources of Pythagorean number theory, for a number which will be most logistically manipulable.

The number finally chosen is the wonderful number 5,040, which has 60 divisors, including the numbers 1 through 12, except 11, which will go into 5,038 (5,040–2).[29]

How did Plato discover the properties of this number? To work this out empirically with the number notation at his disposal would have been an arduous pastime.[30]

Presumably Plato was aware that the total number of divisors of a number is related to the number of prime factors it contains. One way of insuring a large number of divisors (including the number 12, needed to fit the calendar divisions) would have been to multiply successive digits together, beginning with 1 and proceeding until a product suitable in size for the population of a state resulted. As a matter of fact, since 5,040 equals 1·2·3·4·5·6·7 or 7!, this is probably how Plato made his discovery.[31] The other divisors could have then been computed by tabulating the various products of the constituent factors.

The same logistic emphasis is evident not only in the determination of optimum population but in the laws governing standards.[32] Since 12 has 6 divisors, it is a better number than 10 for computation and fits more nearly into the duodecimal calendar periods.[33] Consequently, coinage, measure, and weight are all to be established on the duodecimal system. Analogously, multiples of 12 serve to determine voting districts and representation in the governing assembly.

Thus, though they may dominate the mythology of the state, the mystical numbers 10,000, 100, and 10, with their Pythagorean aura of perfection and their religious connotation, are superseded in actual lawgiving by the less exotic 12 and 5,040, which are greatly superior for purposes of political manipulation.

c. Mathematics in Education for Citizenship

1. *Place of mathematics in the curriculum.* In *Laws* 818A ff. mathematics is given its place in the curriculum of public education. It is given this place in a context in which the Athenian Stranger, admitting that nature has not created men in the pure dialectical types posited in the *Republic,* tries to lessen the discrepancy. This is a context which might be expected to produce several shifts in emphasis, disquieting to anyone for whom "Plato's mathematics" is the pure deductive science described in *Republic* vii as suitable training for philosophers. In fact, several of these details in the *Laws* have puzzled readers and scholars; for example, the strong language in which Greek ignorance of theorems of incommensurability is reprehended, the difficulty in seeing how the distribution of a set of bowls teaches much worth learning about mathematics (the preparation of such special materials would be out of all proportion to their value if the lesson is one of simple computation), and the summary, popularized presentation of the results of astronomical research

(which seem either to ignore or contradict Plato's earlier, more elaborate, astronomy). The central change from the view of mathematics developed in the *Republic,* the fact that *measure* has usurped the place of *axiomatic rigor,* seems not to have been seen as a problem, but is certainly a more striking discrepancy than the minor deviations noted.

This training is part of education for citizenship because the laws of mathematics are necessities against which not even the gods can fight.[34] In ordinary civics, computation will play a part; but, more important, all legislation and applied ethical decision must center around finding the right measure. Measure is the device by which institutions and materials are made subservient to rational control; the statutory code of the state is an attempt to establish fines, maximum expenses, etc., which specify this right measurement. What the curriculum is intended to do is to suggest tactics for social action; it is applied mathematics. Obviously the nature of commensurability will be vitally important in such a technological theory; human errors very often result from not recognizing the incompatibility of alternative plans and motives, and introducing an erroneous compromise.[35]

The study of mathematics as measure is dismissed with a rather brief treatment for two reasons. In the first place, the theory has been exhaustively developed in the other late dialogues; [36] in the second place, the legal code (study of which precedes this course in mathematics) is an elaborate instance and embodiment of metric tactics at work. The course in computation, geometry, and astronomy simply brings out in a more general form tactics that have already been applied in the legislation dealing with population and representation (where the techniques are logistical), with property and penalties (where the tactics are metric), and with state religion and natural theology (where the theory of astronomy is written into the law itself at some length).

The specifications of the nature of this mathematical education are of value for a study of mathematical imagery primarily in the way in which they suggest the sort of principle likely to underlie this state's statistical detail, and the way in which their shifted emphasis confirms generalizations about the sensitivity of "mathematics" in Plato to the dialectical context in which that mathematics is presented.[37]

2. *Astronomy in general education.* As one would expect, the general public in the state described in *Laws* 818A does not study theoretic astronomy; the schools teach chronology, but astronomical theory is studied only by the specially talented group who constitute the state's research academy.

The general citizen is to be informed, however, that the results of this research give scientific weight to the conviction that the world is not haphazard, but designed and run on a rational plan. To correct the popular belief that the irregularities of apparent planetary motion indicate a basic disorder in nature, the citizens will presumably be told, as the Athenian Stranger tells his companions, that these stars each follow "one road, not many." Thus science discovers that these heavenly bodies are not literally "wanderers," as popular belief might suggest.

This remark can cause consternation to scholars who see in it a direct contradiction of Plato's earlier theories of astronomy.[38] But the notion of "one path" instead of many must be understood here as a contrast of the theory of a single planetary orbit (the resultant of three component motions, but perfectly predictable and regular) with the folk-belief that the planet is a wanderer straying down many different paths. The emphasis of the remark falls on the *regularity* of celestial motion, which science demonstrates. In this context, it would be dramatically incongruous to have the speaker suddenly intrude technical theoretic astronomy.

Nor does it seem necessary to explain the Athenian Stranger's remark that he had learned this only recently as evidence that Plato must be presenting some recent discovery of his own. The Stranger, if we study him, is like the Elders of the Nocturnal Council; his time has been spent in administrative and practical affairs, in which he has had considerable experience; there is, therefore, dramatic propriety in having his own experience reproduce that of the members of the Council, who in their old age study geometry and astronomy, and on the basis of this research and their experience give expert advice to the state.

3. *The game of distributing bowls.* School children are to be taught counting and measure in an enjoyable manner which will not destroy their intellectual interest, for example, by pairing wrestlers for games, and by distributing bowls of various materials and sizes to other members of the group.[39] It has been suggested that the objects to be distributed in this lesson were gold, silver, and bronze *coins* in the bowls. This would indeed make it seem that Plato had anticipated one of our modern techniques of progressive education; but one wonders whether the Athenian Stranger would not have detected some illiberal Egyptian or Phoenician implications in this practice of playing store. Still, it must be admitted that the handing around of bowls as a children's game seems somewhat bizarre.

If something about these bowls is deliberately reminiscent of the nested hemispheres of the cosmic model of *Republic* x, however, the game may take on added point. Perhaps the player is supposed to distribute the bowls into matched sets, of assorted sizes, but each set consisting of the same material. (This suggests an analogy to the modern set of identical cubes of different weight used for testing in our child psychology.) The difference in composition would then be determined by the appearance and the weight of each bowl. The art of weighing is one branch of applied mathematics with which the citizen is to be familiar. In this game, it would actually be made the basis of an elementary training in analogy; weight A is to bowl of size B as weight C is to bowl of size D. In addition to familiarizing the students with one of the important branches of measure, this game has merit as an introduction to the study of astronomical dynamics, if that study is conducted with such models as that of *Republic* x. It is even possible that when he comes to choose another incarnation,[40] the student trained in this way will learn more quickly, from his vision of the different colors and sizes of the bowls, of Necessity's spindle. Certainly it will help him to see the cogency of the antiatheistic argument based on the rational order of the cosmos, as revealed through the balanced distribution of its circular motions.[41]

4. *The importance of knowledge of incommensurables.* The Athenian Stranger says sharply that ignorance of solid geometry is "swinish." [42] This is strong language. Perhaps it shows merely that Plato in his old age developed a querulous impatience with the general lack of interest in a branch of mathematics which his Academy had done so much to develop. Perhaps, however, as the context of this remark suggests, it is both literally meant and defensible. The peculiarity of pigs is the domination of their behavior by pure appetite, uncolored by spirit or intelligence. The ability to initiate and perfect social institutions which go beyond the purely appetitive needs of citizens is one of the attributes which differentiate a city of men from a city of pigs. This is Glaucon's criticism of an analysis of the state set up exclusively in terms of supply and demand.[43]

In practical politics, the greatest problem is not so much that of securing agreement about principles, as devising institutions that will put them into effective operation. To do this requires the creation of a structure associated with the function given by the principle to be applied. All technology, including the social, must be dominated by discovering the measure of those constructions which embody a desired formal perfection.[44]

In a later book, we learn that causal efficacy proceeds from a point to a solid.[45] Aristotle reports Plato's use of this same analogy to describe the progression through the faculties of the soul.[46] Reason, which apprehends its object directly, must combine with understanding, which analyzes the associated structure, and opinion, which supplies experience relevant to the materials and techniques suitable to embody this structure, if noetic insight is to issue in social betterment.

d. Statutory Mathematics: Fines and Sumptuary Laws

Plato's general carefulness in the choice of quantitative statements and the legislation relating to population and representation in the *Laws* make it seem, a priori, likely that the details of statutory fines were also carefully considered.

From the fact that the statutes assign different fines to offenders of each property class, one might think that punishment were being equalized simply by making the penalty proportionate to ability to pay. In fact, however, fines based on this principle seldom occur. The fines set reflect the same mixture of oligarchy and democracy that characterizes the system of representation. Members of the wealthier classes are apparently expected to set a good example, and may be punished by an incremental penalty added to their fines when they do not. In certain cases of bad citizenship, the fine levied is intended to reduce the property class of the offender; we are told explicitly that this is to be done with persons not registering their property (*Laws* 775), and we may fairly assume that the extremely heavy fines for members of the upper economic classes who are cowardly [47] or who bring unfounded charges against a state examiner [48] (fines of 1,000 and 720 drachmae, respectively) have the same function. (If this is true, it gives some basis for estimating the actual capital of each class.)

Further, a disproportionately heavy fine is levied in certain other cases. The upper classes are fined 10 drachmae for nonattendance at the assembly (*Laws* 764); the lower two classes pay no fine at all. Failure to confine a madman carries an incremental penalty, above proportionate liability.[49]

On the other hand, the sumptuary laws allow a disproportionate expenditure by the upper classes for such occasions as funerals.[50] The greater civic importance of these groups is recognized in a disproportionate allowance for display; the upper-class funeral is given more importance by statute than the lower-class one. This seems a shrewd device on the part of a legislator bent on harnessing the profit motive to promote social industry; for a certain amount of disproportionate conspicuous display is evidently an added sign of status which will increase the eagerness of the citizens for economic advancement. Further, there seems proportionately less liability for bachelors of the two upper classes,[51] perhaps again reflecting their greater importance to the community.

The device of an incremental penalty, explicitly introduced in cases taken to courts of appeal,[52] is also operative in these statutes, adjusting privilege and penalty to ability to pay and to the social responsibility of persons of each economic group.

In certain cases, where the offense is criminally motivated but where the actual economic damage is not great, a tenfold increase magnifies the penalty to fit the seriousness of the crime. The increase takes effect in cases of petty pilfering[53] and late payment of the bachelor's fine (which is a form of pilfering from the temple treasury).[54] This is the scale on which the gods reward and punish human behavior in the Myth of Er;[55] but in general, except for those cases in which there is so great a discrepancy between motive and damage, such exacting justice is beyond the power of the human legislator.

These arithmetic details of the legal code seem thus to be carefully adjusted to combine the democratic and oligarchic principles basic in the constitution of the state. (See figures on pp. 69–71.)

III. Arithmetic Details in Myth and Chronology

Whatever imperfections time and chance intrude into history may be rectified when that history is recast in a purified form and presented as a myth. The line between Plato's "history" and "mythology" is singularly hard to draw, since his excursions into either field are dominated by a search for episodes which serve a purpose; both the history and the myth have a moral and bring it out more clearly by neglecting accidental distortions of the pattern. Since the value of the myth is its vivid presentation of individual events and careers, radically inaccessible to dialectical abstraction, the myths contain vivid circumstantial details which have often convinced Plato's readers that they were intended as histories.

The rectification of distortions is particularly apparent in mythical mathematical detail. Things happen in periods that are exact "round numbers." The longest such period is represented as a "myriad," and its subperiods as other powers of ten. The "almost as long" or "almost complete" period is projected

Figure 23

FACTORS OF 5,040: A LEGISLATOR'S MANUAL

NO. OF FACTORS	NO. OF GROUPS	NO. OF PERSONS IN EACH GROUP	NO. OF FACTORS	NO. OF GROUPS	NO. OF PERSONS IN EACH GROUP
i	1	5,040	xxxi	72	70
ii	2	2,520	xxxii	80	63
iii	3	1,680	xxxiii	84	60
iv	4	1,260	xxxiv	90	56
v	5	1,008	xxv	105	48
vi	6	840	xxxvi	112	45
vii	7	720	xxxvii	120	42
viii	8	630	xxxviii	126	40
ix	9	560	xxxix	140	36
x	10	504	xl	144	35
xi	12	420	xli	168	30
xii	14	360	xlii	180	28
xiii	15	336	xliii	210	24
xiv	16	315	xliv	240	21
xv	18	280	xlv	252	20
xvi	20	252	xlvi	280	18
xvii	21	240	xlvii	315	16
xviii	24	210	xlviii	336	15
xix	28	180	xlix	360	14
xx	30	168	l	420	12
xxi	35	144	li	504	10
xxii	36	140	lii	560	9
xxiii	40	126	liii	630	8
xxiv	42	120	liv	720	7
xxv	45	112	lv	840	6
xxvi	48	105	lvi	1,008	5
xxvii	56	90	lvii	1,260	4
xxviii	60	84	lviii	1,680	3
xxix	63	80	lix	2,520	2
xxx	70	72	lx	5,040	1

If the suggestion made in a later section as to the intended form of computation of the ratios of the world-soul is correct, one can be certain that Plato used some similar device rather than a tedious trial-and-error series of atomic calculations to establish these sixty factors. Such an atomic procedure would have seemed to him both tedious and inelegant. If the matrix computation pattern suggested for the *Timaeus* passage were applied to logistic, a simple set of tables could produce all of the factors of any given number in a neat, complete, and relatively elegant form, and demonstrate why these were the only factors of that number.

Figure 24

STATUTORY FINES AND SUMPTUARY LAWS * (*Laws*)

OFFENSE	BASIC FINE (LOWEST CLASS)	PROPORTIONATE FINES, OTHER CLASSES			FINES ACTUALLY ASSESSED, OTHER CLASSES		
	1	2	3	4	2	3	4
Not marrying	30	60	90	120	60	70	100
Disrespect of elders	20	40	60	80	30	50	60
Cowardice	60	120	180	240	180	300	1000
Unconfined madman	24	48	72	96	36	48	100
False charges against examiner (accuser gets less than 1/5th of votes)	240	480	720	960	360	480	720
Not voting (see Fig. 25)	10	20	30	40	10	30	40
Absent from assembly	—	—	—	—	—	10	10

SUMPTUARY LAWS

	MAXIMUM EXPENSE ALLOWED LOWEST ECONOMIC CLASS	PROPORTIONATE MAXIMUM FOR OTHER CLASSES			ACTUAL STATUTORY MAXIMUM FOR OTHER CLASSSES		
	1	2	3	4	2	3	4
Wedding feasts	7.5	15	22.5	30	15	30	60
Marriage garments	50	100	150	200	60	90	120
Funeral costs	60	120	180	240	120	180	300

TAX-EXEMPT CAPITAL

FOR LOWEST ECONOMIC CLASS	PROPORTIONATE EXEMPTION FOR OTHER CLASSES			ACTUAL STATUTORY EXEMPTION FOR OTHER CLASSES		
1	2	3	4	2	3	4
60	120	180	240	120	180	240

* Fines, expenses, and exemptions are given in drachmae.

Figure 25
ELECTION STATUTES (*Laws*)
(A council of 360 members is to be elected, 90 from each class.)

DAY	PROCEDURE	FINE FOR NONATTENDANCE			
	(Class)	1	2	3	4
1	Nomination candidates from 1st class	10	10	30	40
2	Nomination candidates from 2nd class	10	10	30	40
3	Nomination candidates from 3rd class	—	10	30	40
4	Nomination candidates from 4th class	—	—	90	160
5	Balloting: 180 of each class elected, 90 selected for council by lot	10	10	30	40

A. E. Taylor * shows that the effect of these differential fines is to give the lower economic classes a majority in the nomination and challenge of representatives of the upper classes and to put the upper economic classes in the majority for the nomination and challenge of candidates from the lower economic classes.

* *Plato, the Man and His Work* (new ed.; New York, 1946), pp. 479–80.

as a power of 10 times 9. This peculiarity of mythical mathematics is apparent in the temporal periods associated with the accounts of transmigration. In *Republic* x the length of human life is 100 years, the cycle of incarnations 1,000.[56] In chronology, the war with Atlantis has taken place 9,000 years before, while the age of Egyptian culture and its records is 8,000 years.[57] In another transmigration myth, the cycle of recurrence is 10,000 years; [58] the foolish lover is bowled around the earth for 9,000; [59] and the philosopher is released after 3,000 years. The state of Atlantis is described as having "myriads" of citizens,[60] troops, and acres.

The "myriad," like our "million," is a large number-word used to express great or indefinite size. It is also a power of ten, which makes it a suitable number for myths in which there are many echoes of Pythagorean tradition. But it is often not enough simply to indicate that a period, while definite, is indefinitely large, if it comprehends subperiods to which it must be shown as proportionately related. In these cases, the smaller powers of ten are assigned to the subperiods.

CHAPTER III

Geometric Metaphor

VERBAL MATRICES: Introductory Remarks

As PROFESSOR BUCHANAN has pointed out in his book *Symbolic Distance,*[1] Plato's techniques of definition and discussion may be visualized as instances of matrix construction. A verbal matrix defines a term by locating it in relation to a set of other systematically ordered terms. If the meaning is not clear from the arrangement, either the matrix is enlarged, or the term in question is itself replaced by a matrix, differentiating the senses in which it is used. Distinctions in scope and field of terms are thus schematized by the spatial relations of the matrix grid structure. The most famous such matrix, which Professor Buchanan cites, is derived from the *Republic.*[2] In each case, the matrix grid presents a graphic and convenient mathematical image for the spatialization of a net of dialectical distinctions.

Since the elements of such matrices are words, which are images, rather than numbers, which are natures,[3] analogies to mathematical matrix theory are misleading, and a logic of verbal matrices cannot possibly be built up simply by applying modern mathematical matrix operations. Verbal matrices, however, have certain formalizable properties and operations of their own which provide images of the formal character of Platonic logic.

The use of matrices for combination diagrams probably was developed by mathematicians and perhaps in the medical tradition before Plato's time.[4] In the *Dialogues,* frequent references to "left" and "right," "vertical" and "horizontal," division suggest that Plato himself was visualizing verbal matrices of the sort described, and referring to them as an aid to the attentive reader. Whatever its original inception, the use of a matrix as a notational device to present clearly relations which are not evident from other conventional notations appealed to the Pythagorean school. A technique so effective in dealing with factors in arithmetic, and with combinations in ratio theory, suggested the desirability of exhibiting terms or concepts in a tabular form, the location of each in the table indicating its resemblance to and difference from the other terms included. Aristotle preserves one such early Pythagorean "table." [5] In this table, ten pairs of contraries are arranged in two columns: terms in the same column are "alike," terms in opposite columns on the same row are "contraries," terms in opposite columns not on the same row are "unlike." This is essentially the same use of spatial orientation to symbolize analogies and differences which Plato adopts in his explicit construction of matrices.

Every matrix has certain conventions in its spatial orientation. Like the divided line, distinctions of being and becoming are arranged from top to bottom, the higher position being reserved for the term most clear to intelligence and akin to reality. The reader's left (the matrix's right) is likewise the more honorific position; and if the differentiation of columns also embodies an application of the reality-appearance distinction, this runs toward the reader's right from his left. The basic orientation of a matrix is therefore that of Figure 29, as in the *Sophist* passage.

The relation of adjacent terms is one of proportion or analogy; thus, in Figure 29, $A:B::C:D$ and $A:C::B:D$. In definition by dialectic, the *proportion* is Plato's favorite mode of presenting an analogy; [6] the terms of every matrix are proportionately related, and a matrix can be constructed from any set of proportions. Often an analogy is developed between sets of terms that are ontologically co-ordinate, as in the analogy of the state to the body in the *Laws* (see Figure 30), and in such cases, the order of columns does not indicate a being-becoming, appearance-reality distinction.

The closest one can come to an analogy to quantitative matrix manipulation is by describing the operations of vertical and lateral "shifting" of terms in the verbal matrix. If a term is shifted along a row, from its proper place to another column, the result is a metaphor. For example, in Figure 30, if we call the councillors in the *Laws* heads of the state, we are using the term "heads" metaphorically. The distance of such a shift gives an index of the vividness of a metaphor, or of its "tension"—the quality which leads us to describe some metaphors as "strained." If a term is shifted between rows within a column, the result is the "collapse" or "reduction" of a distinction. If, for example, "head" is shifted to the row which in contextual columns corresponds to the spirited part of the soul, the shift suggests an erroneous identification of passion and reason; a "being" term is explained by a context inappropriately colored by "becoming." The mechanism of error in the *Timaeus* is developed with the help of optical analogies which give such "shifts" in space precisely this interpretation.[7]

In certain cases, Plato "reduces" his matrices and "multiplies" them. A "reduced" matrix presents a set of combinations in a simple linear listing in which the original rows or columns are arranged successively. Plato "multiplies" matrices in which one set of the distinctions, either that of rows or columns, is the same in both. The two matrices are combined by joining them along the axis representing the distinctions which they have in common, while the different axis of the second is used to represent a projection of the first into the third dimension.

Matrices are also often "schematized." This is done by representing successive matrix positions by ordinal numbers, so that the relations of these numbers summarize the original dialectical proportions which the matrices present. Thus, the operations described above as reduction and multiplication of matrices may be "schematized" as shown in figures 36 and 38.

Plato's method of schematizing matrices by simply numbering successive positions is in principle the same as contemporary matrix technique, which uses subscript positional numbers. The number assigned a given position in Plato's version can be shown to be a function of the subscripts in modern notation.[8]

ΚΕΦΑΛΙΣΜΟΣ

Figure 26

MULTIPLICATION TABLE

			H	HH	HHH	HHHH	𐅅	𐅅H	𐅅HH	𐅅HHH	𐅅HHHH
			Δ	ΔΔ	ΔΔΔ	ΔΔΔΔ	𐅄	𐅄Δ	𐅄ΔΔ	𐅄ΔΔΔ	𐅄ΔΔΔΔ
			I	II	III	IIII	Γ	ΓI	ΓII	ΓIII	ΓIIII
H	Δ	I	I	II	III	IIII	Γ	ΓI	ΓII	ΓIII	ΓIIII
HH	ΔΔ	II	II	IIII	ΓI	ΓIII	Δ	ΔII	ΔIIII	ΔΓI	ΔΓIII
HHH	ΔΔΔ	III	III	ΓI	ΓIIII	ΔII	ΔΓ	ΔΓIII	ΔΔI	ΔΔIIII	ΔΔΓII
HHHH	ΔΔΔΔ	IIII	IIII	ΓIII	ΔII	ΔΓI	ΔΔ	ΔΔIIII	ΔΔΓIII	ΔΔΔII	ΔΔΔΓI
𐅅	𐅄	Γ	Γ	Δ	ΔΓ	ΔΔ	ΔΔΓ	ΔΔΔ	ΔΔΔΓ	ΔΔΔΔ	ΔΔΔΔΓ
𐅅H	𐅄Δ	ΓI	ΓI	ΔII	ΔΓIII	ΔΔIIII	ΔΔΔ	ΔΔΔΓI	ΔΔΔΔII	ΔΔΔΔΓIII	𐅄III
𐅅HH	𐅄ΔΔ	ΓII	ΓII	ΔIIII	ΔΔI	ΔΔΓIII	ΔΔΔΓ	ΔΔΔΔII	ΔΔΔΔΓIIII	𐅄ΓI	𐅄ΔIII
𐅅HHH	𐅄ΔΔΔ	ΓIII	ΓIII	ΔΓI	ΔΔIIII	ΔΔΔII	ΔΔΔΔ	ΔΔΔΔΓIII	𐅄ΓI	𐅄ΔIIII	𐅄ΔΔII
𐅅HHHH	𐅄ΔΔΔΔ	ΓIIII	ΓIIII	ΔΓIII	ΔΔΓII	ΔΔΔΓI	ΔΔΔΔΓ	𐅄IIII	𐅄ΔIII	𐅄ΔΔII	𐅄ΔΔΔI

That such a table as the one in Figure 26 was in use is clear from Aristotle's reference to the κεφαλισμός in *Topics* 163b.17.*

This version is given in the "Herodianic" notation, which was probably what Aristotle was referring to.†

The table is a device for marshalling products under the "headings" representing their factors; by using three rows of headings, as in this figure, a 9 by 9 matrix can be used, as Aristotle says the multiplication table up to ten can, for multiplication up to 1,000. The rules for using such a table to multiply large numbers are, as can be seen, extremely simple in the "Herodianic" notation, which is not the case with the alphabetical number-system; to some extent, this confirms the notion that it was the former which Aristotle had in mind.

No doubt actual computation had led to many short cuts and abbreviations of the "official" Athenian number-system; presumably, however, these would have been taught in a more advanced course of "logistic" than the elementary work in which the student mastered his tables up to ten times ten.

Figure 27

A PYTHAGOREAN TABLE OF CONTRARIES

(Aristotle, *Metaphysics* 986A)

LIMIT	INDETERMINATENESS
ODD	EVEN
UNITY	PLURALITY
RIGHT	LEFT
MALE	FEMALE
REST	MOTION
STRAIGHT	CURVED
LIGHT	DARK
GOOD	BAD
SQUARE	RECTANGULAR

This "table" brings out two points concerning the Pythagorean tradition that are important for the present study. In the first place, many of the analogies suggested reappear in Plato's own mathematical imagery. More important, however, the arrangement in a series of corresponding pairs, such that terms on the same line are

* See Sir Thomas Heath, *Mathematics in Aristotle* (Oxford, 1949), p. 93.

† See Heath, *History*, I, 30 ff.

contrary, those in the same column similar, and those in opposite columns dissimilar, shows exactly, in an elementary form, the use of spatial relations to indicate dialectical ones that characterizes Platonic "verbal matrices." (Aristotle specifically mentions this matter of arrangement of the terms as an integral part of the Pythagorean theory he is presenting here.)

Figure 28

HIPPOCRATIC GENETIC THEORY: A COMBINATION TABLE ‡

"Theoretical combinations of male and female sex-elements. *M* represents a male seed which dominates; *m*, a male seed which is dominated; *F*, a female seed which dominates, and *f*, a female seed which is dominated. The first symbol in each combination represents the seed contributed by the female parent, the second symbol (in italics) that contributed by the male. Combinations marked * are impossible, because they violate the postulate that one and only one seed can be dominant. Combinations marked † are impossible, because they violate the postulate that a male-contributed male seed cannot be dominated by a female-contributed male seed, and vice-versa."

		MALE PARENT			
		M	m	F	f
FEMALE	M	M*M**	M*m*†	M*F**	M*f*
	m	m*M*	m*m**	m*F*	m*f**
PARENT	F	F*M**	F*m*	F*F**	F*f*
	f	f*M*	f*m**	f*F*†	f*f**

‡ From R. S. Brumbaugh, "Early Greek Theories of Sex Determination," *Journal of Heredity*, XL (1944), 50.

The theory as preserved does not use such a matrix, but examines in turn and in systematic fashion each of the theoretically possible combinations of the parental seeds, as though some matrix notation had been used in calculating these combinations. It is hard not to believe that the medical men would have recognized the suitability of mathematical matrix technique for their analogous problem in genetic combination.

In Figure 29 the axes from top to bottom and from left to right represent principles of classification generating combinations, as in the *Sophist* matrix. If three principles of order are used, the matrix may be made three-dimensional and the axis from front to back also used to represent a principle of classification. The dotted lines indicate that this matrix may be extended to include an indefinite number of columns and rows.

Figure 29

STANDARD SPATIAL ORIENTATION OF A PLATONIC MATRIX

```
BEING. . . . . . . . . . . .BECOMING
 .
 .   A  B
 .   C  D
 .
 .
BECOMING
```

Figure 30

A POLITICAL MATRIX FROM THE *LAWS*

PARTS OF SOUL	CLASSES IN STATE	ORGANS OF BODY
Reason	Councillors	Organs of Perception (Head)
Spirit	Citizens	Organs of Locomotion
Appetite	Craftsmen	Organs of Nutrition

Compare figures 34 and 35, following. The craftsmen in this matrix are the resident foreigners who carry on business in the state but who do not share in military duties or other civic "executive" functions.

Some Typical Uses of Matrices in the Scholia

(Figures 31 through 35 from Greene, *Scholia Platonica,* pp. 139, 147, 224, 226)

Figure 31

Gorgias 465C

BODY	SOUL
Gymnastic	Lawmaking
Medicine	Judging
Cookery	Sophistry
Scene Painting	Rhetoric

Figure 32

Gorgias 477A

PARTS OF MAN	DEFECT	EXCELLENCE
Soul	Injustice	Justice
Body	Illness	Health
Externals	Poverty	Wealth

Figure 33

Republic 435B

Justice	Whole State	Whole Man
Wisdom	Rulers	Spirit
Courage	Soldiers	Reason
Temperance	Artisans	Appetite

Figure 34

Republic 440E

IN THE STATE	IN THE SOUL
Productive	Appetitive
Protective	Spirited
Deliberative	Rational

Figure 35

CORRECTION OF THE *REPUBLIC* MATRICES

PART OF SOUL	PROPER VIRTUES OF CLASSES		
	Rulers	*Soldiers*	*Artisans*
REASON	Wisdom	. . . .	. . . .
SPIRIT	Courage	Courage	. . . .
APPETITE	Temperance	Temperance	Temperance

The diagrams of figures 33 and 34 have gained considerable currency, but both distort Plato's definition of "temperance," which is a virtue proper to the function of all classes in the state. The diagram in Figure 35 is more accurate.

Since all classes share in temperance, and since rulers are selected on the basis of their earlier demonstrated courage as well as their wisdom, the scholion figure rather misses the point of the text, in the interest of facility of schematization. If justice is to be included, it will also be a property of all classes, perhaps best schematized as horizontal lines of class separation in the figure.

In general, these figures do not follow Plato's own use of space in their schematization. In Figure 31, the vertical listing has no principle of order; the relation of left-right is just the reverse of that used by Plato in the *Sophist* matrix. Figure 32 is thoroughly Platonic, but in Figure 33 the erroneous schematic presentation of temperance as exclusively the virtue of the artisan class completely throws off the part-whole principle of vertical order. In Figure 34, the typical relations of top and bottom are gratuitously reversed, inverting the picture. This, as well as the inexact schematization of temperance, is corrected in Figure 35.

Figure 36

SCHEMATIZATION AND REDUCTION OF A THREE TIMES THREE PLANE MATRIX

1	2	3									
4	5	6	1	4	7 |	2	5	8 |	3	6	9
7	8	9		row 1			row 2			row 3	

The positions of the terms in such a matrix, if the typical spatial orientation is respected, mean that the position of any term is itself adequate indication of the logical relations of that term to the others in the schema. Hence relational structures may be represented and treated by substituting positional numbers for the terms themselves, as in the square figure. (To do this involves a convention that the left-right principle of order will take priority in assigning the numbers to the positions.) The distance indicates the exact relation of the term in one position to that in another (if the dimensions of the matrix are given): for example, representing the horizontal ordering relation by H, the vertical by V, 1 is in the relation $H \mid H \mid V$ or $H^2 \mid V$ to 6. (Here the stroke ($\mid$) is used, as in relational calculus, to represent a "relational product." $H \mid V$, for example, abbreviates $(\exists x)(\exists y)(\exists z)[H(x,z).\ V(z,y)]$.

If the schema must be changed from two dimensions to one, either because two distinctions must be represented by a single dimension (since four or more principles are being diagrammed), or because such a listing better represents the fact that the two ordering relations are not co-ordinate in importance, the result is the *reduction* of the matrix to a line. The linear series in the figure represents one such reduction, in which the relative importance of the top-bottom relation of the original matrix is emphasized by making this the ordering relation for each group of three.

Within each group, the original left-right relation determines the order. Since the convention of numbering positions is independent of this relative emphasis on the co-ordinates, the resulting list may hide the original diagram from which it was derived.

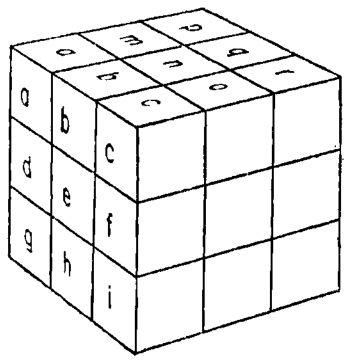
In subsequent discussion, examples will be given of specific uses in Platonic images of reduced, multiplied, and schematized verbal matrices.

Such matrix schematization may be a device for presenting the abstract relations between sets of comparable elements, ordered in terms of two principles simultaneously. With the added concept of "distance," it is a device for stating the relation of any pair of elements in a set ordered in this way. Whenever sets of terms in such relation appear in a dialectical discussion, a matrix schema

Figure 37

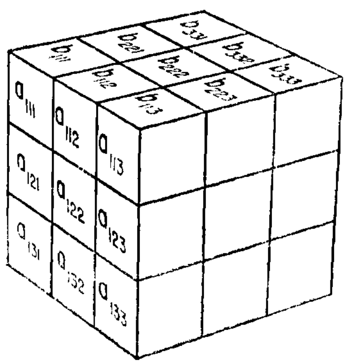
MULTIPLICATION OF TWO SIMILAR MATRICES

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \cdot \begin{vmatrix} a & b & c \\ m & n & o \\ p & q & r \end{vmatrix} =$$



or,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} =$$



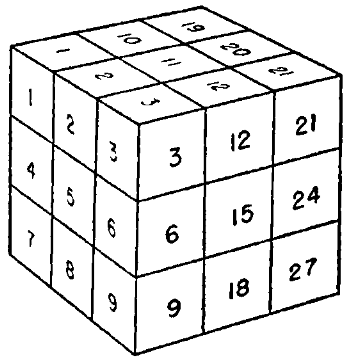
provides an appropriate diagram to clarify their interrelation. What the constitutive elements in the relational scheme are, however, must of course be determined by substituting for the positional numbers the terms which themselves figure in the dialectical context.

Two schemata may be combined into a third, when their dimensions and one set of distinctions are the same, by joining the identical rows and using the second plane as a projection of the first into the third dimension, as shown in Figure 37. The same operation is shown with positional subscripts, indicating plane, row, and column, respectively.

Figure 38

SCHEMATIZED MULTIPLICATION OF TWO SIMILAR MATRICES

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$$



In this figure, the letters and subscripts are replaced by positional numbers; the joining of the planes requires a new assigning of number, on the convention that planes are numbered from front to back, which makes a number of the original plane used for the third-dimensional projection equal to the ordinal number multiplied by 6 times the number of planes minus 1.

I. CONSTRUCTION OF A MATRIX

Sophist 266 *

STRANGER: And so there are two kinds of making and production, the one human and the other divine.

THEAET.: True.

STR.: Then, now, subdivide each of the two sections which we have already.

THEAET.: How do you mean?

STR.: I mean to say that you should make a vertical division of production or invention, as you have already made a lateral one.

THEAET.: I have done so.

STR.: Then, now, there are in all four parts or segments—two of them have reference to us and are human, and two of them have reference to the gods and are divine.

THEAET.: True.

STR.: And again, in the division which was supposed to be made the other way, one part in each subdivision is the making of the things themselves, but the two remaining parts may be called the making of likenesses; and so the productive art is again divided into two parts.

THEAET.: Tell me the divisions once more.

STR.: I suppose that we, and the other animals, and the elements out of which things are made—fire, water, and the like—are known by us to be each and all the creation of God.

THEAET.: True.

STR.: And there are images of them, which are not them, but which correspond to them; and these are also the creation of a wonderful skill.

THEAT.: What are they?

STR.: The appearances which spring up of themselves in sleep or by day, such as a shadow when darkness arises in a fire, or the reflection which is produced when the light in bright and smooth objects meets on their surface with an external light, and creates a perception the opposite of our ordinary sight?

* Trans. Jowett, *Dialogues*, IV, 403–4.

Figure 39

THE SOPHIST MATRIX

	DIVINE CREATION	HUMAN CREATION
Realities	Natural Objects	Artifacts
Appearances	Shadows and Reflections	Mimetic Works of Art

THEAET.: Yes; and the images as well as the creation are equally the work of a divine hand.

STR.: And what shall we say of human art? Do we not make one house by the art of building, and another by the art of drawing, which is a sort of dream created by man for those who are awake?

THEAET.: Quite true.

STR.: And other products of human creation are also twofold and go in pairs; there is the thing, with which the art of making the thing is concerned, and the image, with which imitation is concerned.

THEAET.: Now I begin to understand, and am ready to acknowledge that there are two kinds of production, and each of them twofold; in the lateral division there is both a divine and a human production; in the vertical there are realities and a creation of a kind of similitudes.

IMAGES OF HARMONY AND CYCLE: the *Republic*

II. Mathematical Imagery in the Republic; the State and the Musical Scale

Harmony characterizes those moral and political structures which are closest to perfection. The list of images in the *Republic* marks the crucial stages of the theoretical construction of such social harmonies. The image of harmony in music is inserted as a summary of the discussion of education and art at the conclusion of Book iii,[9] and is again referred to in an explanation at the end of Book iv [10] of the class structure of the state. The way in which education inevitably introduces greater harmony into social cycles is illustrated by the image of the upward spiral in Book v,[11] and the deteriorating effect of time and chance on human institutions, leading to loss of harmony, is presented in the balancing downward spiral marking the transition from the best state to worse states in Book viii.[12] The intellectual ascent to principles is to be accomplished by the largely mathematical curriculum of Book vii,[13] founded on the differentiation of kinds of clear knowledge illustrated by the divided line at the end of Book vi.[14] The greater happiness of the just man, the original point of controversy, is resolved by the calculation of the tyrant's unhappiness in Book ix; [15] and finally given a complete warrant by the mechanism of cosmic justice, shown to the souls in the myth at the end of Book x,[16] in which the two earlier spiral images are combined into a cosmic and moral cycle. It is with this over-all, interconnected pattern in mind that we must proceed to the separate treatment of these constitutive mathematical images of the *Republic.*

Republic 443 *

The just man does not allow the several elements in his soul to usurp one another's functions; he is indeed one who sets his house in order, by self-mastery and discipline coming to be at peace with himself, and bringing into tune those three parts, like the terms in the proportion of a musical scale, the highest

* Trans. Cornford, *Republic,* p. 142.

and lowest notes and the mean between them [lit. νήτη, ὑπάτη, and μέση] with all the intermediate intervals.

Scholion *

The *nete* is the final note of a system of two tetrachords; the *hypate* . . . the first note of the two tetrachord system. The *mese* is the note which is the termination of the first tetrachord and the starting point of the second, as Ptolemy says, and other musicians.

Republic 432A †

. . . Temperance works in a different way; it extends throughout the whole gamut of the state, producing a consonance of all its elements from the weakest to the strongest as measured by any standard you like to take—wisdom, bodily strength, numbers, or wealth. So we are entirely justified in identifying with temperance this unanimity or harmonious agreement between the naturally superior and inferior elements on the question which of the two should govern, whether in the state or the individual.

The image of harmony as an evidence of organization of parts into wholes in which they are internally related is repeatedly suggested in the discussion of education in the *Republic*. It is given one of its most explicit statements in 443D, where the temperate relation of classes in the state is compared to the relation of the fixed notes of a scale.[17] It is important to note that such "harmony" presupposes heterogeneity; the notes do not blend to produce an average pitch midway between them, but each, retaining its own identity reinforces that of the other. Thus a harmony in state or soul does not suggest metaphorically that classes or faculties become identified; on the contrary, one condition of temperance is the presence of justice, which keeps the parts distinct, and insures that each will make its own individual contribution.

The purpose of the science of politics is to construct "harmonies" in the periodic processes which constitute the temporal careers of state and soul. The imagery of the *Republic* reflects

* Greene, *Scholia Platonica*, p. 226.

† Trans. Cornford, *Republic*, p. 126.

this enterprise in its initial introduction of disparate metaphors of harmony and cycle, which in the later images are gradually shown combined.

Figure 40

CHANGES OF TOPIC AND LOCATIONS OF MATHEMATICAL IMAGES IN THE *REPUBLIC*

Asterisks in the left column mark the locations of mathematical references; it will be noted that these fall for the most part at places where either a summary or an anticipatory schematism would be reasonably expected.

LOCATION (BOOK)	TOPIC OR IMAGE
I	Criticism of popular definitions of justice
II–IV	Origin and functional class-structure of the state
**	Analogy of temperance and harmony
V	Marriage and family regulations
**	Image of cyclic growth of state
V	Conduct of war
VI	Education for rulers
**	Kinds of knowledge: the Divided Line
VII	Mathematical curriculum; dialectic
VIII	Variant forms of state and character
**	The Nuptial Number: cyclic decline of the state
VIII–IX	Variant forms of state and character
**	The Tyrant's Number: summary of desirability of lives of each type
X	Summing up: tendency of experience to confirm the argument
X	Mythical postscript: the afterlife
**	Astronomical model, Myth of Er: justice in the cosmos

III. The Cycle of Social Progress

Republic 424A *

And, moreover, said I, the state, if it once starts well, proceeds as it were in a cycle of growth. I mean that a sound nurture and education if kept up creates good natures in the state, and sound natures in turn receiving an education of this sort develop into

* Trans. Shorey, *Republic,* I, 331.

better men than their predecessors both for other purposes and for the production of offspring as among animals also.

This reference to the state "growing like a circle" is non-technical and brief. In Shorey's translation, just quoted, the circle is thought of as a kind of civic life-cycle, just as animals have a cycle of growth. Jowett's translation embodies his interpretation; the state gains in momentum as a heavy wheel would, alternately pushed at *A* and *B* (See Fig. 41).[18] J. Adam interprets the passage as referring to a circle being drawn with a compass; the state pursues an upward arc for a time, passes its acme, and continues through a downward one (as shown in Fig. 42). In context, it seems quite clear that the correct interpretation of this "circular" effect of nature and training must take account of some sort of *reciprocity,* and Jowett's notion of the reciprocal effect of impulsions applied to a circle which is rotating seems to do this best. But this judgment is contingent on the interpretation of the "nuptial number," a passage which Cornford and Adam seem to have in mind as suggesting interpretations here.[19]

The idea that the cycle of political history is being drawn in this metaphor of social evolution, which underlies Adam's interpretation—that the state is growing as a circle does when it is being drawn with a compass—certainly brings out a relevance between this entire section of the discussion and the subsequent uses of imagery of cycle.[20] It does so, however, at the expense of a rather unlikely interpretation of the metaphor given by the text. In this context "cyclic increase" in social excellence stresses the reciprocity of *nature* and *nurture,* a reciprocity of which no adequate account is given by this interpretation. In effect, Adam has substituted an alternative and plausible metaphor for the one which Plato really employed. It should be noted also that Plato's image of a cyclic development conditioned by the interaction of two contributory factors is not (as, according to Adam's interpretation, it should be) an arc of a circle, but rather a spiral.

To say that this metaphor refers to the spreading out of circles from a center of disturbance, like the rings in water, completely overlooks the element of reciprocity which in this context any appropriate metaphor must include. It explains the function of growth, but gives no functional significance to the textual stress on circularity.

Professor Shorey's note on this passage denies that the "circularity" involved is intended to suggest any specific metaphorical illustration.[21] But all the passages cited by him in this connection involve the same basic metaphor of the growing momentum of a wheel spun by successive impulsions to its rim.

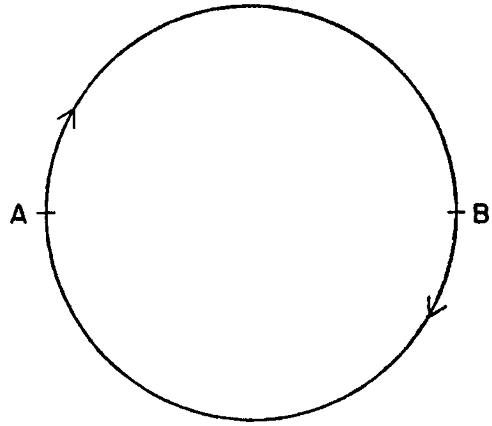
Aside from this point, since all citations involve the same basic metaphor, the only question that remains is whether the passage itself is clearer or more vivid if the metaphor is taken in a concrete rather than in the proposed abstract and generic sense. Both the precedent of Plato's other illustrations from techniques of the arts and his predilection for concrete vividness of illustrative metaphor would lead one to believe that he here intended the passage to suggest the more concrete interpretation.

The intuited fact that motion communicated to a point on a wheel is distributed evenly to all points on its periphery is prominent in the picture of the Fates turning the celestial hemispheres in the Myth of Er.[22] Some clue to the relation of cosmic mechanism and political history is latent in the similarity of the mechanics of social evolutionary process to the mythical account of the cosmic turning.

Plato has a habit of introducing references and phrases from familiar processes of the arts and crafts. His assumption that it will help his reader to visualize things as being "like an eeltrap" or "like a reinforced trireme hull" was no doubt a valid one for readers who were his contemporaries; but it is not valid for readers today. Unfamiliar as we are with the everyday sights of ancient Athens, and limited in our ability to reconstruct and visualize Athenian technology, the allusions become more of a hindrance and puzzle than a help.

The passage from *Republic* 424A, in discussing which I would accept Jowett's interpretation, is a case in point. A pot-

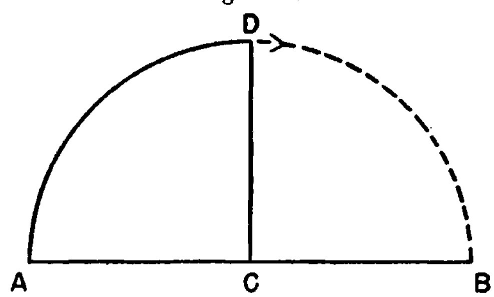
Figure 41



GROWTH OF THE STATE: CIRCULAR IMPULSION

This figure interprets the metaphor as "increase, like a wheel being impelled from two directions in its rotation." In other words, a force applied at *A* will increase the momentum of the whole circle, and a second impulse applied at *B* will act reciprocally to increase the velocity at *A*. The central point is that nurture and education act cumulatively in accelerating the improvement of the state. An analogous use of momentum and impulsion appears in the Myth of Er.

Figure 42



GROWTH OF THE STATE: CIRCLE UNDER CONSTRUCTION

In this interpretation, *AB* represents a time axis, *CD* an axis of perfection. The circular growth of the state is thus envisaged as a progress from *A* to *D*, a decline from *D* to *B*. The chief objection is that this figure completely ignores the stress in the context on reciprocity, and the absence of any reference to the decline indicated by *D–B*.

ter's wheel is heavy (like the flywheel of an old Ford) and therefore requires a good many impulses before it takes on high-speed motion. Further, once it is started, its momentum carries it on its revolution, and, like the Ford flywheel, it is very difficult to stop. This would be immediately suggested, by the phrase Plato uses, to anyone who had watched potters at work with hand-powered wheels; the increasing speed with each impulsion and the inevitable persistence, once started, of the moving wheel is far less self-evident to us, who buy our pottery ready-made, than it would be if we watched the potter wedge, throw, center, and shape his clay.

This brief passage constitutes the first of a balanced set of three *cyclic* images in the *Republic.* The inevitable upward spiral of the state, following the introduction of temperance as the result of improved education, exactly balances the inevitable cyclic downward spiral, presented in the nuptial number image of Book viii. The upward and downward spirals are balanced against each other, and combined into genuinely circular images and periods in the eschatological and cosmological imagery which in Book x presents the relation of cosmic justice and human freedom.

It is reasonable to suggest that for Plato there probably is an intentional resemblance between this metaphor and the description of the successive impulsions by which the cosmic spheres are given their irresistible momentum in the Myth of Er.

IV. The Divided Line

In dialectical illustration, as in lyric poetry, it frequently happens that many metaphors are developed simultaneously. It is this very simultaneous development of many significant metaphorical references that gives aesthetic interest to poetry; but in the attempt to carry out many image-class developments at once there is bound to be what might be called "interference of metaphor." The concrete image required for one line of thought may not coincide exactly with the image which would be most appropriate to another. A poet must then choose which line of imagery is to be primary in his development, and within that line retain as much relevance to the alternative lines of development as possible. The actual choice of imagery can be explained only by seeing the chosen development in its context of aesthetically relevant alternatives.

Suppose, for example, that a poet begins a work with the statement that "Josephine is like a violet. . . ." What possibilities does this establish for the next line? Violets are pretty to look at, seclusive in behavior, perfumed, ephemeral, wild, dressed in green, etc. The basic simile may be expanded in any of these directions, or in any variant of them suggested by the subjective associations of violets for the poet. The different possibilities of expansion are not, however, all compatible; for example, in the next line the author cannot combine the notions that "she will wither in the fall" and that "she is always fair to see." Permanent and transitory properties of these alternatives reciprocally interfere, and equal aesthetic emphasis cannot be placed on both. However, whatever line of development is chosen as central, the noncentral properties of violets remain in the background, constituting a context which colors (if in fact it does not establish) the "meanings" of subsequent images in the poem.

The development of metaphorical mathematical imagery renders itself most intelligible if we analyze it as analogous to a poetic enterprise, and here the phenomenon of interference may be even more marked, since the alternatives are even more clearly disparate, than it is in the work of the lyric poet. If, for example, four things in a set are unequal in respect to property P, but tightly connected by proportion in respect to a related property, Q, a linear representation of these four things as segments cannot adequately present P and Q as simultaneous principles of order. The metaphor of proportion would require the equality of two segments, whereas the metaphor of disparity in intensity would absolutely preclude it. The diagrammatician must then choose which of the two properties, P or Q, he intends to make central and which peripheral to his illustration. The peripheral property is relegated to the contextual background and is effective in the figure actually constructed as a felt, contextual alternative of relevant development.

A peculiarity of discussions to be illuminated by diagrams, however, is that the property which is central in importance may change as the discussion progresses. (Analogously, in a sonnet, the property chosen for expansion is often replaced by an alternative in the final quatrain.) An accurate diagram can reflect this change in emphasis by unfolding in sequential stages, so related that the property which is central at each stage of illustration is the one that was central in the corresponding stage of discussion. If, as may be the case, the schematic properties interfere, the unity of the schematism can be discovered only by analyzing the sequential stages of construction, not the final product of the process.

Actually, there are relatively few passages in Plato in which shifts in emphasis require the reader, having started with one central mathematical diagram, to shift to another. The suggestion of the existence of such passages is put forward with some reluctance, since it so easily lends itself to misuse. If we allow the concept of separate but connected figures to dominate our reading of mathematical passages, the way is open to bypass all intended relations that present problems of interpretation by treating each part of the diagram atomically, and radically disconnecting it from the others. It is precisely against this tendency to disconnect the "mathematical" passages from their contexts that the present study is entirely directed. Consequently, very careful controls must be set up to prevent this technique from being overused. In defense of its being used at all, however, the following two considerations seem conclusive. First, there are frequent simultaneous uses of several mathematical metaphors in Plato's text, and it is impossible that this should not produce some "interference"; second, there are passages in which incompatible specifications of figures are supplied. In addition, there are many cases in which diagrams develop from the same elements in sequential stages, as shifts in context suggest new interpretations of the relations of these elements; and this creates, a priori, the expectation that a similar device may be used in a single passage developing a complex image or set of images.

This use of a postulate of "interference" seems necessary and not harmful so long as two conditions, to be stated, govern its use. In the first place, a mathematical passage can shift its emphasis by including more than one figure only if the context it illustrates contains an analogous shift of emphasis in content. (In Section 6a of this chapter a more detailed discussion is given of the way to determine to what context the elements of an image correspond.) The several figures must have to one another a relation exactly analogous to that of the portions of context to which they correspond. In the second place, the notion of interference *should not* be employed unless the passages in question have produced antinomies of interpretation in the Platonic tradition. By this is meant that there are alternatives which, though inconsistent, are firmly grounded in the text and defended by scholars. If in such a case the incompatibility cannot be resolved, use of the notion of "interference" to account for the antinomy seems justified. These two conditions insure that there will not be an evasion of real problems by alleging incompatibilities where they are not explicable from the context illustrated, and not already discovered by students of the texts in question.

As an instance of sequential evolution of imagery made up of the same elements, the shifted meaning of an image of concentric circles in the contexts of the *Republic-Timaeus-Critias* trilogy, discussed in Appendix A, is a concrete case in point.

The discussion of interference has been presented here because the divided line is the first case of a geometrical image in which the use of interfering metaphors appears.

Republic 509–11 *

Now take a line which has been cut into two unequal parts, and divide each of them again in the same proportion, and suppose the two main divisions to answer, one to the visible and the

* Trans. Jowett, *Dialogues*, III, 211–13.

other to the intelligible, and then compare the subdivisions in respect of their clearness and want of clearness, and you will find that the first section [*AB*, Fig. 43] in the sphere of the visible consists of images. And by images I mean, in the first place, shadows, and in the second place, reflections in water and in solid, smooth and polished bodies and the like: do you understand?

Yes, I understand.

Imagine, now, the other section [*BC*, Fig. 43], of which this is only the resemblance, to include the animals which we see, and everything that grows or is made.

Very good.

Would you not admit that both the sections of this division have different degrees of truth, and that the copy is to the original as the sphere of opinion is to the sphere of knowledge? [i.e., $AB : BC :: CD + DE : AB + BC$]

Most undoubtedly.

Next proceed to consider the manner in which the sphere of the intellectual is to be divided.

In what manner?

Thus:—There are two subdivisions, in the lower of which [*CD*, Fig. 43] the soul uses the figures given by the former division as images; the enquiry can only be hypothetical, and instead of going upwards to a principle descends to the other end. In the higher of the two [*DC*, Fig. 43], the soul passes out of hypothesis, and goes up to a principle which is above hypothesis, making no use of images as in the former case, but proceeding only in and through the ideas themselves.

I do not quite understand your meaning, he said.

Then I will try again; you will understand me better when I have made some preliminary remarks. You are aware that students of geometry, arithmetic, and the kindred sciences assume the odd and the even and the figures and three kinds of angles and the like in their several branches of science; these are their hypotheses, which they and everybody are supposed to know, and therefore they do not deign to give any account of them either to themselves or others; but they begin with them, and go on until they arrive at last, and in a consistent manner, at their conclusion?

Yes, he said, I know.

And do you not know also that although they make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble; not of the figures which they draw, but of the absolute square and the absolute diameter, and so on—the forms which they draw or make, and which have shadows and reflections in water of their own, are converted by them into images, but they are really seeking to behold the things themselves, which can only be seen with the eye of the mind?

That is true.

And of this kind I spoke as the intelligible, although in the search after it the soul is compelled to use hypotheses, not ascending to a first principle, because she is unable to rise above the region of hypothesis, but employing the objects of which the shadows below are resemblances in their turn as images, they having in relation to the shadows and reflections of them a greater distinctness, and therefore a higher value.

I understand, he said, that you are speaking of the province of geometry and the sister arts.

And when I speak of the other division of the intelligible, you will understand me to speak of that other sort of knowledge which reason herself attains by the power of dialectic, using the hypotheses not as first principles, but only as hypotheses—that is to say, as steps and points of departure into a world which is above hypotheses, in order that she may soar beyond them to the first principle of the whole; and clinging to this and then to that which depends on this, by successive steps she descends again without the aid of any sensible object, from ideas, through ideas, and in ideas she ends.

I understand you, he replied; not perfectly, for you seem to me to be describing a task which is really tremendous; but, at any rate, I understand you to say that knowledge and being, which the science of dialectic contemplates, are clearer than the notions of the arts, as they are termed, which proceed from hypotheses only: these are also contemplated by the understanding, and not by the senses: yet, because they start from hypotheses and do not ascend to a principle, those who contemplate them appear to you not to exercise the higher reason upon them, although when a first principle is added to them they are cognizable by the higher reason. And the habit which is concerned with geometry and the cognate sciences I suppose that you would term understanding and not reason, as being intermediate between opinion and reason.

You have quite conceived my meaning, I said; and now, corresponding to these four divisions, let there be four faculties in the soul—reason answering to the highest, understanding to the second, faith (or conviction) to the third, and perception of shadows to the last—and let there be a scale of them, and let us suppose that the several faculties have clearness in the same degree that their objects have truth.

I understand, he replied, and give my assent, and accept your arrangement.

Republic 534A *

We shall be satisfied, then, with the names we gave earlier to our four divisions: first, knowledge; second, thinking; third, belief; and fourth, imagining. The last two taken together constitute the apprehension of appearances in the world of Becoming; the first two, intelligence concerned with true Being. Finally, as Being is to Becoming, so is intelligence to the apprehension of appearances; and in the same relation again stand knowledge to belief, and thinking to imagining. We had better not discuss the corresponding objects, the intelligible world and the world of appearance, or the twofold division of each of these provinces and the proportion in which the divisions stand. We might be involved in a discussion many times as long as the one we have already had.

Scholion from Iamblichus

The initial bisection of the line, whether it is divided into equal segments . . . as Iamblichus says, or into unequal, as some commentaries say, represents an initial division of all things into two classes, intelligible and sensible. If the segments are equal . . . then the relation of things participated in to those participating is similar, and the proportion [which is the next stage of Plato's construction] applies in the same way to both. If unequal, as others say, then the relation of intelligibles to sensibles will also be in excess [of a 1 : 1 ratio], and dissimilar. . . . [The remainder of the scholion recapitulates the proportions of the

* Trans. Cornford, *Republic*, p. 254.

divided line, and the things which are represented as objects of knowledge by each of the four segments.] *

Evidently, these commentators assume that the greater reality and clarity are to be represented by the *longer* line. If the interpretation of the *De Anima* figure given below is correct, this is another proof that no continuous tradition connects the figures of Hellenistic scholars with Plato's original designs.

The point at issue is whether the analogies are to be construed as representing only relations, in which case those of intelligibles and sensibles are similar (i.e., mathematics : imitations of forms : : shadows : imitations of sensible objects), or whether the difference in status of relata is also reflected in the properties of the figure. Our own contemporary practice seems to be to draw the diagram in one way, then interpret it in the other.

The figure referred to here has properties which cannot be combined by any geometrical construction. Either we can make all four of the segments unequal, or cut the two divisions in the same ratio; but not both.[23] Since the relation schematized seems too complex for a single figure, two figures are in effect needed: the one would bring out differences in clarity and adequacy of knowledge by differentiating lengths of segments; the other would bring out analogical relations between knowledge of these kinds by proportioning lengths of segments. Evidently this is a case of interference; the trans-spatial, trans-temporal characteristics of the forms lend themselves rather inadequately to spatialized representation. This double representation parallels a contextual shift in emphasis. To establish the character and proper wisdom of the philosophic rulers in Book vi, it is necessary to *contrast* their knowledge to the erratic opinions and hypotheses of nonphilosophic statesmen. But to achieve a curriculum for the training of these rulers, in Book vii, it is necessary to provide a transition based on the *resemblance* of knowledge of each kind.

The metaphor of unequal segments was used by Plato else-

* *Appendix Platonica*, ed. C. F. Hermann (Leipzig, 1885), pp. 350–51. Other scholia will be found with the figures at the end of this section.

where to underscore the *contrast*,[24] but if this contrast is the final word that can be said, and if we construct the divided line with all segments unequal, the only interpretation one can give the figure is that of a Stoic or Parmenidean division of the ignorant from the wise; and education seems impossible. In fact no interpreter, in the face of the stress on proportions in the passage, has had the temerity to propose this as the intended construction. On the other hand, if the reference to inequality is functional and deliberate, its function must be to call attention to an alternative line of metaphorical development, needed to understand the relation of the line chosen to the full situation, one aspect of which it characterizes. It is the relative position, not the relative length, of segments in the proportion figure on which the mathematical statements of analogy are based. "Higher" and "lower" are the key spatial concepts in this analogical interpretation. Actually, this is so evident in context that relatively few Plato students have been bothered by the fact that if we interpret the proportions as relating absolute lengths, the second and third segments of the line come out equal. They are clearly unequal in respect to higher-lower position.[25]

The metaphor of inequality seems intended, in its other uses, to represent a distance between the knower and the object of knowledge; the longer the segment, the greater the distance so represented, and the more diffuse the unity of the form to be apprehended. On this basis, the longer segment must represent the less adequate knowledge, and insofar as the metaphor of inequality is retained (in the final figure) in its initial division into unequal segments, the longer segment is not honorific, but significant of a greater negative factor. (Similarly, the Pythagoreans tended to assign greater value to small rather than large numbers.) [26] This consideration is decisive in determining the relative lengths of the "knowledge" and "opinion" segments of the constructible proportion figure, and it is hard to see how any editor could have thought that a modern metaphor involving the notion "the bigger, the better" could conceivably have been what a Greek of Plato's time intended.

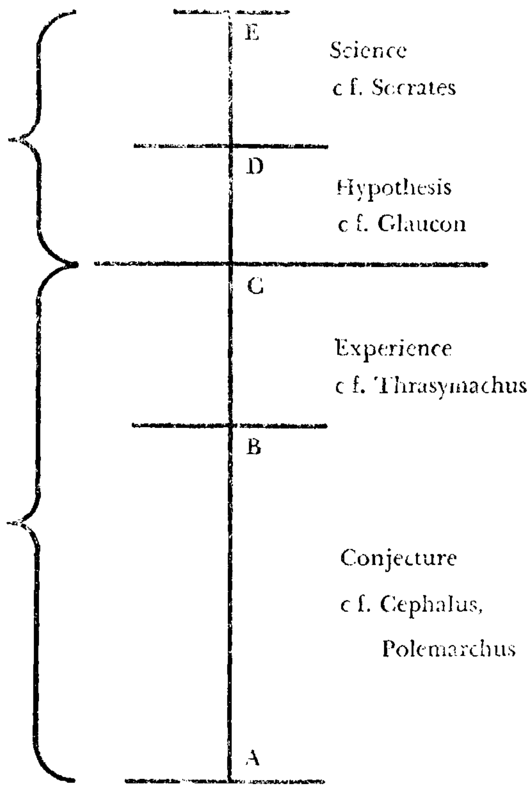
This last paragraph reminds us that the interference of the two principles of spatial representation is not a complete one, for as between the major divisions of the proportion figure, and within each division, the metaphor of inequality may be, and is, retained without geometrical impossibility.[27]

Since the ratios given will work equally well in terms of higher-lower if the segments are made all the same length, the figure is often drawn in this way; and it will be suggested that Plato himself so represents it in later synthetic imagery, where stress is only on the *number of stages* relevant to classifying kinds of wisdom. However, this representation, though it brings out the similarities of different kinds of knowing central to Book vii, seems to ignore the context of Book vi, in which the total difference in kind between these sorts of knowledge is underscored.

In summary, then, the present contention is that two alternative diagrams are relevant to schematizing the differences and similarities of kinds of knowledge at a point of transition at which the present figure appears. On the one hand, the relative separation of knower and known can best be represented by a schematization in which unequal lengths characterize knowledge of each kind, and the relative length of the segments is the property underlying the metaphor (which is one of *distance*). On the other hand, analogical connection can best bring out the common nature of all knowledge, which is fundamental to the scheme of education, and this analogy is based on the metaphor of the relative position of segments in respect to height on the line. The two figures cannot be completely combined; at best, the property of inequality of length can be transferred only to some of the proportionate relations. The metaphor of proportion is chosen for full development because, for the transition at hand, it is the most important. The metaphor of inequality is, however, retained in the figure as a reminder of a needed qualification, which Book vi has made clear. The situation is somewhat analogous to the relation of temperance and justice in the state; the similarities between knowledge of different kinds must be seen and the differences not confused.[28]

The vexed question of the relation of the two upper segments of the divided line, in terms of an adequate determination of the relations of concepts and schematisms, will not be separately discussed here.[29] The reason is that this entire present study of Plato's use of schematisms to explain and illuminate concepts is a philosophic inquiry into precisely that problem, and nothing briefer would be adequate to such a discussion.

Figure 13



THE DIVIDED LINE

This figure is lettered to correspond to the references in the preceding text. In parentheses are indicated the names of speakers who have appeared earlier in the dialogue and who seem to have knowledge of the nature of justice at each level of adequacy. There is no clear agreement about what exactly constitutes "conjecture";

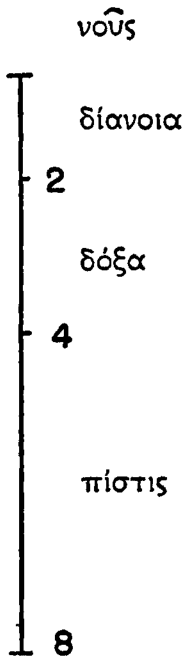
it may be knowledge in the sense of the Kantian manifold of representation; or it may be, in view of the stress on images and reflections, that Plato had in mind the sort of thinking analyzed by Vergil C. Aldrich in his article "Pictorial Meaning and Picture Thinking," *Kenyon Review*, 5, 1943 (pp. 175–181 in Feigl and Sellars, *Readings in Philosophical Analysis*, New York, 1949). In the latter case, the criterion of meaning would be imaginability, the inadequacy a lack of empirical verification of the image constructed; in the former, the inadequacy would lie in the absence of any concept to unify the subjective representations into objects. The second segment of the line, from Plato's later account of the game of guessing at sequences of shadows in the cave, seems quite clearly to involve a pure empiricism, similar to Hume's interpretation of causality (as an expectation, based on past experience, that impressions which have been contiguous in space and time will be so in the present case). The third segment is quite clearly a kind of axiomatic, deductive knowledge best exemplified in mathematics; by suggesting in the figure that Glaucon may be said to have had in mind knowledge of this type in his explanation of human behavior by a deductive development of the social contract theory, I intend to suggest that any subject-matter is capable of such axiomatic formulation, the proper excellence of which is its coherence in bringing out relations and causal connections, the weakness of which is that such coherence never guarantees the rightness of the axioms (or at least does not seem to a Platonist to do so). In that case, the final segment of the line shares the clarity of deductive structure with the third, but is based on "better" hypotheses; better either in the sense that there is no possibility of deducing a contradiction from them (Plato will later say that the man who has knowledge of the good must be able to discuss it "without tripping"), or that they take account of a wider range of data to be explained.

The representation in Figure 44 brings out the differences in clarity of knowledge of these types but loses the similarities basic to a system of education. (For example, the ratio of knowledge to opinion in this figure is 6:1; that of conjecture to experience, 2:1.)

The figure is described by Plato in the *Laws* to show the diffusion of causal efficacy from the form to its space-time embodiment.

In *De anima* 404b, Aristotle points out that "[Plato] says . . . that the universe is derived from the idea of the one and the first

Figure 44



ALTERNATIVE CONSTRUCTION OF THE DIVIDED LINE, CARRYING OUT THE INEQUALITY OF SEGMENTS

length, breadth, and depth, and all other things in the same way. And further, νοῦς is one, and ἐπιστήμη is two; for the latter is generated from the one; and the number representing δόξα is a plane, that representing αἴσθησις a solid; and he [Plato] says these numbers are the forms and principles, but they are composed of elements. Sometimes things are known by reason, sometimes by understanding, sometimes by opinion, and sometimes by perception; but the form of things is these numbers."

Compare *Laws* 894A: "What must happen to anything to cause its generation? It is as if a principle being augmented comes to a second type of change, and after this goes on to the next, becoming perceptible through this third."

In each of these analogies, the notion of a diffusion of a principle from its existence in an act of noetic insight to its embodiment in a sensible medium is represented on the analogy of an arithmetical progression. In the passage from Aristotle, this progression is $1:x:x^2:x^3$. In the *Laws* passage, the same notion of different dimensions separates the "increases" of each type.

V. Mathematics in Higher Education

The discussion of the mathematical curriculum in *Republic* vii is more relevant to a treatment of the philosophy of mathematics or of the history of mathematics than to the present enterprise of interpreting mathematical imagery.

However, certain comments indicating the relevance of this curriculum are in order. In the first place, whatever the basic philosophical detail intended, the statement is clear that objects studied by the mathematician are the images of the ideas investigated by the dialectician.[30] Apparently the idea, in an understanding of which there is a grasp of some perfection or function, has associated with it a quantitative schematism or structure, in which an image of the form itself can be recognized, and through which the form would be embodied in space and time [31] as copies of it were constructed. (This is very reminiscent of the rôle of the Kantian schemata, through which connection is established between the concepts of the understanding and the forms of intuition.) The theoretical statement corresponds exactly to the relation of dialectic and mathematics that we find Plato using in practice. The various relations of ideas are schematized in mathematical imagery, with a transposition of their nature into that of quantitative analogues, to illustrate and clarify a contextual, dialectical method or schema.

A second point important for the present investigation is that the final mathematical study is harmonics, a study of the concord and discord of numbers themselves.[32] The concords of music will furnish images and illustrations of this type of mathematical structure.[33] As Socrates suggests in a later remark, the goal of this education is to enable students to see things "synoptically," and the order of the curriculum is one in which

Figures from Scholia

(Greene, *Scholia Platonica*, pp. 245–46, 253)

Figure 45

Republic 510B

3	4
Mathematics from hypothesis to final conclusion through sensible images	Dialectic from hypothesis to non-hypothetical principles through the forms themselves
διανοια	νους

Figure 46

Republic 508A

Good	Sun	Sun's Light	Darkness
Νοῦς	Sight	Truth	Likeness
Intelligibles	Sensibles	Knowledge	Opinion
Place of Forms	Place of Visibles	Being	Becoming
Knowledge	Vision	Sight	Blindness

Figure 47

Republic 510D

Vision Opinion Sun		Knowledge Intelligence The Good	
Shadows Conjecture	Nature Opinion	Mathematics Understanding	Divine Things Knowledge
Becoming		*Being*	

Figure 48

Republic 534A *

R	1	3	O
E	KNOWLEDGE	BELIEF	P
A	BEING	BECOMING	I
S	UNDERSTANDING	CONJECTURE	N
O	2	4	I
N			O
			N

* Rows of dots represent horizontal brackets in the original figure.

each successive science presupposes the principles of the one preceding, but integrates them with new attributes peculiar to its own treatment.[34] In this order, harmonics comes last, mediating the transition from mathematical to dialectical study. Although he goes into some technical detail in specifying the method and content of the other curricular studies, Socrates dismisses this final subject with no technical information other than the statements (1) that its images are the concords of music and (2) that it must not be modelled on the empirical study of music of various instrumentalists and Pythagoreans. Consequently, a good deal of commentary has been devoted to remedying this brevity of technical treatment, much of it centering on the problem of what mathematical concept "numbers concordant and discordant in themselves" can possibly mean. The explanation for this lack of technical description is relatively easy, however; Socrates goes on to a detailed discussion of "dialectic," and there is almost nothing to be said about harmonics to which his subsequent remarks do not apply. A difference in principles (quantity in the one case, nature in the other) is the only point of separation between the study establishing mathematical concords and the establishing of harmony in states and souls, which is the politically relevant objective of the course in dialectic in this plan of higher education. After

Figure 49

PRINCIPLES OF THE MATHEMATICAL SCIENCES

Subject Matter	Principles	
	SAME	OTHER
Sets of units	prime factors	composite factors
Numbers	odd	even
Figures and solids	circle & sphere	straight
Straight figures and solids	regular	irregular
Motions	rotation	translation
Translative motions	regular	irregular
Proportions	geometric	nongeometric
Nongeometric ratios	harmonic	arithmetic

the study of the structural features of dynamic systems in astronomy, one may safely conjecture, in view of the ideal of synopticity, that the interrelations of numbers, figures, solids, and motions already studied will be taken as one example of systematic organization.

It is entirely in keeping with the important place given to analogies of harmony throughout the *Republic* that a study of the theory of harmonic structures be the final point of a student's instrumental education, immediately preceding the more complex inquiry into the conditions by which harmony can be realized in human careers and in the administration of cities.

VI. The Nuptial Number

a. Introductory Remarks

A good schematic illustration of a discussion must meet two conditions: in the first place, there should be no doubt as to what the geometrical or arithmetic structure of the schematism is; and, in the second, there should be some clear indication of what element of the discussion each part of the illustration is intended to represent. In our contemporary use of diagrams, there are typically two distinct vocabularies at work: a mathematical vocabulary of *construction,* and a nonmathematical vocabulary (taken from context) of *interpretation.* If, however, the mathematical vocabulary has been borrowed from nonmathematical fields, such as biology or physics, and if a feeling for the basic sense of the borrowed term is still operative, there is evidently no need in every case to separate the two vocabularies of interpretation and construction. A modern mathematician, applying topological concepts to neurology, might well speak of "nerves" meaning both the technical topological element establishing a mutual relevance of areas, and the physiological element supplying connections between parts of an organism. Insofar as he had borrowed his nomenclature from biology because topological and biological "nerves" have the same basic properties, his metaphorical terminology would work equally well whether it were the abstract aspect of the diagram or the concrete function of its intended interpretation that the reader supplied.[35]

This point seems fairly evident when we take a very fresh mathematical term (and we would not be surprised to find the medical topologist giving a diagram of "nerves" without a separate vocabulary of interpretation appended), but as mathematical terms become more shopworn and are more frequently used in their exclusively structural sense, their nonmathematical reference is deliberately and effectively bleached out of them, so that a term of long standing will not be felt to carry with it any suggestion of interpretation. This point, of course, has been already suggested in the Introduction, where the postulate of a tradition of mathematical metaphor in Plato's time is presented.

What is peculiarly relevant, however, to the passage about to be interpreted is more than this general sense of metaphor in the mathematician's vocabulary: it is the ability of a given mathematical term to indicate at the same time an intended construction and an intended interpretation.

This use of terms with a deliberate selection which makes them significant on any of several levels of meaning is, of course, an aesthetic device that is familiar to contemporary authors and readers.[36] It is only in an exceptional case, however, that a contemporary scientist or mathematician would feel at all comfortable when confronted with it. Such an exception, if it occurs, will do so with a newly appropriated term which still retains a felt extra-technical reference.

However, the mathematical vocabulary of Plato's time had very recently appropriated words from the general vocabulary for its special needs of designation. With this in mind, it does not seem unlikely that a writer of the period might make a single term do the work both of technical specification and interpretation. Why, for example, would it be necessary to say that an "irrational" line is like an "irrational" man, when the resemblance is already embedded in the etymology of the name of the line? If the metaphor that seems natural to the mathematical giver-of-names still remains apt for the later constructor of diagrams, there may be no need felt to append a key to interpretation, since the metaphors embodied in the technical names themselves provide an adequate statement of the interpretation which these structural elements are meant to represent.

This practice of making names do double work carries over a feeling of affinity between figure and interpretation, and probably gives a reader in the same tradition a sense of natural fitness or translucence in the construction of the diagram.

It may also explain the intended interpretations of passages where descriptions of construction are given, with no separate dictionary of interpretations appended. The suggested rule of interpretation for dealing with this device is that if no indication of interpretation is given, a term occurring in a technical mathematical sense in a construction may be intended to stand for the concept represented by that same term when it occurs in a less restricted sense in the relevant context.

b. Problems of Translation and Paraphrase

Prologue to the Number *

In what way, then, will our city be moved, and in what manner will the two classes of auxiliaries and rulers disagree among themselves or with one another? Shall we, after the manner of Homer, pray the Muses to tell us 'how discord first arose'? Shall we imagine them in solemn mockery, to play and jest with us as if we were children, and to address us in a lofty tragic vein, making believe to be in earnest?

How would they address us?

After this manner:—A city which is thus constituted can hardly be shaken; but, seeing that everything which has a beginning has also an end, even a constitution such as yours will not last forever, but will in time be dissolved. And this is the dissolution: —In plants that grow in the earth as well as in animals that move on the earth's surface, fertility and sterility of soul and body occur when the circumferences of the circles of each are com-

* Trans. Jowett, *Dialogues*, III, 803.

pleted, which in short-lived existences pass over a short space, and in long-lived ones over a long space. But to the knowledge of human fecundity and sterility all the wisdom and education of your rulers will not attain; the laws which regulate them will not be discovered by an intelligence which is alloyed with sense, but will escape them, and they will bring children into the world when they ought not. . . .

Translation of the Number

No "neutral" translation of the rest of this passage can be found. In the two translations that follow the literal sense most relevant to interpretation is presented in the first, and in the second, the meaning most relevant to mathematical construction. Both these levels of meaning operate in the passage simultaneously. An appended note gives the translation which results if the text as accepted by Proclus and later editors (except Martin) is used.

Level I: Interpretation *

For divine becoming, there is a period comprehended by a perfect number; but for human, by the first in which developing capacities, dominating and dominated, on realizing three stages determined by four points [in the field of these processes of] becoming like and unlike, growing and declining, make all things conversable with and rational in respect to one another. Of these [the element representing referents in] a ratio of 4 to 3, in lowest terms, married to the pempad, produces two harmonies when thrice augmented. The one [harmony] is a resultant of equal components, the same arithmetically in each dimension; the other, while equal to the first in one dimension, is unequal in the other. Each side of the former [equals] numbers produced by squaring "the diagonal lines" that represent a "rational" component of the pempad, each diminished by one; [the other side equals numbers produced by squaring the "diagonals" that represent a component of the pempad which is] "irrational," [each of these diminished by] two. The latter [harmony] equals one hundred cubes of three. And this whole thing is a geometrical number. . . .

* General meanings are emphasized here.

Level II: Construction *

. . . but for human, by the first [number] in which increases of roots and squares, when they reach three stages with four bounding points [in the field of processes] of becoming like and unlike, increasing and decreasing, show all things related and in integral ratio to one another. Of these, [that part represented by] the integers 4 and 3, joined to five, produces two products when made three-dimensional. The first is square, equal in each dimension; the other is equal to the square in one dimension, but rectangular. Each side of the first [equals] numbers [representing] squares on rational diagonals of five, each minus one; or irrational, [each minus] two. The second is one hundred cubes of three. And this whole thing is a number summarizing the dimensions of a figure. . . .

If we accept the version of the text given below in the notes to the Greek text of this passage, the translation is changed:

The first harmony becomes "equal to 100*x* in each dimension," instead of "the same arithmetically in each dimension." This harmony has its side produced from "one hundred numbers [representing] squares on rational diagonals of five, each minus one, or irrational, each minus two" instead of "Each side of the first [equals]. . . ."

Republic 546A–D

(Chambray, *République*)

Ἔστι δὲ θείῳ μὲν γεννητῷ περίοδος ἣν ἀριθμὸς περιλαμβάνει τέλειος, ἀνθρωπείῳ δὲ ἐν ᾧ πρώτῳ αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι, τρεῖς ἀποστάσεις, τέτταρας δὲ ὅρους λαβοῦσαι ὁμοιούντων τε καὶ ἀνομοιούντων καὶ αὐξόντων καὶ φθινόντων, πάντα προσήγορα καὶ ῥητὰ πρὸς ἄλληλα ἀπέφηναν· ὧν ἐπίτριτος πυθμὴν πεμπάδι συζυγεὶς δύο ἁρμονίας παρέχεται τρὶς αὐξηθείς, τὴν μὲν ἴσην ἰσάκις, ἕκαστον τοσαυτάκις, τὴν δὲ ἰσομήκη μὲν τῃ, προμήκη δέ, ἕκαστον μὲν ἀριθμῶν ἀπὸ διαμέτρων ῥητῶν πεμπάδος, δεομένων ἑνὸς ἑκάστων, ἀρρήτων δὲ δυοῖν, ἑκατὸν δὲ κύβων τριάδος.

ἕκαστον AF: ἑκατὸν A²Proclus
ἕκαστον AF Pap. 4: ἑκατὸν A²

* Here specifically mathematical meanings are taken.

(A = cod. Parisinus 1807; A^2 = later correction of A; F = cod. Vindobonensis 55: Pap. 4 = Papyrus Oxyrhynchus XV 1808, column 4.)

If Plato had anticipated the controversies and interpretations occasioned by this passage explaining the principle of political decline, he would probably also have anticipated the example of some of his recent translators, and deleted it from his text.[37] For the purpose of a mathematical image in Plato's writing is to schematize and clarify dialectical relations in the context, and if it fails to do so, it is dialectically nonfunctional. Although this conclusion follows immediately from an examination of the entire set of mathematical illustrations used in the dialogues, accidental factors have combined to produce a few specific cases which are apparent exceptions. These exceptions have been so disproportionately stressed that it is they which have led to frequently repeated scholarly generalizations about Plato's use of illustrative problems and diagrams. It must seem paradoxical that the author capable of describing and explaining the diagram of the divided line in Book vi of the *Republic* should by Book viii have decided to "throw a little mathematical dust in our eyes," with an elaborate enigma designating some simple biological period, or signifying nothing; or that the author who worked out the sequential development of the argument of the *Republic* could not resist interpolating a compressed, technical, and digressive footnote on cosmological embryology at a point of transition in his text where a summarizing diagram of the antecedent discussion would have been more naturally called for.

c. The Philosophy of History

There is no doubt that Plato's imagery here is intended to be applied to human genetics as cause of the decline of the state and is to be considered a concrete example illustrating a general principle of the philosophy of history. But the moral of the philosophy present in this passage, as contrasted with the earlier remark about the inevitable cyclic progress of the state instituting educational reform, comes as something of a shock to the tradition that Plato is an arch-conservative, and stands in apparent opposition to his repeated prohibitions of innovation in the arts. The inability of the rulers to master completely the recalcitrant auxiliary causes which upset the clear mathematical genetic theory is typical of the basic moral, that no rigid attempt at retention of form can endure in time. Adaptation, not crystallization, of institutions and classes seems to be the precondition of progress. (This of course is why the rulers in each generation must know the nature of the good, so that they do not confuse tradition and value.) Just as cosmic process leads each living organism to overshoot its mature acme in the trajectory of a life cycle, so the flux of time and change will deform any structure literally imposed in its plastic medium, and here in the *Republic,* as well as in the later dialogues, some intermediate tactics must intervene to secure the preservation of intelligible structures in their embodiments as objects in the world of change. If we read Plato's strictures on innovation as attempts to avert changes in tactics rather than as attempts literally to embalm specific institutional structures, the contradiction between the notion of a life-cycle of the state and the ban on innovation seems to disappear.

d. The Rulers' Problem

The basic problem of the rulers in planning marriages is presumably not (as many interpreters have thought) astrological, since it has already been decided that dates of marriage festivals are to be set whenever needed to maintain the stable population of the state. Neither is it likely that the problem referred to here is that of determining the proper age for marriage, for simple observation of normal human maturing has already lead to a definite decision on that point. Nevertheless, the context clearly indicates that this problem is closely associated with some property of the human life-cycle. The nature of this problem is suggested by a comparison of earlier specifications in the text of the respective ages of procreative and intellectual maturity. The stages of human growth are such that couples must be selected to marry and have children before they can have completed the training and selection which finally determine who are best fitted by nature to be rulers. Nature thus forces the hand of the arrangers of marriage, since their calculations can be based only on the estimated future success of the couples they select. By the time they are old enough to marry, the auxiliaries will have been tested and graded on the basis of their elementary and military-mathematical education—in terms, in other words, of their demonstrated abilities in control of appetite and obedient exercise of spirit. How reasonable they will become as they mature intellectually can be decided only by scoring their performance in the course of theoretic dialectic and applied dialectic (civil administration), but by the time this training is completed, they are too old to have healthy children.

The rulers, therefore, must decide suitability of parents on the basis of demonstrated military and mathematical ability, which may not be a fair index of intellectual and administrative capacity passed on to the children. Under these circumstances, practical problems of genetics as well as theoretic considerations of possible constitutions lead to the conclusion that later generations of rulers may be born who will be too congenitally dominated by spirit to carry on the tradition of responsible rule in the aristocratic state.

e. Why the Muses Speak Playfully

The playfulness of the Muses may be taken, together with the mathematical joke in the *Statesman*,[38] to provide instances of what might be called Plato's "mathematical irony." When history or immediate experience is schematized mathematically, there is a great discrepancy between the neatness of the theory and the confusion of our perception of the immediate fact to which it applies. There is an ironic inappropriateness in the formulation of a neat textbook of genetics and eugenics which is sure to be empirically inapplicable and transitory in human history. There is a "tragic over-pretentiousness" in developing an elaborate intellectual schematism of phenomena which, because of their inconstant character, will not behave as schematized. This does not mean that such schemata are false or not useful; they are in fact the only way in which the interacting factors of a complex situation can be intelligibly exhibited and analyzed. But the Muses' neat schematic structure of the causes of social degradation, though understandable and true, still leaves its possessor helpless to attribute to data dependent on perception and calculation an accuracy that these do not possess, and hence helpless to avert the schematized historical progression.

f. Detailed Interpretation

A paraphrase of the passage follows. Numbers in parentheses refer to passages in which the statements made in the paraphrase are discussed in detail. These passages are identified by means of numbered side headings.

Indexed Paraphrase

The various factors which interact to determine the nature of a human life are not related as invariant, proportionate recurring cycles, as the motions of the heavens are. (1) Although calculation and perception are never absolutely accurate, astronomic calculation of celestial periods attains considerable accuracy; but for human maturity and reproduction, the simplest logistical representation must take account of a process of growth (as opposed to simple locomotion) in which potentialities are realized in different orders of dominance, a realization taking place in constant interaction with a fluctuating environment. (2) When this process of growth has attained four successive points, through three stages (like a line of three segments, divided and bounded by four points), (3) the organism that has grown and the number representing it will constitute a rational and commensurable pattern. (4) This rationality is attained at the acme of the period involved.

Two proportions which determine the periods underlying marriage regulation (5) are produced from (6) the joining of the line representing the three stages of growth of this organism to another line, in 4:3 ratio, by a line of length 5 (7). (The joining thus produces a figure uniting the original line of three segments with the two other lines.) (8) The image which is a geometric representation of these two proportions has one square and one rectangular plane surface. (9) The length of the side of the square surface is equal (numerically) (10) to (11) (the number representing) (12) the square of (13) the "rational diagonal (14) of five" (15) minus one (16); and this length is also equal to the number of the square of the "irrational diagonal" (17) minus two (18). [This completes the specification of the square image (19).] The rectangle is equal to the square in one dimension (20); the "rectangular harmony" is represented, arithmetically, as 27 x 100 (21).

This whole number, summarizing the proportionate relations of the geometrical image which specifies the number's interpretation, gives the formula or pattern of the interacting factors which the rulers must apply, by calculation and perception, to each individual couple in their arrangement of marriages; and if they make mistakes, the children born are not naturally well endowed and do not have good fortune.

1. *Divine vs. human generation.* The basis of the μὲν-δὲ contrast between periods is not only that one cycle is divine and the other human, but, more important, that the former can be reliably observed and calculated (and its analogy to the "concordant ratios" of harmonics is recognizable, so that the astronomer can see these periods comprehended by a τέλειος number), whereas the human periods cannot be observed or calculated so adequately because the number comprehending *them* involves a multiplicity of processes and factors which require elaborate geometric schemata, far removed from the simple proportions of music. The *common* feature of divine and human becoming —its periodicity—is reintroduced and emphasized again at the end of the number; but until that reintroduction, the stress is not on some *proportional connection* or *similarity* of the two periods, but on their totally *different* degree of observability and regularity.

2. *Mathematical metaphors of growth.* ". . . αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι" introduces a mathematical metaphor which is not associated with a specific diagram until later in the

passage. The period of human life is a process of growth in which potentialities are realized, and these realizations then serve as potentialities for further development. The most relevant aspect of this process for the rulers to observe is the *growth* to the *acme* of the period, but the two moments that complement each other in the complete cycle are noted in αὐξόντων καὶ φθινόντων. The contrasted pairs of participles emphasize the fact that this cycle takes place in a matrix of qualitative and quantitative change. The relation of "assimilation and dissimilation" to "growth and decay" is one of external to internal relations, which recalls the description of growth in the *Timaeus* (in which the flux pouring through the body disrupts the natural movements of the soul, while at the same time a more intricate and better organization emerges as the organism grows and progressively disengages itself from the immediate impact of this flow).[39]

3. *The line of three segments.* On encountering this phrase, the reader of Plato's time, who did not expect his text to contain illustrations, would probably have reached for his tablet and stylus and noted the indicated "figure." ". . . τρεῖς ἀποστάσεις . . ." is precisely the sort of *direction for construction* that Plato incorporates in mathematical passages to assist the reader in visualizing or drawing intended diagrams.[40] A more extended "direction" or "description" of the same kind is given by the phrase discussed in (9), following.

4. *Conversable and commensurable.* The full development of the period to its *acme* is marked by the emergence and dominance of intelligence. (This also parallels the account in the *Timaeus,* in which the study of philosophy and cosmology are intended eventually to re-establish the circles in the soul in their proper natural functioning.)[41] The "things" which are "rendered conversable and commensurable" are both the structure of the organism, which attains full actualization, and the relations of the objects of knowledge which are seen in their true relation only by an intelligence fully perfected and "akin," but which exist in and constitute such an intelligence.

A further mathematical metaphor is also introduced by this phrase: proportions and commensurables are related to roots in the same way that intellectual insight or maturity is related to native capacity.

5. *The two harmonies.* A detailed schematism, based on the mathematical metaphors introduced generally in the first part of the passage, now begins to emerge. Just as the number comprehending the period of divine becoming is a product of its constitutive proportionate subperiods, so the number of human generation will be. In both cases, the final "number" is dependent for its significance on the geometrical and chronological relations which it summarizes; [42] the preliminary statement has already indicated how much less exactly these can be determined in human affairs than in astronomy.

6. *The mode of production of the harmonies.* These periods, or "ratios," are the numerical representation of a three-dimensional diagram which is "produced from" the triangular image discussed in (7) and illustrated in Figure 53, following. They are produced from it in the sense that the same *elements* are recombined in its construction, and the same basic *relationships* illustrated by it. Hence Aristotle feels that he can ignore this later elaboration completely in formulating a criticism applying generally to the *elements* and *principles* of "Plato's theory of revolutions." His paraphrase, "meaning by τρὶς αὐξηθείς when the number of the diagram is solid [στερεός] . . ." simply indicates that the later development leaves unaltered [43] the principle being criticized.

The actual mode of production, however, is by a *combinatorial* technique elsewhere applied to elements by Plato (most literally and spectacularly in the stereometrical element theory of the *Timaeus*).[44] Since the real elements in this problem are the factors in human generation, these must be held constant, but the same necessity need not apply to the mode of their quantitative schematization.

7. *Altitude of the acme and the 3–4–5 triangle.* The initial part of his passage having indicated two aspects of human generation—the process which constitutes progress toward maturity, and the insight which constitutes its natural acme—

Plato borrows an earlier image by which the height of this acme was measured in Book vi, and joins it to the "three-segment" representation of growth already introduced in the passage. The hypotenuse of 5, which connects the extremities of these segments, illustrates a simultaneous progress toward maturity and intelligence (see Fig. 53) that anticipates the discussion of the five modes in which actual maturity may be related to man's "natural" perfection, reflected in individual character and social organization (see Figs. 51 and 52). Both capacity and maturity may fall short of this natural human perfection; the successive dominance of higher parts of the soul may never be established, or the strength of the higher faculties may not be great.

The triangular image represents "character" as a function or, to follow Platonic metaphor, as "the offspring" of intellectual ability and psychological normalcy. (An "abnormality" is the continued *domination* of the psyche by a part which should have been *subordinated* in natural or normal development.)

The 3–4–5 triangle was used as a "symbol" of marriage by the Pythagoreans,[45] and Plato's choice of diagram was no doubt made with full awareness of the aptness of this tradition.[46] Not only is the suggested metaphor of character as "offspring" reminiscent of the Pythagorean interpretation, but the central aspect of the rulers' problem (hence of this passage) is that the capacities of an individual child are inherited from his parents. It is on the basis of his perception of the parents' mature natures that the ruler must calculate their children's probable capacities. The triangular diagram thus serves two functions and has two simultaneous interpretations. The exhibition of the natural basis of character and maturity, which the rulers must observe, is given by the immediate dialectical interpretation, while the relevance of these to reproduction (the mechanism of heredity) is also consciously incorporated through the Pythagorean tradition.

8. *Dialectical interpretation.* It should be noted that the dialectical interpretation of the triangular image (Fig. 53, following) constitutes a perfect diagram of the dialectical transi-

tion from Book vii to Book viii in the total argument of the *Republic*. Books ii through iv have set up a definition of virtues based on the proper, harmonious development of the three "parts of the soul"; Books v through vii have culminated in the discovery of an education which will lead to the maximum development of the natural capacities of the rational "part." The discussion of Books viii and ix is about to trace the degradations of character and society from a model aristocracy through the two deviations caused by wrong dominance of the parts of the soul, and through a subdivision of the second of these deviations into three others, depending generically on the intelligence shown in discriminating between more and less real objects of the dominant appetitive part. The conclusion, in Book ix, that the tyrant is neither better, wiser, nor happier than the just man, returns to the beginning of Book ii. An alternative way of conducting the same discussion, proceeding from the tripartite soul in Book iv through kinds of character to kinds of knowledge, is also indicated by the diagram in Figure 53. This explains why Socrates, whose dominant interest is in the definition of virtues and proving the identity of virtue and knowledge, starts to follow this *alternative* plan of argument in Book iv, only to be recalled by the objections of "impracticability" raised by his audience—a recall which diverts the *Republic* from a linear dialectical progression from becoming to being into a cyclical progression, both beginning and ending in becoming.

This reflection of the argument of the whole dialogue in miniature is briefly indicated in Figure 53.

9. *The shape of the figure.* The following description is designed primarily to help the reader who must draw or visualize the diagrams as he goes along. It gives a purely geometrical description of the shape of the image, the elements and dimensions of which are about to be specified, and from which the final schematic "number" will be derived as the product of its dimensions. The description is dialectically functional, because the emphasis on *equal* and *unequal* anticipates and suggests a differentiation of the components of the final image into a

"being" component and a "becoming" component, which actually emerge as a "nature" component and a "career" component when the interpretation has been supplied. But with a published diagram this phrase would not be indispensable, and its presence argues strongly against any notion that Plato has here deliberately compressed his text with the intention of being esoteric or enigmatic. On the contrary, he seems here to be taking pains to clarify his procedure by instructing the hypothetical reader with the wax tablet to set down, for his guidance in visualizing the final diagram, the geometric figure shown in Figure 54. This evidence against deliberate obscurantism by compression substantiates the implicit thesis of the present interpretation, that a detailed statement of dialectical referents and antecedents of the geometric elements is omitted in this passage simply because, in the immediate context of this same dialogue, an extensive interpretation of those same elements has been supplied.

10. *An arithmetical ambiguity.* The fact that the equality referred to is *arithmetic* must be stressed here, since although the significance of the numbers is determined by the structure of the diagram, the structure of the diagram need not be determined specifically by the arithmetical properties of its geometrical elements. Thus, though it is arithmetically a matter of indifference whether the two lines that combine to bound the square represent the same or different elements in the cycle of human life, it may be of great significance for the interpretation. This suggests—for the first time—that the *alternative* constructions of the square are functional in the passage, the alternatives indicating *different* constituents which are, however, represented in this image by lines that are *quantitatively* identical. Any other assumption leaves this alternative specification a mystery, no less surely than the leftover gear is a mystery to the amateur clock repairer (a mystery not really lessened by the latter's recollection that he has already put in one gear just like the part that is still left over).

11. *The correct text of* Republic *546A ff.* The reason why the reading ἑκατὸν is usually accepted at this point (and further on)

instead of ἕκαστον, and the probable origin of the reading ἑκατὸν, is the Neo-Platonic conviction that this whole passage must be about astrology—a conviction that is lent some credibility if the numbers literally specified are sufficiently frequent and large (and the rest of the text unintelligible), but that is patently impossible on the basis of the text which has the best manuscript authority and the least evidence of "improvement" in the light of the Neo-Platonic tradition.[47] The Platonic analogy of microcosm and macrocosm, which holds for Plato only insofar as both are explicable by a principle of perfection, is transmuted by later astrology into a simple and quantitative one-to-one correlation (so that each planet must correspond to some part of the body, each proportion of parts of the heavenly cycle must be *identical* with a proportion of parts of the human life-cycle, and every mathematical image illustrating a dialectical analogy must simultaneously have a valid and verifiable, *literally astronomical* interpretation). That this assumption of rigid quantitative correspondence is not always the happiest basis for a sound textual interpretation is spectacularly illustrated by Proclus' "older and better" text of the astronomical section of the Myth of Er—which is certainly not older, which is not the text, and which is "better" only in that it avoids a sheer impossibility for some pre-Proclean scholar who interpreted Plato literally and astrologically.

This assumption—that microcosm and macrocosm are quantitatively alike, and that this passage is simple astrology—has uniformly led interpreters willing to treat a mathematical image seriously to expect a literal "cosmic number" representing the time periods of astral influences on human embryology, or perhaps human cultural history, or perhaps reincarnation. This expectation, in turn, has led to the assumptions that the first part of the passage specifies the same number as the second and that the contrast of cosmic and human periods is intended only as one of duration, not one of kind. Neither assumption, however, is definitively established by this passage in its context or by Plato's treatment of the same analogy elsewhere.

The acceptance of ἑκατὸν in its first two occurrences has hinged on the assertion of higher criticism that the alternative would make intelligible interpretation impossible. Once the bare possibility of any alternative has been demonstrated, the textual question must be re-examined in the light of proper philological criteria, with less emphasis on what the passage must have said, and more on what it did say. Examined in this way, the evidence is very strongly in favor of ἕκαστον.

12. *The identity of two diagonals reconsidered.* The geometrical identity of the two "diagonals," one of the two or three phrases which most interpreters have been sure they understood, and with which they have begun "decipherments" of the rest of the passage, is not clear and cannot be the one traditionally assigned.

Since it is the two *harmonies*, arithmetically represented, that the final "geometrical number" comprehends and the construction specifies, the final number mentioned (27 x 100) is that of the second *harmony*, and the δὲ refers back to τὴν δέ, not to προμήκη δέ. A "harmony" is arithmetically represented as the product of its components unless otherwise specified. Plato's ἕκαστον, which seems unnecessary, is actually directing attention to the fact that he is *not* specifying the *first* harmony in the normal arithmetical way. This qualification does not apply at all to the *second* harmony, which *is* represented by a simple arithmetical statement. Consequently, the number 2,700 represents the product of the two dimensions of the rectangle, not its longer side.

It will be argued in the discussion following that multiplication by 100 represents a final part of the projection of the square figure into the third dimension, and therefore is a factor only in the second harmony.

These two observations clear up the *arithmetical* dimensions of the entire figure, and of the "diameters" that generate it. The ratios are integral, and the relation of the equal side, x, to $1/100$ the unequal side, y, is given as $xy = 27$. Hence one of these factors is 3, the other 9. If x were 9, then the two "diameters" would equal 9 plus 1 and 9 plus 2, or 10 and 11, respectively; whereas if x is 3, they are $\sqrt{3}$ plus 1 and $\sqrt{3}$ plus 2, or 2 and

$\sqrt{5}$, respectively. Of these two exclusive arithmetical possibilities, the metaphor of "rational" and "irrational" clearly points to the latter. (From an argument given in subsequent discussion, it is possible to show that these must have been the intended values of the two diameters, without taking account of the fact that these values are actually established by the arithmetic of the passage.)

The failure of previous interpretations to take account of the relevance of 2,700 as the number of the second harmony is explained by two factors. First, the text had been altered to ἑκατὸν in the belief that the numbers involved were all astronomical, and in the corollary belief that the method of statement of the two harmonies was the same; second, by Theon's time the vocabulary of geometry had settled into a literal, technical language, with loss of awareness of the metaphors in which its terms originated, and in this later language Plato's phrase had come to have a technical interpretation which yields a different arithmetical value.

13. *Parents and children: Pythagorean genetics.* The linear side of the "square harmony" is really itself the product of two factors. The analogy of squaring to human procreation has already appeared twice in the passage. The generic metaphor of the process of human life is a growth of roots and their squares; the first specific application is involved in the traditional Pythagorean interpretation of the 3–4–5 triangle (in which, presumably, the invariance of total human endowment, and the joint parental contributions to heredity, were illustrated by the equality of the squares of the sides—the "actualizations" of the δύναμις contributed by each parent—to the square of the hypotenuse—the actualization of the composite δύναμις inherited by the child.) [48]

This introduction of squaring gives a separate diagram representing the parent-offspring relationship, which leaves the schematism of the natures, characters, and fortunes of individual parents and children—the schematism by which the rulers select couples for marriage—for separate treatment in the final parallelepipedon diagram (which, with only three dimensions, is sufficiently complex without attempting to represent parentage as well as nature, character, and fortune). The factor of *parentage* as determining a human life is indicated by this separate diagram, which is made to show the transition from the schematic triangle to the more accurate and specific solid image.

To some readers, Plato's assumption that children inherit the natures of their parents seems to mean, in practice, that the apparatus of competitive tests in the educational system is only a fraud, designed to give the artisan class an illusion of equal opportunity.

In fact, however, the concept of inherited aptitude as an average, which Plato's symbolism in this passage takes over from Pythagorean genetic theory, carries with it the necessary consequence that some provision for shifting hereditary class status be made.

A man (or woman) may fail to qualify as a ruler through several sorts of imbalance or deficiency. Weakness, lack of courage, stupidity, or constitutional intemperance may lead to disqualification. There is always, therefore, the possibility that a marriage such as that of a brilliant but irresponsible artisan with a steady, moderately intelligent woman of his class will produce children who have just that balance of qualities which the state seeks in its rulers.

Again, it would follow (even if the cycle of human fertility did not complicate the problem) that children of the ruling class might inherit an imbalance which would exclude them from holding rule.

The apparatus of examinations and equal opportunity is not contradicted by the genetic theory, though it appears to be a consequence of that theory that favorable matches between parents of the selected upper classes are most likely to result in an improvement of the breed.[49]

14. *"Rational diagonals."* The generic metaphor of "rational" and "irrational" has already appeared twice in this passage. In its first occurrence, "rationality" was the attribute of the mature man who had developed his capacity for reason, and was opposed to the fluctuation and imperfect integration of the stages of

growth through which this maturity is attained. "Rationality" in that context was identified with the "altitude of the *acme*" of a period of human life (as opposed to the chord subtending the period, representing the successive dominance of faculties emerging with the passage of time). In the triangular diagram, this metaphor is used more specifically in the representation of the harmony of the "moral" and "intellectual" excellences by the two sides of a right triangle, in concordant ratio, which jointly determine the types of character represented by the hypotenuse of 5, which is the "offspring" of their "marriage." The line of 4 in this triangle represents rational or intellectual attainment, the other two lines representing the submissiveness to reason of the other two, "irrational," parts of the soul.

As the metaphor of squaring indicates, the present diagram exhibits generation as the inheritance of capacities, and the contrasting characterizations of the two diagonals parallel the contrasting character of the capacities which each represents. The inheritance of intelligence is illustrated separately from the inheritance of predispositions toward a given moral character—a separation maintained throughout the preceding discussion in the *Republic,* and emphasized again in Books ix and x. In each case, the image is that of the joining of the contributions of the two parents, represented linearly, to produce a square. The squares thus produced can be identified with the lines of the triangular diagram, which are a simplified representation of the squares, because the lines have the same number of units of length as the squares have units of area. This is a standard Platonic way of changing from plane to linear representation.

15. *The diameter of five.* The intended significance of this designation apparently was dependent on some further, commonly known Pythagorean or technologically inspired diagram, which can now only be conjectured. The "pempad" of which these lines are diagonals must be the number 5, the arithmetical representation of the pempad of the triangular image. As the later tradition still clearly indicated, the problem inspiring the differentiation of "rational" and "irrational" diagonals is the problem of representing irrational magnitudes as num-

bers. The "rational" diagonal is the closest integral approximation. The "diagonals of 5" must have represented this number in some familiar geometric presentation, analogous to Theaetetus' geometrical classification of numbers. In this connection, it is worth noting that the simplest representation of marriage as a product of male and female is a 1 x 2 rectangle, the diagonal of which is, of course, $\sqrt{5}$.

Although this "diameter" is the "δύναμις" of five, it is not possible for Plato simply to say so. For to have said it in that way would have made the use of the metaphor of rationality (which specifies his intended interpretation of the final figure) impossible. It is clear that a δύναμις which does not produce a given number, but some other, is really that other number's δύναμις. Hence the "rational" approximation to the diagonal of a 1 x 2 rectangle is not "the rational δύναμις of five" at all, but simply "the δύναμις of four."

The "marriages" represented by squaring (hence in an arithmetical statement by the numbers representing squares) are used to show the *generation* of the lines representing character and intelligence in the triangular diagram. The differentiation of the two components of heredity will be discussed in (17).

It is possible that this phrase is an architect's familiar abbreviation of a designation, originally more complete, of one of the lines which appear frequently in temple construction. The complete description of such a line would be "the diagonal which, squared, equals an area of five." In contemporary writing on "dynamic symmetry," we can observe a similar emergence of an architect's geometrical vocabulary which has undergone abbreviation. Thus, "whirling squares," "root five rectangles," etc., are substituted for full geometrical designations. The studies of the dynamic symmetry of the Greek temple, paralleled by references to Hindu temple construction, indicate that, in any geometrical construction of temple plans, the 1 x $\sqrt{5}$ rectangle would appear often enough to make the $\sqrt{5}$ a frequent line, for which such a shortened designation might be employed.

16. *Subtracting one.* In other words, only three, not four,

hereditary levels of intellectual capacity are to be used in the final diagram. Actually, only three such levels are functional in determining classes in the ideal state, stages in education, and types of human character, elsewhere in the *Republic;* apparently, in Plato's view, any man in a favorable environment has the capacity to go beyond εἰκασία to some form of δόξα, as he matures and acquires experience.[50]

17. *Function of the alternative statements.* To keep his interpretation clear, Plato cannot simply present the first harmony in a numerical statement. It is important in understanding that interpretation to notice that the two lines joining to determine the "equal harmony" represent *different* elements in the context; but this difference is *arithmetically* irrelevant, and would not be made clear by the statement that "*nine* is the number comprehending the first harmony." On the other hand, the difference between integral approximations to roots (which partake of conversability and commensurability but do not completely exhaust the extension of the approximated lines themselves) and roots (which cannot be treated arithmetically until they have been developed into their squares) provides a geometrical analogy very appropriate to the distinction at hand.

18. *Subtraction of two.* The number of hereditary predispositions in respect to moral character is also represented, in the final diagram, as less than the enumerations elsewhere would at first lead us to expect. In fact, however, the divergences of character among oligarch, democrat, and tyrant have a common appetitive predisposition as their native basis; insight and environment are responsible for their differentiation.

19. *The final square harmony.* Figure 59, following, is the square matrix which results from combining the linear representations of basic types of moral predisposition and degrees of intellectual ability.[51] The same matrix appears in the tyrant's number in Book ix, and the list of lovers in the *Phaedrus* is also the same, with the rows reading from top to bottom, then from left to right, successively. The identifications of character from the *Phaedrus* list are indicated in figures 59 and 60. This diagram is so lucid and so relevant to the architectonic of the

Republic that, in the last analysis, it is the best evidence that can be offered that the interpretation here proposed is actually a rediscovery of the intention of Plato.

20. *The other dimension.* The function of the third dimension in other Platonic images and analogies is often that of representing the actualized spatial and temporal characteristics of the embodiment of a structure the nature of which has been indicated by the two-dimensional diagram.

The development of the present passage in *Republic* viii has been, to this point, one of making explicit the abstract constants which are imitated and developed in actual "periods of human becoming." The number as a whole is to be determinative of better and worse births, and the rulers' errors in calculation lead to children who are "neither well endowed nor fortunate." The projection of the square matrix into the third dimension, represented by the unequal element in the προμήκη harmony, is given as the product of two factors, the dialectical counterparts of which represent those additional factors which interact with an inherited "natural disposition" to produce a particular individual career.

In the schematization of ἦθος as a function of a rational and an irrational φύσις, abstraction has been made both from the effect of τροφή, and that of τύχη. But in fact the nurture of an individual is as instrumental as his heredity in determining the kind of life he will lead, and education has been treated as coordinate with genetics in importance in all other parts of the *Republic.* Fortune, on the other hand, has not played a major role in the antecedent discussion. The sequence of physical coincidences which presents opportunity to one man and denies it to another has been treated principally as a variable to be provided against in any plan—whether the plan is that of a young man bent on becoming a tyrant, or that of a legislator bent on seeing to it that the guardians function as disinterested administrators. Only in the last two books of the *Republic* does it begin to appear that the man who deserves opportunities is the one who finds them; and even this is a general rule which does not hold of every given case in any single incarnation.

The variations in nurture which lead to different developments of native capacities have in fact already been indicated in the schematism of types of character shown in figures 59 and 60. For since the state is the individual written large, the kinds of individuals who dominate determine also the possible number and kinds of environments. The effect of training can thus be schematized simply by applying the 9-celled matrix of character to itself in the third dimension.

The resulting image, without the added factor of 100, is a simple progression through powers of 3, from the 3 parts of the soul to 9 natural endowments to the 81 types of life resulting from possible interactions of endowment and environment. This final representation is a parallelepipedon, with one plane surface a square 3×3, the other a rectangle with an area of 3^3, and sides 3×9. (It will be shown in subsequent discussion that the progression 9, 9^2, 9^3 in the tyrant's number in Book ix uses an interpretation analogous to that of the present passage, and that the compression of the tyrant's number may be explained by Plato's assumption that the images of Book viii would already be familiar to his readers.) The elements composing this figure are all identifiable and central in the context of the passage, and the combination of variables represented is exactly what a ruler's handbook of abstract genetics would discuss—leaving to the calculation and perception of each ruler the determination of the application of these general rules to specific cases, the point at which time and chance may play havoc with even the best reasoned social plan.

21. *The human cycle.* The diagram, like the *Republic* as a whole, reverts in its final step to the images of cycle with which it began, and in which must appear the spatio-temporal particularizations of the pure, atemporal dialectical proportions, the construction and systematization of which intervene.

The periods in Plato's myths are more often than not represented as powers of ten. The largest period, determinate but indefinitely great, appears as a "myriad," and its component parts are hundreds, thousands, or ten. Since the function of a myth is the illustration of arguments by fictions embodying true principles but exhibiting them with the aesthetic concreteness and individuality which argument and equation can never reach, the astronomy and mathematics of Plato's mythology are definite, but neither scientifically nor empirically accurate. In the description of a divine cosmic justice built into the mechanism of the universe, in the great myth with which the *Republic* closes, a period of one hundred years is given as the temporal span of human life. The sections of the *Republic* which intervene between this final myth and the present passage fill in testimony from experience regarding the usual correlation of good luck and good lives; but in the dialogue as a whole, as well as in the present passage, the display of general dialectical relations at work in individual sensible careers requires a mythical projection—whether as a separately stated "chronological" factor in a multiplication, or as a vision of The Fates spinning human destiny. In this passage and the whole dialogue alike, cycles are recognized as aiming at structures which they imperfectly realize; and the elaborations of the relations of these structures constitute the best means of rational description and approximation to the unique individual trajectory marked out by a career in space and time.

g. Interpretation Established from Text of the Passage Alone

It has been traditionally assumed that the crucial requirements for any interpretation of this passage (*Republic* 546A-B) are that it make sense philosophically and that it be faithful to the precise terms and syntax of the text, apart from intruded assumptions of higher criticism. But previous interpretations have usually not been rigorous in applying this second test: philologists commenting on the passage have intruded assumptions derived from sources as remote as Chaldean astrology and Hellenistic embryology, while claiming to deal only with the text.

If a stricter interpretation is attempted, two basic points must be emphasized. First, except for the "harmonies," which usage quite clearly associates with multiplication, there is no reason to think that all the elements of the nuptial number must be related by addition or multiplication. On the contrary, in a "geometrical number," we should expect the relation to be given by geometrical construction, which would not necessarily be represented arithmetically as a product or sum. When scholars have asked, therefore, "whether the numbers 3, 4, and 5 are to be added and cubed, or multiplied and cubed, or cubed and added," they have overlooked the possibility that these numbers refer to geometric elements, and that the construction need not limit itself to simple addition or multiplication. Second, in a passage as elliptical as this one, a careful author will use co-ordinating particles very carefully to indicate elision and organization, so that a careful interpreter should start with these, rather than beginning, as some have, by "deciphering" 10,000 or 4,800 or 7,500 as a number referred to in one part of the passage. It is interesting to note in this connection that some ancient commentators, e.g. Nestorius (cited by Proclus, *In Rem Publicam,* ed. Kroll, II, 49, 409–11), recognized that "the number" might be a geometrical rather than arithmetical function of the elements given.

The co-ordinating particles set up a carefully balanced pattern, in which are contrasted with one another (1) the periods of divine and human generation, (2) the four termini and three segments which the latter attains, (3) the shape of the first harmony as opposed to the shape of the second, (4) the construction of the first harmony with rational or with irrational "diagonals of the pempad"; (5) the final phrase, 100·27. This construction seems to require that (5) in the foregoing balance (4) by giving the dimensions of the second harmony. This will not give as large a number as some commentators, who assume a priori that the number here is astronomical, are lead to expect, but the construction will fit better the actual use of connectives. Also, the division of (5) into two separate factors, 100 and 27, may well balance the subdivision of (4) into its two alternative or component constructions.

We can agree to assume that a "number" here means an integer, that the passage describes a three-dimensional figure $x.x.y$, and that x and y are unequal and are both derived, by some geometrical or arithmetical elaboration, from the elements that are represented by 4, 3, and 5 early in the passage.

For my own interpretation, I should also want to add the assumption that the 100 in the final phrase is a factor incorporated in the third dimension of the figure as a semimythical "temporal periodicity" factor, not derived from the antecedent construction. But it will be seen that this assumption is needed only to distinguish the present interpretation from a class of other mathematically possible "solutions," none of which has ever been seriously proposed.

Working rather informally, and using the notion "function of" as meaning an integer resulting from arithmetical operations or summarizing a geometrical construction with given elements, we shall see that it is possible to set up the passage as a set of equations, and to examine the interpretations which represent solutions of them.

We are told first that there is a number which equals the period of human generation. We can write this

$$(1)\ n = P$$

At some limit, this period attains four termini bounding three intervals.

$$(2)\ P = 4t,3i \text{ (at some relevant limit)}$$

Two "harmonies" (which in standard Platonic use are products of at least two factors) are derived from (2), and another element:

$$(3)\ h_1,h_2 = f(4t,3i)$$

In (3), "f" represents "function of" as defined above. The derivation of the harmonies includes two other items, but first, the use of *pythmen* has the effect of making t and i in (3) arithmetically irrelevant, by taking their 4:3 ratio in its lowest terms. Arithmetically we can state this as

$$(4)\ t = i = 1$$

This ratio is then "joined to" a line or set or integer of five to help form the harmonies. We can represent this "joining" as F, another function; this is cautious, and avoids assuming that an arithmetical sum or product *must* be intended by Plato's non-technical language here.

$$(5)\ h_1,h_2 = F(f[4,3], 5)$$

Yet another construction step, a "triple augmentation," is involved in deriving the harmonies from their elements. The exact meaning of this is also ambiguous (as we can see by looking at past explications), and again it is safest to say only that it represents a functional relation, indicated by g.

$$(6)\ h_1,h_2 = g(F[f[4,3], 5)$$

Now we are given a preliminary picture of the shapes of the geometrical representations of the "harmonies"; the first is a square figure, hence arithmetically a square number. Using x as a variable to represent any "number" (in Plato's sense, i.e., any integer), we can write

$$(7)\ h_1 = x^2$$

The second harmony has x as one dimension, but is unequal in the other.

$$(8)\ h_2 = x \cdot y,\ x \neq y$$

Probably we could assert also that y is greater than x, but we will not risk doing so just yet.

We are now told the exact values, or the exact constructions giving the values, of the first harmony.

$$(9)\ x = (d5)^2 - 2$$
$$(10)\ x = (dr5)^2 - 1$$

From which,

$$(11)\ (d5)^2 - 2 = (dr5)^2 - 1$$

In the preceding equations x is defined in terms of the rational and irrational diagonals of "the pempad," $d5$ is irrational, and $dr5$ is the rational approximation. (It requires very complex formulation to get the exact mathematical statement that $dr5$ is the nearest rational approximation to $d5$, unless we use the form in (11).)

Finally, the second harmony is identified as "one hundred cubes of three." The identification of this with factors which give h_2, not y alone, is based on the use of connectives already discussed in the text:

$$(12)\ h_2 = 100 \cdot 3 \cdot 3 \cdot 3$$

If (9) and (10) are the two dimensions of the first harmony (arithmetically equal, but resulting from different constructions), the separation of 100, 3, and "cubed" in (12) may be exactly parallel. Let us assume, at any rate, that the factor 100 is not a function of x, so that the second harmony $= 100 \cdot x$ multiplied by some third integer, z:

$$(13)\ h_2 = 100xz$$

Since $h_2 = 2{,}700$, we are given the value of xz:

$$(14)\ xz = 27$$

Since x and z are integers, x must be either 3 or 9. Working backward and substituting, we see that the distinction of "rational" and "irrational" diagonals in (9) and (10) makes no sense for $x = 9$. For $x = 3$, however, we get

$$(15)\ 3 = (d5)^2 - 2 = (dr5)^2 - 1$$

This checks with (11) in the following:

$$(16)\ 2 \times 2 - 1 = \sqrt{5} \times \sqrt{5} - 2$$

By further substitutions, we arrive at a complete solution:

$$(17)\ h_1 = 9$$
$$(18)\ z = 9$$
$$(19)\ y = 9 \times 100$$
$$(20)\ h_2 = 2{,}700 = 3y = xy$$
$$(21)\ 3 \times 3,\ 3 \times 9 \times 100 = g\,(F[f[4,3],\ 5)$$

Of various functions which give this result, the choice is limited by the two derivations of x in (9) and (10). Here, among these possibilities, are solutions which fit the interpretation given in preceding discussion:

$$(22)\ f(u,v) = (u - 1) \cdot v$$
$$(23)\ f(4,3) = 9$$
$$(24)\ F(u,v) = 1 \cdot (v - 2)^2$$
$$(25)\ F(f[4,3], 5) = 9$$
$$(26)\ g(u) = 300u$$
$$(27)\ g(F[f[4,3], 5) = g(9) = 2{,}700$$

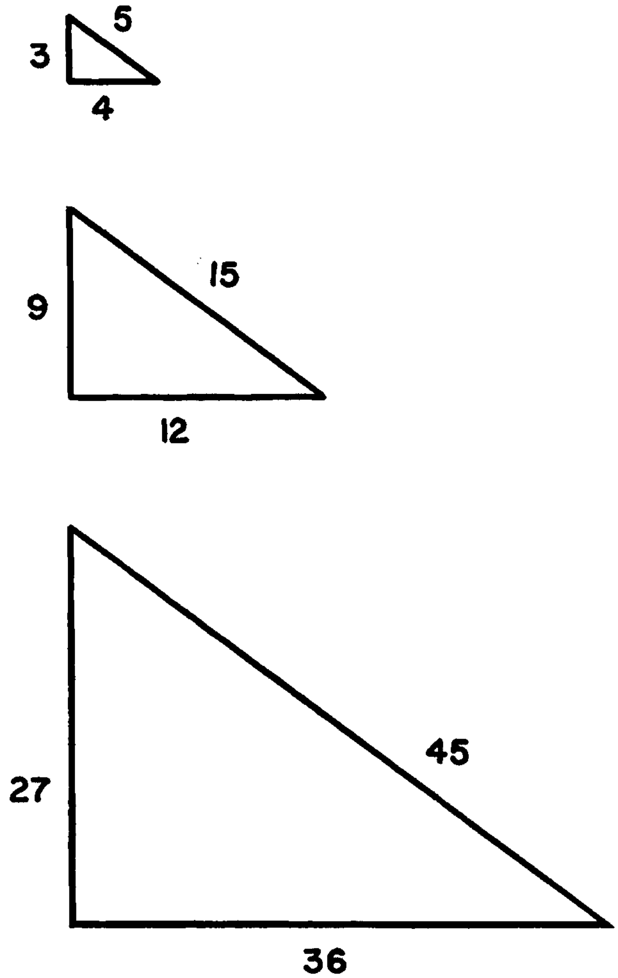
These are not uniquely determined by the text of the passage alone, but from equations 12–21 it does follow necessarily that

$$(28)\ P = 9 \times 2{,}700 = 243 \times 100 = 24{,}300$$

This is "the number" summarizing dimensions of the final geometric figure, and this solution seems to follow quite strictly from the passage out of context, with the possible exception of the isolation of 100 as a factor in (13). But that step does not prevent us from showing that previous interpretations will not fit the present analysis of the text; for, if $xy = 2{,}700$, x must be less than 2,700 or equal to it, and standard prior interpretations uniformly make it larger.

Proclus approaches the passage with three assumptions: (1) that the macrocosm-microcosm analogy is literal and the two careers completely isomorphic (in effect, that astrology is a science); (2) that Platonic mathematical images are constructed on the single principle of (*a*) giving a schematization of method, (*b*) exhibiting that method in pure mathematics, and (*c*) showing the empirical application; he is convinced that there must be a significant geometric, arithmetic, astronomical, and cosmological (dialectical) sense to be had from the text; unfortunately, he assumes also (*d*) that a single type of dialectic, that used by Plato in the empirical science section of the *Timaeus,* is always the method operative in any context; (3) that the "triple augmentation" producing the harmonies describes a reconstitution of the initial triangle into one with the same ratio of sides, but bounded by "solid" numbers. Figure 50, following, represents Proclus' version of the "thrice augmented" phrase of the text; reading the first section as a general statement of a method of ratio construction, the solid numbers of the third triangle in this figure are shown to yield ratios by that method. The

Figure 50



SCHOLION FROM PROCLUS
(Greene, *Scholia Platonica*, p. 257)

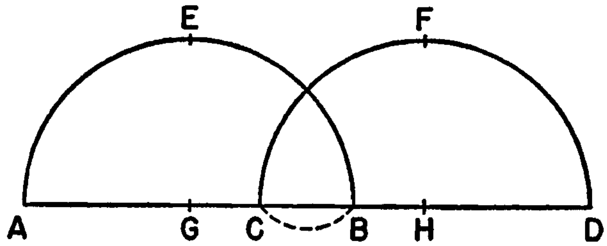
reading ἑκατὸν naturally appealed to Proclus as more amenable to astrological and astronomical interpretation than any other. See A. G. Laird, *Plato's Geometrical Number and the Comment of Proclus*, Madison, Wis., 1918; F. Hultsch, "Excursus," in Proclus *In Platonis rem publicam commentarii*, ed. G. Kroll, 2 vols. (Leipzig, 1889–1901), end of Vol. II.

Figures 51–61: The Nuptial Number

The interpretation suggested is, in outline, the following:

(1) There is a period of human generation, divided into three stages of maturity, which, in contrast to the simple periods of astronomy, is highly complex because it involves growth as well as simple locomotion.

Figure 51

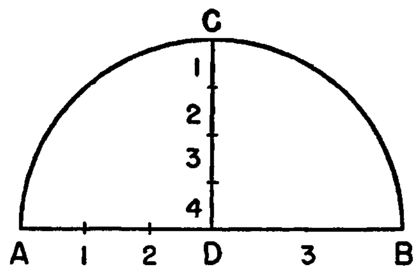


LIFE CYCLE

In Figure 51, arc *AB* = the period of a human life; arc *EBCF* = the period of a human generation (from the maturity of a parent to the maturity of his child); segments *CB, BH,* and *HD* represent the stages of maturity (marked by the successive dominance of the appetitive, spirited, and rational parts of the soul); and points *C, B, H, D* mark the critical periods of a human life. The influence of heredity is shown by the similarity of arcs *AEB* and *CFD.*

(2) These periods differ in height of acme as the individual approaches more closely the human norm of intelligence.

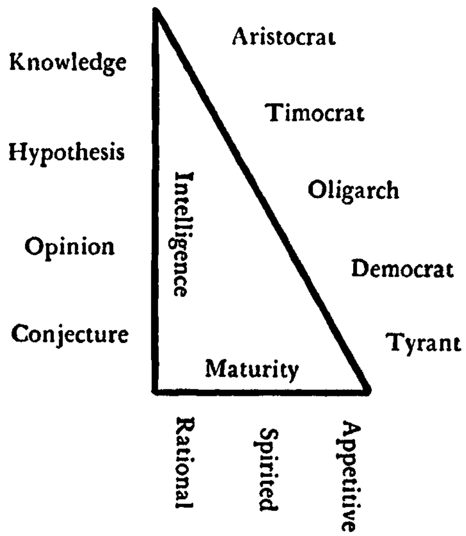
Figure 52



THE ACME OF THE HUMAN LIFE CYCLE

Figure 53 shows these lines joined, with their interpretation. Just as in the Pythagorean 3–4–5 symbol the child (square of the hypotenuse) is the sum of the contributions of the parents (squares of odd and even sides), so here the character of an individual is represented as the "offspring" of intelligence and maturity. Read from the lower right to the left, the terms that appear are the key concepts, in order, of Books i–iv, v–vii, and viii–ix of the *Republic*. Thus the diagram shows the dialectical structure of the *Republic* as well as the factors determining the characters of prospective parents.

Figure 53



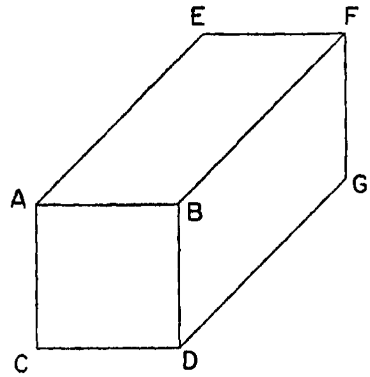
THE 3–4–5 TRIANGLE

The altitude of the acme *CD* is indicated by the divided line—a line of four segments representing possible levels of intellectual ability.

(3) The planning of marriages is determined by taking the "line in 4:3 ratio" (line *CD* in Figure 52) and joining it, in some way not here specified, to a line of five segments. The simplest junction gives Figure 53, but does not produce the final diagram intended.

(4) For the benefit of the reader, since there were no figures in the manuscript, Plato next "sketches" in the general structure of his final diagram.

Figure 54

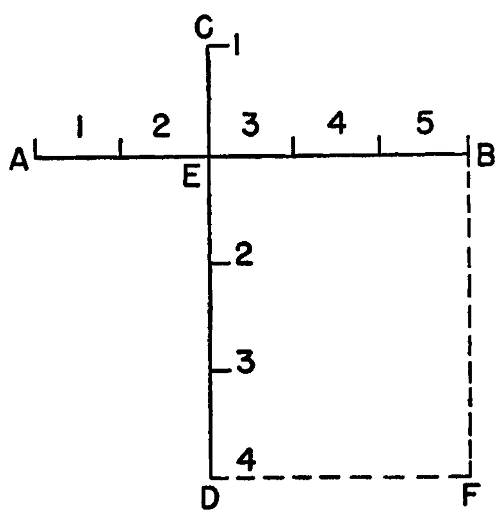


SHAPE OF THE FINAL FIGURE

In Figure 54 *ABDC* is a square, but *ABFE* is a rectangle ($AC = AB$). The "two harmonies" will be $AB{\cdot}AC = AB^2$, and $AB{\cdot}AE = 2{,}700$.

(5) Plato now modifies the original lines of 4 and 5 and combines them into the square *BFDE* to form Figure 55.

Figure 55

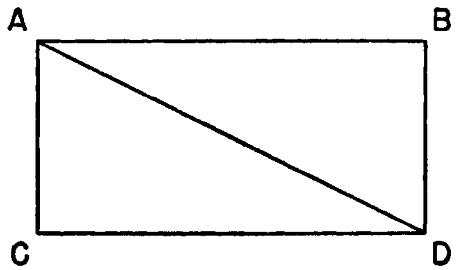


LINES OF GENETIC TRIANGLE RECOMBINED INTO SQUARES

Lines AB and CD in Figure 55 equal lines CD and AB, respectively, in Figure 52. $EDFB$ is a matrix showing the different combinations of motivation and intelligence. AE indicates two types of character (democrat and tyrant) which do not represent distinct types of motivation, since the appetitive part of the soul is dominant in the democrat and tyrant just as in the oligarch. CE represents a type of intelligence, εἰκασία, so low that it is transformed into δόξα in any normal human experience. Εἰκασία is like the infant's blooming, buzzing confusion or the madman's confusion of fact and hallucination.

(6) Figure 56 is derived from Figure 53 by the metaphor of "squaring," which shows the hereditary nature of both motivation and intelligence. Here one must posit a hypothetical Pythagorean construction.

Figure 56



THE IDENTITY OF THE "DIAGONALS"

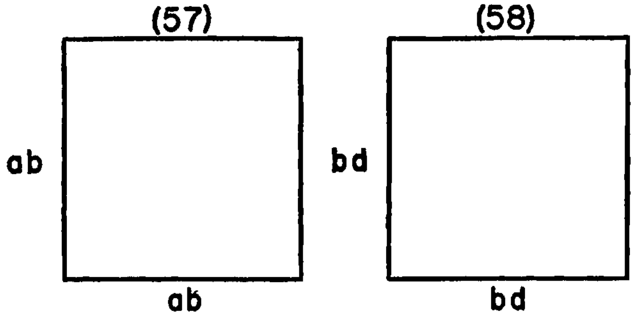
In Figure 56, $AC = 1$, $AB =$ the rational approximation to $AD = 2$, $AD =$ the "irrational diameter of five" $= \sqrt{5}$. The derivation of these values from the algebra of the passage is given in "Detailed Interpretation" (Sec. *f*), preceding. Line CD in Figure 55 is generated by squaring and subtracting appropriate segments from either AD or CD in Figure 56. (*See* 17–19 of Sec. *f*.)

Line CD of Figure 55 is equal to the area of Figure 58, the same factor being represented by the line and the figure. In Figures 57–58

$$(ab)^2 - 1 = CD - CE \text{ (in Fig. 55)}$$
$$(bd)^2 - 2 = AB - AE \text{ (in Fig. 55)}$$

The equation holds between the number of units of area of each square and the number of units of length of each line.

Figures 57–58



SQUARES ON DIAGONALS

Figure 59

PHAEDRUS MATRIX OF LIVES

	ΝΟΥΣ	ΔΙΑΝΟΙΑ	ΔΟΞΑ
RATIONAL	Philosopher	Auxiliary	Artisan
SPIRITED	General	[Prophet]	Sophist
APPETITIVE	Merchant	Democrat	Tyrant
	Soul	*Body*	*Artifacts*

Here we can confirm this diagram from two other passages in Plato. The list of lovers in *Phaedrus* 248 who vary in their "vice and forgetfulness" is shown in Figure 60.

Figure 60

LIST FORM OF MATRIX USED IN THE *PHAEDRUS*

Part of Soul Dominant		*Strength of Recollection of Forms*
R	Philosopher	Lovers of things of the mind
S	General	(least forgetful)
A	Merchant	
R	Gymnast	Lovers of the body
S	Prophet	(more forgetful)
A	Poet	
R	Artisan	Lovers of artifacts
S	Sophist	(most forgetful)
A	Tyrant	

The "difficult computation" in *Republic* ix, where the distance between aristocrat and tyrant is given as $3 + 2 = 9$, is explained by Figure 61.

Figure 61

Aristocrat	1	2	3
Timocrat	4	5	6
Oligarch	7	8	9

MATRIX UNDERLYING "REPUBLIC" IX

Here the proportion to be derived is Aristocrat : Tyrant :: 1 : 9.

h. Historical Comments: Other Interpretations

J. Dupuis, in his "Definitive Memoir" on this passage (*Republic* 546A) has included a résumé of sixteenth-century through nineteenth-century interpretations which has considerable interest as a history of discussions of the passage. A translation follows.

Dupuis' History of Interpretations [52]

I. (BAROCIUS [53]) The title indicated what Barozzi thinks of the difficulty of this passage. Among the authors who have attempted to explain it, he cites Iamblichus, Thermacides the Pythagorean, Sebastien Fox, Raphael Volterranus (Maffei), St. Thomas, Jacques Lefebvre d'Etaples, and Donatus Accaciolaus. . . .

For him, the number of the period is 1728, the cube of 12: "Geometricū itaque numerum vocat Plato ipsum cubū mille septingenta vigenti octo."

A large number of commentators have thought to see, with Barocius, in the base of the epitrite the number seven, 3 + 4. This, added to five, would be 3 + 4 + 5, or 12, which raised to its cube (τρὶς αὐξηθείς) gives 1728.

The dissertation of Barocius is one of the most distinguished. The literal Latin version of the passage . . . is one of the best.

II. (BODIN [54]) René Herpin was a native of the town of Angers. Bodin used the name to give himself more freedom in writing his *Apologie* himself. . . . Bodin does not propose any number.

III. (PEUCER [55]) [Does not propose any interpretation.]

IV. (MERSENNE [56]) Mersenne believed that "the Platonic number is 729," which he obtained by an error in calculation. "The hundred numbers of commensurable diagonals mean 3 and 4, which, being multiplied by 100 give 700, to which the cube of 3, that is to say, 29 [*sic*] being added, makes 729, which is the number meant by the enigma of Plato."

V. (T. TAYLOR [57]) The author devotes a chapter of his study to the geometric number. He believes the two harmonies are 10,000 and 1,000,000 and that "the whole geometric number is a million."

VI. (LE CLERC [58]) M. Le Clerc does not explain his translation nor propose any number.

VII. (SCHNEIDER [59]) These two dissertations by Schneider are very distinguished. The first is a thesis, a disputation. The author believes, with reason, that there are two numbers, and that the description of the second commences with ὧν, the actual *numerus fatalis;* and he is convinced that συζυγείς indicated addition. Like Barozzi, he recognizes that by ἑκατὸν ἀριθμῶν ἀπὸ διαμέτρων it is necessary to understand one hundred squares of diagonals, and not one hundred diagonals. He thinks that the number of destiny contains the factors 8 and 27, the final terms of the series 1, 2, 4, 8, and 1, 3, 9, 27, but that Plato has deliberately left the data needed to find the number incomplete. The second dissertation contains the opinions of preceding commentators: Barozzi, Boulliau, Cardan, Peucer, Melanchthon, Matthias Lauterwald, Bartholomée Bredell, Bodin, Kleuker, Lefebvre d'Etaples, etc.

VIII. (VINCENT [60]) Vincent tries next to justify his translation, and concludes thus: "In summary, the answer to this enigma proposed by Plato is the number 216, the cube of 6, and fourth term of the proportion 1 : 8 :: 27 : 216." His first takes 216, or 3×72, as the small side of a right triangle, of which the two other sides are 4 times 72 and 5 times 72, the perimeter 12 times 72, or 864, which seems to him to satisfy the conditions given, provided one adopts the corrections that he has proposed. [In his earlier discussion of the text and translation given by Vincent, Dupuis says: "Vincent replaces ἑκατὸν δὲ κύβων τρίαδος by ἕκτου δὲ κύβου τρίαδος.]

IX. (MARTIN [61]) Martin has adopted the solution of Vincent with certain modifications.

X. (MYNAS [62]) The second book being undiscoverable in the public libraries, we believe that it was never published. Minoïde Mynas, a Greek philologist, died in 1860. In his preface, he says that the solution of the theorem of Plato was more precious to him than the discovery of Babrias (which he had made in 1840 in a monastery on Mount Athos). Here is this solution [page 119 of the memoir]: "The creation of the world, divine primogeniture, is comprised in a perfect number; for that of man, it is otherwise: in the beginning of its development, it passes, under the influence of dominating and dominated stars, by the three dimensions which, combined with the four elements in affinity and in opposition more or less great, set in proportion and harmony all the parts of the nascent being. In effect, the first quaternary epitrite joined to the pempad and tripled, yields two harmonies; the one, in double ratio, perfectly equal, goes just up to a hundred and something; the other, in triple ratio, is combined proportionately with the first. Each term (100) of this harmony has for diameters (*i.e.,* factors) round digits of the pempad, the ones less great than the others by a unit. Among these terms, which give one hundred ternary cubes [*sic*] there are two incommensurables. The entire number existing in geometrical proportion indicates the relation of better and worse generation."

The explanation of the language of the passage, which Plato had intended "to obscure," [see p. 131] stops at the words dominating and dominated, [p. 159] which, according to the commentator, relate to the planets.

The solution of Minoïde Mynas, who was notwithstanding very erudite, and who knew Greek even better than French, is a remarkable example of the strange divagations to which, even in our day, the interpretation of the number of Plato has given rise.

XI. (ZELLER [63]) In this very remarkable work, the celebrated historian of Greek philosophy says that the cosmic period is 10,000 years, and that this is $4/3$ of the political period, so that the latter would be $3/4$ of the former, and takes 7,500 years.

XII. (WEBER [64]) In discussing the problem, he criticizes with some animus the interpretation of Vincent and Martin. The latter, in comparing to a right triangle in which the sides are 3, 4, 5, and area 6, the triangle of which the sides are 72 times greater, gives to this latter triangle an area of 6×72, although it is 6×72^2. Weber notes this *lapsus* by saying: "Aream trianguli rectanguli, cujus latera sunt 216, 280, 360, non 432, sed 72×432

vel $72^2 \times 6$ velere, nemo nisi mathematicae elementorum imperitus nescit. . . !"

He believes, with Hermann and Rettig, that ἀρρήτων δὲ δυοῖν stands for ἀρρήτων δὲ δεομένων δυοῖν ἑκάστων. Vincent and Martin had made ἴσην ἰσάκις relate to the first harmony, and ἑκατὸν τοσαυτάκις to the second; he finds this interpretation unsatisfactory: "pessimam." He thinks, with reason, that ἑκατὸν τοσαυτάκις is short for ἑκατὸν ἑκατοντάκις. He takes as his harmonies the two numbers 10,000 and 7,500, with the reading προμήκη δέ, but he presents no other arithmetical conclusion.

From Dupuis' summary it will be seen that from 1600 to 1850, interpreters of this passage (*Republic* 546A) seem generally to have assumed that the reference intended was some specific number important in another Platonic context, or in pure mathematics (e.g., Cardan). This conviction seems to have operated as an unquestioned assumption, regarding which there was a consensus of scholarly opinion. The details of the text were generally assumed to result from Plato's desire to write in a deliberately enigmatic form.

In the second half of the nineteenth century the idea became current among Plato scholars that this number was intended to represent the relation of the Great Year, or cosmic cycle, and the cycle of human constitutions, as society declined. The first phrase was then construed to mean the 10,000-year cycle of the *Phaedrus* myth, with which the political cycle was assumed to be in 4:3 ratio (hence equal to 7,500 years). Considerably more attention was paid to the detail of the passage than had been in previous interpretations, apparently on the assumption that though an enigma, the answer should somehow be written into the riddle. Many variant details of interpretation were suggested, but the consensus as to the numbers Plato had intended was so marked that Tannery, who thought the key to the text had been lost, believed that the loss was not too serious, since there was general agreement that this was what the passage must have meant.[65]

With the twentieth century, a new "common sense" is reflected in Plato scholarship. The impact on philosophy of new empirical, scientific discoveries and the relatively greater awareness of the complexities of astronomy (against which the humanistic philologists of the previous half-century had been self-insulated), awakened a sensitivity to the same elements in the culture of Greece; and Plato's text comes to be appproached with the assumption that what had before been taken as an enigma presenting an extraordinarily naïve notion about history, may in fact have been meant as a technical, scientific passage presenting some complex and conjectural scientific hypothesis, with some discoverable empirical reference. The "obscurity" is then no longer seen as the result of a nineteenth-century romantic passion for *Tiefsinnigkeit* on the part of a Plato who was a German philologist in spirit, but as the result of our loss of the technical, scientific vocabulary used by a Plato who was in fact more of a twentieth-century laboratory scientist. (Compare A. E. Taylor's interpretation of the meaning of Plato's "likely story" in the *Timaeus*.) Being ignorant of this technical vocabulary, certain scholars have advanced two types of argument: (1) they have emphasized the technical use, by post-Platonic scholars, of terms found in Plato, and (2) they have attempted to find in other Platonic dialogues mention of some empirical phenomenon which this passage might be thought to explain. A Plato recreated in the image of the early twentieth century is evidently capable of framing hypotheses of vastly greater complexity than his nineteenth-century counterpart, and indeed one mark of a great philosopher lies in his capacity to grasp and formulate such a complex hypothesis, so that a priori the text is approached with the expectation that "the number" described may be an extremely large one. Perhaps also this "scientific" Plato is thought of as sharing the obvious penchant of his twentieth-century counterpart for using small standard units of measure—the gram, the second, etc.—even when measuring large quantities.

Unfortunately, the later technical meanings of terms which are employed in this passage represent a distinct post-Platonic development of mathematics as a separate science, with a set of univocal technical terms. These later usages become progressively less reliable guides to interpretation as we move back toward their prespecialized origin. Further, the attempts to find the referent of this number in other Platonic dialogues are left fairly uncontrolled, and overlook the fact that a difference in dialectical context will be reflected as a genuine constitutive difference in mathematical imagery. In any case, the number representing the scholarly consensus of this period (12,960,000) [66] cannot be established in either of these ways, without extrapolation from tradition and other texts to Babylonian astronomy and Neo-Pythagorean astrology. But if one is allowed this spatial and temporal latitude in extrapolation, almost any large number could probably be defended as an "interpretation" of the text about as plausibly as any other. (For example, we could very well defend Thomas Taylor's thesis that the number is 1,000,000 by multiplying the great year by 100 to fit our taste for larger numbers. This could be eruditely supported by citations from the Neo-Platonists to show that any cosmic periodic cycle ought to be represented as a power of the perfect number, 10.)

The development in the first half of the twentieth century of a more critical attitude toward method and its rôle in history and philosophy, the availability of the results of critical philological studies, and work on the history of science suggest a new "common sense" for the scholarly consensus that should underlie an interpretation of Plato's text. This common sense hinges on the idea that the method by which a doctrine is constituted should be the method by which it is interpreted. In other words, in interpreting Plato we should self-consciously adopt the methods which Plato himself advocates for textual construction and interpretation. This attitude, if we retain the fairly hardheaded notion of philosophy inherited from the past generation of scholars, takes away the mysterious fascination of the mathematical imagery in the dialogues, and views it as a pedagogical adjunct to the dialectical enterprise. It seems that the very presuppositions which underlay the interest of the nineteenth century in these images operated to prevent a correct interpretation of them, since the texts were Platonic, but not the maxims of interpretation.

The present interpretation is put forward as an example of this reflexive application of Platonic methods to Platonic texts. The enterprise is not a facile or easy one; ten years of intermittent research lie behind the present section. In spite of this investment of effort, I have no illusion that the interpretation as here put forward is definitive or complete. I am convinced, however, that it is the only right beginning. It aims to avoid the effect of indeterminacy which is created when the interpreter, taking his equipment from external sources, intrudes expectations and tactics that deform the text into a meaning acceptable only to scholars of his own generation. These scholars, unaware of their latent assumptions (since these are part of their common sense), arrive at a consensus that the passage really means what, had they themselves written it, they might have intended.

Adam's interpretation is currently the most widely accepted. It will be found that my proposed interpretation agrees with Adam's philological conclusions, except for (*a*) the reading ἕκαστον and (*b*) the interpretation of τρὶς αὐξηθείς. Since the final number in my interpretation is of the form x^4, there would be no necessary disagreement with Adam. His case for τρὶς αὐξηθείς as meaning "raised to the fourth power" has been attacked as the weakest point in his treatment. The scholion on *Republic* 587 may seem, however, to give some plausibility to his unorthodox construction. Apart from these points, the objections to the interpretation are philosophic and aesthetic rather than philological.

In the first place, Adam violates the aesthetic principle that parts should be functional, by making the alternative constructions with the two diameters of the pempad an unnecessary and confusing duplication. Further, he does violence to the dramatic consistency of the character of Glaucon, who in Book ix is baffled by a simple multiplication up to 729, but is not similarly puzzled by the elaborate computation of 4,800 $\times$ 3,600, which is attributed to the Muses by Adam. Finally, as I think has been shown in my calculation of the two harmonies, if the number 2,700 is the second harmony, Adam's arithmetic is impossible.

The major objection, however, seems to lie in the fact that

Adam's reconstructed diagrams confound, rather than elucidate, the relation of this passage to Plato's similar images and to its immediate dialectical context. At best, on this view, the nuptial number becomes an elaborate digression, not an image giving new pedagogical insight into the architectonic of the *Republic,* and must seem to be a very unworthy and unlikely counterpart of the brilliant contextual image of the divided line.

The earlier interpretation which I proposed (in *The Rôle of Mathematics in Plato's Dialectic,* Appendix A) had the merit of simplicity for the postulated construction. I took the harmonies as produced from a line of four (hence in 4:3 ratio with the line of three stages of human growth), and from the line of five. The square harmony I construed as a square bounded by the line of four, minus one, and the line of five, minus two. This entailed identifying the line of four, which suggests the divided line, as the "rational diameter of five," and the line of five, representing types of character or state, as the "irrational diameter." The phrase "thrice augmented" could have been interpreted as a projection to an equal distance in the third dimension, giving a cube 3 x 3 x 3, and the final phrase as a description of the entire figure, which could be "one hundred cubes of three" viewed from the side. But the pattern of connectives suggested so strongly that 2,700 was the area of the rectangular surface, not the entire figure, that I used a more complex projection, making the "triple augmentation" stand for *three* projections to distance three in the third dimension. The fatal defect of this, so far as I can see, is that it requires "the irrational diameter of the pempad" to be the pempad itself, and that is indefensible. A further objection is that the parallel imagery of the passage really requires a mathematical "irrational" to represent the "irrational" contextual referent. I anticipate, however, that some later interpretation will synthesize the various factors and references of this passage into some such simple, intended construction.

VII. The Tyrant's Number

Republic 587 *

". . . then the tyrant's place, I think, will be fixed at the furthest remove from the true and proper pleasure, and the king's at the least." "Necessarily." "Then the tyrant's life will be least pleasurable and the king's most." "There is every necessity of that." "Do you know, then," said I, "how much less pleasurably the tyrant lives than the king?" "I'll know if you tell me," he said. "There being as it appears three pleasures, one genuine and two spurious, the tyrant in his flight from law and reason crosses the border beyond the spurious, cohabits with certain slavish, mercenary pleasures, and the measure of his inferiority is not easy to express except perhaps thus." "How?" he said. "The tyrant, I believe, we found at the third remove from the oligarch, for the democrat came in between." "Yes." "And would he not also dwell with a phantom of pleasure in respect of reality three stages removed from that other, if all that we have said is true?" "That is so." "And the oligarch in turn is at the third remove from the royal man if we assume the identity of the aristocrat and the king." "Yes, the third." "Three times three, then, by numerical measure is the interval that separates the tyrant from true pleasure." "Apparently." "The phantom [*eidolon*] of the tyrant's pleasure is then by longitudinal mensuration a plane number." "Quite so." "But by squaring and cubing it is clear what the interval of this separation becomes." "It is clear," he said, "to a reckoner." "Then taking it the other way about, if one tries to express the extent of the interval between the king and the tyrant in respect of true pleasure he will find on completion of the multiplication that he lives 729 times as happily and that the tyrant's life is more painful by the same distance." "An overwhelming and baffling calculation," he said, "of the difference between the just and the unjust man in respect of pleasure and pain." "And what is more, it is a true number and pertinent to the lives of men if days and nights and months and years pertain to them." "They certainly do," he said.

In *Republic* ix. 587D, Socrates begins his "marvelous and baffling calculation" of the unhappiness of the tyrant by stating

* Trans. Shorey, *Republic*, II, 397, 399.

that the "distance in linear measure between the aristocrat and the tyrant is nine." This is surprising since in his list of lives there are only five, the aristocrat being the first and the tyrant the fifth. He explains this by saying that the oligarch is third from the aristocrat and the tyrant third from the oligarch. This has been variously regarded by Plato's readers as pleasantry, nonsense, or arithmetical sophistry, but no one has contended that in its context it is not a *non sequitur*.[67] Adam, in his notes (*Republic,* II, 360–61), suggests that we may assume some intermediate characters which were not previously mentioned to account for this "shift to a larger modulus," but he does not suggest how or why this might be done, since he believes that in any case the "real purpose of the computation is to arrive at the number 729," chosen for astronomical reasons.[68] There the matter stands.

I should like to suggest that the computation in question, while compressed, is perfectly sequential; that Adam's conjectured "intermediate constitutions and characters" are in fact clearly implied in Plato's dialectical context; and that from a similar context (that of the list of lovers in *Phaedrus* 248) one can say why there is a total of nine such characters, and what the "intermediates" are.

The simplest mathematical image illustrating the combinations of two independent factors is a square subdivided into cells. In the classification of divine and human objects and imitations at the end of the *Sophist,* Plato uses precisely such an image for this purpose,[69] and it is a usage which has precedent in both mathematical and medical writers before Plato's time.

When in the *Republic* we are told that the distinction between aristocrat, timocrat, and oligarch lies in the dominance of the parts of the soul,[70] and that the distinction between oligarch, democrat, and tyrant can be stated only in terms of the reality of the objects of their desire,[71] it is clear that the list of five types of state about which discussion centers in Books viii and ix involves the introduction of two distinct principles of degradation successively. In other words, the five constitutions and characters discussed represent a gnomon-section of that matrix which would illustrate the complete result of these factors in their simultaneous independent operation.[72] The total matrix would have to provide, for example, for the distinction between two characters both of whom are guided by the rational part of the soul, but in one of whom the natural strength of this part permits contact with realities, whereas in the other its weakness limits such contact to (artifacts (thrice removed imitations).) But the path of degradation selected for schematic treatment shows each step to be the same as or worse than the preceding constitution in respect to *both* factors, going from best to worst in a continuous movement. The absence of the distinction of rulers-auxiliaries-artisans [73] in this scheme, a distinction of nature and character basic to the structure of the argument of the *Republic* and to the aristocratic state, is a sign of the inadequacy of the five-step path of degradation as a schematic summary of the entire relevant argument.

In *Phaedrus* 245, nine classes of lovers are hierarchically arranged in terms of their degradation from the best lover through the "double load of ignorance and vice." [74] The nature of each kind of lover is indicated by a description of the typical vocation or preoccupation of a man of the given type. When there is not a unique or exact correspondence between vocation and character, more than one characterizing term must be used; and in fact there is only one case in which a single vocation is the only one typical of a given type of lover, hence only one case for which a single term will suffice—that of the tyrant.[75] When this list is compared with the list of kinds of character in Books viii and ix of the *Republic,* or with the list of kinds of citizen in Books iii–vii, certain correspondences are very striking. (See Fig. 63.) Within each triad, these characters are arranged in a descending order of *virtue* defined by the dominance of the rational, spirited, or appetitive part of the soul. Hence the correspondence between the first three "kinds of lovers" and the first three "types of human character" is exact and complete. The successive lists of threes in the *Phaedrus* are arranged in a descending order of *memory,* since the soul sinks through forgetfulness as well as vice. Lovers who recognize things of the soul as objects of love differ from those who ascend no higher than love of the body, and from those who can see beauty only in artifacts, in the adequacy of their recollection of the forms.

This accounts for the identity of the three classes in the ideal state with the first, fourth, and seventh kinds of lovers in the *Phaedrus* list, for these classes differ in natural intelligence, but are alike "virtuous," and hence have the same relative dominance of parts of the soul.

If the joint variation of these two factors, ignorance and vice, is represented by a square, the resultant image is Figure 59, preceding.

If one accepts the equivalence, which Socrates emphasizes in *Republic* ix,[76] of the democratic character and the mimetic artist's occupation, the degradations of the best constitution presented in the five-part scheme of *Republic* viii–ix are seen to follow the gnomon formed by the left side and bottom of this matrix. The reason for this selection is, as has been said, that the other "intermediates" would not represent the progression from best to worst in direct and unqualified form, since the ideal artisan, for example, though inferior to the general or timocrat in wisdom, is his superior in temperance. The differentiation of philosopher, athlete, and artisan has already been developed in Books iii and iv of the *Republic;* the separation of Sophist-prophet and statesman-general is left to a later trilogy of dialogues for its development.

From the analogy of the *Republic* and *Phaedrus* images exhibited above, certain definite conclusions follow. The first is that the distance from aristocrat to tyrant equals 9 because, as Socrates says, the distance being represented linearly is "the number of a plane figure." [77] The "plane figure" in question is the matrix showing the types of human character as combinations of intelligence and motivation determine them. As Adam suggests,[78] therefore, the peculiarity of computation results from the recognition of intermediate types of character, necessitated by the dialectical schema represented. This final image, summarizing the judgment of the relative happiness of the good man and the tyrant, extends in scope beyond Books viii and ix to embrace the entire antecedent discussion of the *Republic.* The "intermediates" suggested by the *Phaedrus* diagram include the soldiers and artisans central in the discussion of Books ii–iv, and the successive speakers in Book i were a merchant, his son, a lover of the poets, and the Sophist Thrasymachus, speaking in praise of tyrants.

The reduction of a plane image to a linear distance is a natural step in the construction of a mathematical image which can be at most three-dimensional, but which must represent the interaction of more than three variable factors in the dialectical context.

Given this representation of the nine types of character as the serially ordered segments of a line, what is the significance of the "squaring and cubing" by which the real distance in pleasure between best and worst "is rendered clear"?

Evidently, there is more than one "dimension" to be considered in estimating relative pleasure, and in each of these dimensions, the same nine-term order obtains, with the tyrant at the end. From the earlier statements of the criteria needed in forming a judgment on this matter, and from the organization of Book ix itself, in which these three criteria are in fact successively employed, we derive a clear indication of what these intended dimensions are.[79]

In addition to the intrinsic excellence and reality of the pleasure sought by a man of a given type, relative *prudence* and *opportunity for experience* are factors which determine whether his life will be a pleasant one, or the reverse.[80] Φρόνησις in other Platonic contexts is associated with the ability to select means which do in fact realize the ends for which they were intended. In the present discussion, this takes the form of choosing appropriate pleasures for each of the parts of the soul. If a love of victory or a love of gain is made the criterion by which courses of action are selected, no part of the soul realizes to the fullest degree its own appropriate pleasure.[81] Since the possible criteria for prudential choice are identical with the possible characters of men, the same nine-term list which serves as a schematism of character may be used as a schematism of prudence as well. The interaction of νοῦς and φρόνησις may therefore be schematically represented by a 9 x 9 square matrix. In this matrix, each cell will represent the relative pleasure of a life in which a man with given intelligence employs one of the various possible criteria in his prudential choice.[82] Whether or not all of these combinations do exist in fact (it is perhaps debatable how far the wise man can be misguided in respect to his own practical interests), such a schematism is an effective way to represent the simultaneous disparity in respect to choice of ends and efficacious selection of means to implement them which exists between the philosophic monarch and the tyrant.

In addition to prudence and intelligence, *opportunity* is needed for a life which is in fact to be pleasant.[83] Here the earlier argument from experience, in which it is shown that the tyrant has of all men the least opportunity to act on his decisions, is relevant. As in the nuptial number, the extension of the image into the third dimension introduces a temporal factor. The final cube of 729 cells is a three-dimensional matrix, each constituent cube of which represents a combination of nature-prudence-opportunity, which are the three factors relevant to a comparison of the happiness of the life of the best man and that of the worst. The distance between the first cell and the last, computed by arranging them in serial order, is, as Socrates says, 729.

It will be noted that certain elements of this image repeat imagery found in the interpretation of the nuptial number already proposed. This gives an explanation of the extreme compression of the present passage, since these elements retain the same interpretation assigned them in their earlier introduction.

Glaucon is bewildered when Socrates states this interval as a definite arithmetical number. He does not express this same bewilderment at the sudden introduction of squaring and cubing, which seems to us the most baffling part of the entire passage. This suggests that Socrates' number (729) is a logistical device to designate an important geometrical relation in a diagram, and that it is this passage from geometry back to calculation that especially troubles Glaucon. This is made particularly plausible by the fact that the other differences between the just man and tyrant are dismissed as ἀμήχανον immediately following this calculation of relative pleasure.

Socrates' remark that this is a true number if days and nights and months and years pertain to human life has again been made an argument for the astronomical interpretation, that the entire purpose of Socrates' geometrical gyrations has been to arrive at this number, 729, which equals the number of days plus the number of nights in a year. If the remarks made on the general purpose of the squaring and cubing are correct, the correctness of such an astronomical interpretation would not be ruled out, but it would be necessary to say that the primary purpose of Socrates' calculations is not to indicate that this is the measure of careers which actually take place in time. The truth of the number would depend upon its inclusion of some comparison of such temporal careers. The mention of four periods of time rather than one seems gratuitous, unless these are periods relevant to different kinds or degrees of pleasure. (Thus the tyrant's dreams are more terrifying than those of the just man,[84] as we find later; the life of the unjust man does not end in honor after his early years are past; [85] his daily pleasures will be less; and the months in this list may suggest seasons in some similar connection.) To interpret the passage as a calculation of number of days and years still leaves most of the passage unaccounted for, though it adds a new insight to the function of the third dimension in the foregoing interpretation.

The notion that the metaphor here is that of computing the area of the shadow of the tyrant's unhappiness, which increases as the square of the distance, leaves the connection so loose between this image and the antecedent discussion, that the presentation of the image as though it were a summary of antecedent dialectic is not intelligible.[86] In effect, the connection between the image thus interpreted and its contextual passages is of the sort Coleridge describes as "fanciful" rather than "imaginative"—the fanciful connection is based on a single, shared, external relation, the imaginative on a likeness of the properties of wholes. But the fanciful diagram is of very little pedagogical use, and seems not to be typical of other instances of mathematical imagery in Plato. This interpretation unjustifiably attributes to Plato so loose a sense of dialectic that he could substitute a bad pun on the word "shadow" for a helpful illustration of the serious point at hand in the argument. Perhaps the interpreter would here again invoke the "freakish" sense of humor attributed to Plato by readers who want to explain away the *Parmenides*.

Figure 62

SCHOLION FIGURE TO ILLUSTRATE THE TYRANT'S NUMBER

(Greene, *Scholia Platonica,* p. 269)

ARISTOCRAT	TIMOCRAT	OLIGARCH	DEMOCRAT	TYRANT
1	2	3	6	9

This figure suggests (as does the other scholion on the passage taken from Proclus) that as the distances become greater, some shift of modulus is required, introducing a greater unit for their measurement. (This is essentially Adam's interpretation, discussed in the text.) There is no suggestion of the fact that the number is square because it refers to a square plane figure.

Figure 63A

PARALLELS AMONG *PHAEDRUS* 248, *REPUBLIC* VIII–IX, AND *REPUBLIC* III–VII

	"PHAEDRUS" 248		"REPUBLIC" VIII–IX		"REPUBLIC" III–VII
1	Philosopher	1	Aristocrat	1	Ruler
2	General	2	Timocrat		
3	Citizen-Merchant	3	Oligarch		
4	Gymnast			2	Soldier
5	Prophet				
6	Poet	4	Democrat		
7	Artisan			3	Artisan
8	Sophist				
9	Tyrant	5	Tyrant		

The *Phaedrus* list is arranged within each section in an order of relative virtue, the sections in an order of relative ignorance.

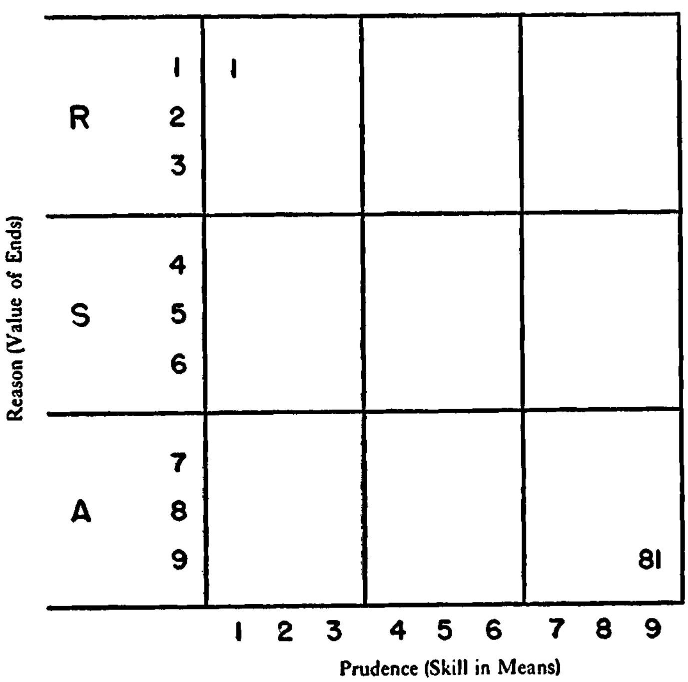
Figure 63B

TYRANT'S NUMBER: FIRST STEP OF CALCULATION

	RATIONAL	1 Philosopher	4 Gymnast	7 Artisan
VICE	SPIRITED	2 General	5 Prophet	8 Sophist
	APPETITIVE	3 Merchant	6 Poet	9 Tyrant
		SOUL	BODY	ARTIFACTS
			IGNORANCE	

This is the *Phaedrus* matrix reconstructed as the geometrical figure underlying the use of the number nine in the nuptial number passage. Its present relevance is to show why the linear measure of the distance from the aristocrat to the tyrant is a square number; the interaction of the two three-part principles produces 9 combinations, and a reduction to a single list thus comes out with 9 as the distance between the first cell and the last.

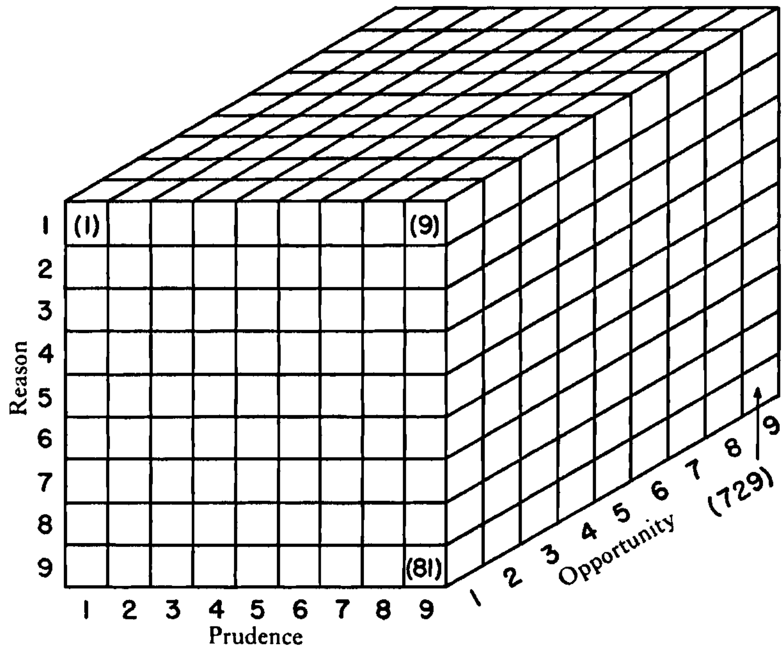
Figure 64



TYRANT'S NUMBER: SECOND STEP OF CALCULATION (SQUARING)

The linear representation of Figure 64 is used here as the basic line from which the final image can be constructed by "squaring and cubing." The present figure shows the first step in expansion, in which the ordered list of kinds of character is used to differentiate pursuers of pleasure both in respect to their relative reason and their relative prudence.

Figure 65



TYRANT'S NUMBER: THE FINAL CALCULATION

The final stage of calculation again uses the nine-step scheme as an axis to represent the relative opportunity for enacting the prudential conclusions which men of each type have. Each cell in the figure now characterizes the pleasantness of a career in terms of (1) reality of the ends to which it is oriented, (2) relative skill in determining means to those ends, and (3) probability that there will be a chance to employ those means. The summary computation of Socrates shows the distance between first and last cubes in this matrix to equal 729.

VIII. The Myth of Er—Astronomy

a. The Unity of *Republic* x

Book x has frequently been referred to as the most loosely constructed and least unified book of the *Republic.* It is apparently easy to overlook the tight, underlying tripartite structure of the argument, which finds frequent analogues in the *Timaeus* cosmology and reflects the levels of the divided line. The condemnation of mimetic artists is expounded in a recapitulation of the abstract dialectical proportions which summarize the aspects of the theory of ideas that appeared in earlier matrices. In particular, it seems to repair the omission, in Book vi, of an investigation into the "objective correlates" of the stages of the divided line.[87] The transition from form to process, from pure dialectic to personal history, is mediated by the second section, devoted to the contemplative and moving activities of the immortal soul, which would be simple in its own nature but as we know it is multiform.[88] Finally, in the history of the journeys that individual souls have made, in the vivid story of the return from death to life, we find an artist at work depicting the varied trajectories which individuals endowed with intelligence and immortality have actually marked out in space and time.

When in the *Timaeus* the perfection of the eternal model is embodied in a beautiful, projected material process through the mediation of the world soul, no one has ever accused the argument of loose connection. But it is this same tripartite argument which dominates the tenth book of the *Republic.* Thus the much resented condemnation of poetry is the ultimate judgment of pure reason; but it is a judgment by which souls embedded in time and change cannot be guided, for their insight requires such metaphysically inferior imaginative embodiments as Plato has given his conception of cosmic justice in his great poetic myth of immortality.

In Plato's Myth of Er[89] the properties of the model of the universe, which all the souls are shown before they choose their next lives, have been inadequately understood. The purpose of this model is to show how divine justice operates throughout the world-order; the closing myth of a dialogue on justice very properly deals with cosmic justice in its relation to individual choice. The justice of destiny operates in such a way that the pattern of life which each man chooses freely is the pattern that he himself grows to resemble;[90] if erroneous choices lead to catastrophe, the fault is in the chooser, not in the natural order.[91]

The moment of choice hangs balanced, as it were, between the attraction of the eternal forms, always visible to reason, and the habits and conditioning developed during the individual's previous career. It is these antecedent habits which constitute each man's "genius" and impel him to the choice which leads him to meet his "destiny."

In the same way, in the universe, the motions of the present and the past both operate and balance, producing a system in which there is a neutral moment in the progress from past to future: a timeless "now" in which future and past are alike ingressive. In the cosmic scheme, the motions of same and other are held in equilibrium, so that "justice" is served by each sphere holding to its constant path of motion. The similarities between the working of the model of the heavenly motions and the cycle of human life lie in the polarities of future and past, of same and other, and in their point of momentary balance, which the Law of Nines discloses.[92]

Thus the sizes, colors, and velocities of the moving hemispheres all have a neutral equilibrium zone or point of balance of the hemispheres following the motion of the same, and those moved (with constantly decreasing retrograde velocity as the outside is approached) by the motion of the other.

This image is entirely in the spirit of the Ionian philosophers who, like Anaximander with his chariot wheels, turned their talents as technologists to the invention of a physical mechanism duplicating astronomical motion, and in this context of Necessity that spirit is appropriate. If, as the argument has shown, virtue is a healthy and natural condition of the soul and the external natural order is a just one, we may expect the concrete experience of the man of virtue to be more pleasant than that of his opposite. But this expectation is dependent on our belief in the justice of nature. A purely rational argument for the perfection of the cosmos, based on its exemplification of natural law in its behavior, cannot of itself create the conviction that is needed to translate pure intellectual assent into applied pursuit of justice. As the importance of education in poetry has shown, true knowledge involves presentation of concrete, particular embodiments of principles as well as their apprehension in the abstract. The physiology of appetite in the *Timaeus* makes clear Plato's conviction that it is only through the projection of vivid, associated imagery that concepts can move the appetitive part of the soul.[93]

The myth represents such a concrete presentation of principles in a form designed to carry conviction to the appetite. A valid myth not only carries with it the sensuous vividness which secures attention but also has an intelligible order in its parts which is recognized and is the basis of aesthetic satisfaction.

For the appetites, seeing is believing. The prophet is able to devise myths which make his hearers see the concrete aesthetic value of the principles of morality. Before the conviction needed to secure harmony of interest and reason can be complete, however, the imagery of such myth must be checked against the content of normal sense-experience. The myth shows how concrete things should seem if they are in fact in accord with the intelligible principles of order. Perception shows whether or not the mythical imagery coincides with the events of our concrete experience. The final appeal of reason to appetite must therefore be empirical; when perception coincides with the images accompanying thought, conviction results.

Various factors interfere with a sound judgment of this coincidence of predicted application and perception.[94] This cannot be, therefore, a completely naïve empiricism; it must be safeguarded by conditions which insure that the perceiver is perceiving rightly. These conditions are carefully met in the training in music and gymnastics which is to be the basis of public education in an ideal state.

If in fact the cosmos is reasonably ordered and run, even in its sheer appearance there should be discernible both a beauty and a perceptible differentiation and integration of parts which, when hindrances to right perception are removed, will exactly correspond to mythical prediction and will carry complete practical conviction to the perceiver.

The myth at the end of the *Republic* has the double function of showing how a just cosmos should look, and of indicating that the appearance of the world we live in coincides clearly with this expectation, so that mere detached inspection must carry belief that this world is a just one.

In this connection, mere vividness is beside the point; aesthetic satisfaction demands further a perception in that vividness of a functional order, a perception which reason alone can finally verify but which is immediately felt as satisfaction by an appetite habituated to the kind of order that the projected and associated images of thought in its experience always possess.[95]

The vision of the universe,[96] shown to all souls before each chooses its next life, is presented mythically as justification for the responsibility of each soul for the consequences of its choice; each has been given a chance to see what sort of world it will be living in, and the prudent soul takes account of this in choosing.

The elaborate detail of the description of this cosmic model is functional and intended to show the latent order of its appearances. This latent order, through its impact on the appetitive part of the soul, creates conviction that we can see empirically that the world is justly run. The soul with an aesthetic sense actively functioning will at once be aware of this latent order as an accompaniment of its immediate perception; the less sensitive perceiver may need to have recourse to the structural formulae exemplified in this concrete instance before its functionality is clear. Plato's account is presented by a soul of the first type, but by the detailed report he gives of his observations, it is expanded in such a way that the latent structure of the system will be apparent to the less sensitive observer, willing to reflect on the type of structure which these details of appearance exemplify.

The phenomena treated in detail are the relative volumes, colors, and velocities of the zones of the stars and planets. The significance of these lists has never been quite functionally explained. Apparently, since the order of listing differs from simple distance from the center, some latent feature of structure is operative in each list to provide its principle of order. The cumulative effect of these details should then be a clear indication of the type of functional structure which the apparent behavior of the model is exemplifying. Since the myth is a true one, astronomical observation should suggest an empirical basis for these principles of order. But no interpreter has been able to reconcile these reasonable expectations with the text to show how in fact this bringing out of relevant structure has been done.

Perhaps, however, interpreters have generally tended to approach this passage with erroneous presuppositions. Any phenomenon which can be empirically perceived will be observed as an interaction of parts in a material space-time matrix.[97] Any form latent in this process is invisible and cannot be an object of direct perception.[98] So far as pure observation can guide one, astronomy deals with the properties of a world machine, activated by mechanical interplay of its constituent parts.[99] If, to derive a theoretical astronomy from the data of observation, reason infers the self-moving, divine character of planets and stars, such an inference is observable only insofar as the mechanical behavior provides a model of the sort of motions that would be expected.

Perception and appetite are, except for the feeling of aesthetic satisfaction, insensitive to evidence of purpose or over-all plan. We perceive separate objects influencing one another by mechanical contact; and, insofar as the bare imaginative picture fits the thing itself, every object of experience must be interpreted as a machine. Certain physical properties of such machinery, however, are associated with the presence of a directive function and integration in their design.

Among these perceptible evidences that some reason has created a mechanical system one might mention (*a*) a sharp differentiation of the individual parts, consequent upon each having an appropriate contributory function; (*b*) an adaptation of adjacent parts; (*c*) a total equilibrium of some kind in the dynamics of the mechanism, resolving various disparate forces into some more simple resultant motion. These are the perceivable properties from which we infer a latent reason and purpose as present in a mechanism.

The detailed appearance of the cosmic model seems to be described in a way which underscores these properties. The hemispheres are differentiated by mechanical restraint; each has its own boundary, walling it off sharply from the adjoining zones.[100] It has been shown that the order of the lists of colors and volumes presents a kind of balance which would correspond to the observable projection of adaptation of adjacent parts, although this interpretation has never been proposed as the explanation of the balancing observed. And the dynamic effect of the machine is to translate a set of irregular and contrary impulsions into a resultant of two simple circular motions with a balanced distribution of forward and retrograde momentum per unit of impulsive force.

Even for a disembodied soul, whose perception is entirely accurate as long as that soul perceives without calculation or reflection, all that one can legitimately expect is this sort of observation of properties that are perceptible; and such properties cannot, in themselves, serve as adequate basis for a theoretical astronomy. This vision is supplemented in a myth by the personifications of other attributes which are not perceptible. Goddesses of harmony and justice appear who control and inhabit the machine. But to the soul less privileged than Er to see such supernatural personages, their presence must be inferred by reason, or felt by appetite on the basis of those spatial configurations of volume, mass, velocity, and color, to which alone the senses can respond.

Though the general significance of this myth, and the various details other than those of the astronomical model, have been subjected to extensive and satisfactory interpretation, the model itself, which plays a crucial rôle, has not been very well explained. An attempt to give a detailed and functional account of the properties of this model is therefore still needed, and seems to be the aspect of the present myth which falls most naturally within the province of the interpreter of Platonic mathematical imagery.

b. The Allegorical Intention of the Myth

The careers of the souls between incarnations, as Er describes them, show a curious parallel to the career of the argument of the antecedent books of the dialogue. The parallel seems to begin with the pilgrimage across the treeless plain that follows the experience of punishment and reward. If we note that it is this fact of reward and punishment in the after-life that has become the preoccupation of Cephalus, at the opening of the dialogue, this parallel carries out the cyclical structure of the *Republic,* which begins with the old man's speculations on the passage of the soul from life to death, and ends with the return of the soul from death to life. These transitions link the problem of justice during a given life-span with the justice of the cosmic order, which controls the greater cycle of reincarnation.

Life has been referred to earlier in the dialogue as a "race" and as a "journey"; in the anticipatory enactment by the souls of their next careers, the journey across the plain toward a place where new insight is attained sounds like an allegorical counterpart of the whole historical career—birth, primary education, marriage, and war—from which men at last reach the insight, either in the career of higher education of Book vii, or in the reasoned judgment of the man of wide experience in Book ix, that justice is natural and best. After a four-stage journey, ending in a place which clearly is above and beyond the realm of becoming and which offers them a vantage point from which they have the philosopher's view as "spectators of all time and all existence," the souls are shown completing the final courses of higher education prescribed in Book vii for rulers. In fact, by a little forcing of the correspondences, an interpreter might find them undergoing experiences in insight which are counterparts of the entire curriculum outlined earlier.

The inspection of the cosmic model gives direct insight into the celestial motions studied in astronomy; the presence and song of the Sirens bring them face to face with a personification of the harmony which is the subject matter of harmonics; the dialectical linking of these insights with the problems of human affairs is presented through the personified goddesses of destiny, who conduct the course of instruction and finally sum up the lesson of the curriculum through their spokesman, the prophet.

The next episode after this education is its application to practical affairs, just as, in the earlier discussion, the model educational and administrative system of an ideal state having been set up, the next consideration is the degradations of such a state when actually transposed into the context of contingencies and erosions of a career in time. Just as varieties of polity among which a man must choose his allegiance occupy Book viii and Book ix of the *Republic,* so a showcase of sample lives of all varieties is presented to the souls following their instruction, and each must choose its next career. The factors differentiating these lives offered for choice are a list familiar from the previous contexts of the discussion, and represent combinations of factors appealing to each part of the tripartite soul. Just as, in the earlier context, opportunities for action beyond the control of the individual are one relevant component of the pleasure of a given life, so, in this recapitulation, the fortune of the lot is one relevant factor in the opportunity of each soul for choice.

To underscore this parallel, Er reports a list of cases of actual choosing in which there are examples of choosers dominated by each of the parts of soul, ranging from the inexperienced voracity of the holder of the first lot to the toil-hardened, dearly bought sagacity of Odysseus, who chooses last and whose rôle is clearly intended to symbolize the confirmation in concrete experience of the advantages of justice established earlier in abstract argument.

In describing the souls as drinking different amounts from Lethe, Plato includes a mythical detail to "explain" the variation of individual intelligence, which is also recognized in the

Phaedrus myth, where Plato says that lives differ in kind according to their possessors' "ignorance and vice." [101]

Apart from this detail, the allegorical parallel seems to end with the pageant of souls transforming themselves into animals, and the machinery of ratification and reincarnation completes the frame proper to the allegorical picture.

Among its other advantages, such an allegorical interpretation of the myth permits us to explain many of its details which on any other interpretation seem somewhat out of place and puzzling. The foregoing treatments of mathematical imagery are evidence that Plato's intuitive sensitivity leads him to choose significant details for which we can reconstruct an intellectual explanation by abstract analysis of the connections which cause the chosen affinity to be felt.

The details chosen for explanation in the present treatment are those of mathematical imagery. The arithmetical descriptions of periods and the elaborate description of the dimensions and properties of the astronomical model that is shown to the souls are the problems peculiarly relevant to the present study; but there seems to be no reason to exclude the possibility that an attempt to interpret other portions of the myth along these lines might finally disclose their intended significance as well.

As confirmation of this approach, it is worth noting that Plato himself has done everything he could have, short of a hopelessly anticlimactic explanation by Socrates, to indicate to his contemporary reader that the passage at hand is intended to be read as allegory. In the first place, the subject matter places this story in a religious tradition in which death and judgment after death are explained to the initiate by allegorical pageantry. As Plato himself was aware, mythology gains in effect and solemnity by retaining the established forms of religious tradition. The revelation of doctrine to the mystery-religion initiate took the form, as we have noted, of a vision of an allegorical pageant or tableau. Plato's myth is told so successfully, in a way which seems to use every device of artistry to present the scenes described as successive *visual images* to the mind of the reader, that it is almost impossible to read the passage without such a visualization. Further, the relative importance of the things the souls see is frequently indicated by giving a visual vividness to their description, an equation of importance with brilliance rather than with magnitude or priority in order that is peculiarly intelligible if the author is consciously restricting himself to the tactics of the tradition in his choice of devices for indicating the distinctions he wants to make. The reader finds himself much in the position of the souls in the story; a direct vision replaces inferential construction (as inspection of the model of the heavens within the myth replaces study of theoretical astronomy); but a correct inferential construction results from applying metaphorical interpretation to the content of this direct aesthetic vision. (This point is later illustrated in the discussion of the souls' intended reaction to the cosmic model.)

As confirmation of the notion that this property of the myth is deliberate, evidence can be cited to show that it is not typically so central a device of Platonic myth-construction. For illustration, to choose a relatively trivial example, the reader is not invited by the Platonic account to speculate on the color of the clothing which Theuth, Thamus, the demigods of the *Timaeus,* God at the helm of the cosmos in the *Statesman,* or Prometheus in the *Protagoras* happen to be wearing, nor on the color of the landscape in which they stand. Only in the myth of the "true earth," at the end of the *Phaedo,* where the world is observed from a great height above, as it is by the souls in the present myth, is there an analogue to this importance of details of color. We may note further that the device so regulates the tactics of the description of Er's vision that the identification of the planets and stars in the astronomical model is made only on the basis of their color.

The myth of the cave, earlier in the dialogue, uses allegorical illustration, with a point-by-point interpretation appended, thus leading one to expect allegory from the earlier context. One should note also that abstract concepts are presented to the souls by personification, just as, we may suppose, such concepts would have been displayed in the mystery pageant.

What seems to be conclusive evidence for this allegorical intention, and for the reading of the astronomical part of the allegory given below in my detailed interpretation, is that in the *Epinomis* precisely this material occurs. Whether written directly by Plato, or by a student of Plato who was thoroughly at home in Plato's philosophy, this dialogue has exactly the relation to the *Laws* that the Myth of Er bears to the *Republic.* Once again, the rule of law is shown to extend throughout the cosmic order. But in the *Epinomis,* flat literal statement replaces the earlier mythical account of justice in the heavens.

In reading the detailed comments on the myth, which follows, it is worth noting specifically that (1) in the *Epinomis* there are statements concerning the very large masses of the sun and planets, (2) that from these is shown the absurdity of any person or planet setting his or its course against the vast momentum of nature, (3) that the motive principle postulated for this system must be some sort of "soul," and that "mind" and "volition" also have their analogues. (The third point intrudes more of the world seen *qua* organism than the Myth of Er specifically introduces.)

It may be suggested here that the analogy of *Laws: Epinomis* : : Myth of Er : *Republic* deserves further study. One objection to the possibility that the *Epinomis* is Plato's own, the reference to "memoranda," can be neatly met if we identify mnemonic diagrams of the sort recovered in this chapter as the type of "memorandum" in question, and discover such diagrams in the *Laws.*

c. Detailed Interpretation

Republic 617 *

Now when each company had spent seven days in the Meadow, on the eighth they had to rise up and journey on. And on the fourth day afterwards they came to a place whence they could see a straight shaft of light, like a pillar, stretching from above throughout heaven and earth, more like the rainbow than anything else, but brighter and purer. To this they came after a day's

* Trans. Cornford, *Republic,* pp. 353–55.

journey, and there, at the middle of the light, they saw stretching from heaven the extremities of its chains; for this light binds the heavens, holding together all the revolving firmament, like the under girths of a ship of war.

And from the extremities stretched the Spindle of Necessity, by means of which all the circles revolve. The shaft of the Spindle and the hook were of adamant, and the whorl partly of adamant and partly of other substances. The whorl was of this fashion. In shape it was like an ordinary whorl; but from Er's account we must imagine it as a large whorl with the inside completely scooped out, and within it a second smaller whorl, and a third and a fourth and four more, fitting into one another like a nest of bowls. For there were in all eight whorls, set one within another, with their rims showing above as circles and making up the continuous surface of a single whorl round the shaft, which pierces right through the center of the eighth. The circle forming the rim of the first and outermost whorl [Fixed Stars] * is the broadest; next in breadth is the sixth [Venus]; then the fourth [Mars]; then the eighth [Moon]; then the seventh [Sun]; then the fifth [Mercury]; then the third [Jupiter]; and the second [Saturn] is narrowest of all. The rim of the largest whorl [Fixed Stars] was spangled; the seventh [Sun] brightest; the eighth [Moon] colored by the reflected light of the seventh; the second and fifth [Saturn, Mercury] like each other and yellower; the third [Jupiter] whitest; the fourth [Mars] somewhat ruddy; the sixth [Venus] second in whiteness. The Spindle revolved as a whole with one motion; but, within the whole as it turned, the seven inner circles revolved slowly in the opposite direction; and of these the eighth [Moon] moved most swiftly; second in speed and all moving together, the seventh, sixth, and fifth [Sun, Venus, Mercury]; next in speed moved the fourth † [Mars] with what appeared to them to be a counter-revolution; next the third [Jupiter], and slowest of all the second [Saturn].

* The names in brackets are supplied by Cornford; Plato himself did not give them in his text.

† Cornford has here followed, as have almost all editors, the text quoted by Theon; the MS text, which reads "the third seemed to them to move in an opposite direction to the fourth," makes sense neither astronomically nor mechanically in this context. The differences between the manuscripts and Theon's quoted version are discussed in the final part of this section.

> The Spindle turned on the knees of Necessity. Upon each of its circles stood a Siren, who was carried round with its movement, uttering a single sound on one note, so that all the eight made up the concords of a single scale. Round about, at equal distances, were seated, each on a throne, the three daughters of Necessity, the Fates, robed in white with garlands on their heads, Lachesis, Clotho, and Atropos, chanting to the Sirens' music, Lachesis of things past, Clotho of the present and Atropos of things to come. And from time to time Clotho lays her right hand on the outer rim of the Spindle and helps to turn it, while Atropos turns the inner circles likewise with her left, and Lachesis with either hand takes hold of inner and outer alternately.

Proclus reports an "older and better" version of the list of sizes of hemispheres in *Republic* 617E. The diagram in Figure 68 is based on his "older" version and shows that the manuscript version is preferable. The list of sizes is here given as Proclus' "older and better" text presents it, with the corresponding terms, marked MSS, of the manuscript version added in brackets where the two differ.

> The circle forming the rim of the first outermost hemisphere is the broadest; next in breadth is the seventh [sixth, MSS]; then the eighth [fourth, MSS]; then the sixth [eighth, MSS]; then the fourth [seventh, MSS]; then the third [fifth, MSS]; then the second [third, MSS]; and the fifth [second, MSS] was narrowest.

Republic 616D–617D

(Chambray, *République*)

Τὴν δὲ σφονδύλου φύσιν εἶναι τοιάνδε· τὸ μὲν σχῆμα οἵαπερ ἡ τοῦ ἐνθάδε, νοῆσαι δὲ δεῖ ἐξ ὧν ἔλεγεν τοιόνδε αὐτὸν εἶναι, ὥσπερ ἂν εἰ ἐν ἑνὶ μεγάλῳ σφονδύλῳ κοίλῳ καὶ ἐξεγλυμμένῳ διαμπερές ἄλλος τοιοῦτος ἐλάττων ἐγκέοιτο ἁρμόττων, καθάπερ οἱ κάδοι οἱ εἰς ἀλλήλους ἁρμόττοντες, καὶ οὕτω δὴ τρίτον ἄλλον καὶ τέταρτον καὶ ἄλλους τέτταρας. Ὀκτὼ γὰρ εἶναι τοὺς ξύμπαντας σφονδύλους, ἐν ἀλλήλοις ἐγκειμένους, κύκλους ἄνωθεν τὰ χείλη φαίνοντας, νῶτον συνεχὲς ἑνὸς σφονδύλου ἀπεργαζομένους περὶ τὴν ἠλακάτην· ἐκείνην δὲ διὰ μέσου τοῦ ὀγδόου διαμπερὲς ἐληλάσθαι. Τὸν μὲν οὖν πρῶτόν τε καὶ ἐξωτάτω σφόνδυλον πλατύτατον τὸν τοῦ χείλους κύκλον ἔχειν, τὸν δὲ τοῦ ἕκτου δεύτερον,* τρίτον δὲ τὸν τοῦ τετάρτου, τέταρτον δὲ τὸν τοῦ ὀγδόου, πέμπτον δὲ τὸν τοῦ ἑβδόμου, ἕκτον δὲ τὸν τοῦ πέμπτου, ἕβδομον δὲ τὸν τοῦ τρίτου, ὄγδοον δὲ τὸν τοῦ δευτέρου. Καὶ τὸν μὲν τοῦ μεγίστου ποικίλον, τὸν δὲ τοῦ ἑβδόμου λαμπρότατον, τὸν δὲ τοῦ ὀγδόου τὸ χρῶμα ἀπὸ τοῦ ἑβδόμου ἔχειν προσλάμποντος, τὸν δὲ τοῦ δευτέρου καὶ πέμπτου παραπλήσια ἀλλήλοις, ξανθότερα ἐκείνων, τρίτον δὲ λευκότατον χρῶμα ἔχειν, τέταρτον δὲ ὑπέρυθρον, δεύτερον δὲ λευκότητι τὸν ἕκτον. Κυκλεῖσθαι δὲ δὴ στρεφόμενον τὸν ἄτρακτον ὅλον μὲν τὴν αὐτὴν φοράν, ἐν δὲ τῷ ὅλῳ περιφερομένῳ τοὺς μὲν ἐντὸς ἑπτὰ κύκλους τὴν ἐναντίαν τῷ ὅλῳ ἠρέμα περιφέρεσθαι, αὐτῶν δὲ τούτων τάχιστα μὲν ἰέναι τὸν ὄγδοον, δευτέρους δὲ καὶ ἅμα ἀλλήλοις τόν τε ἕβδομον καὶ ἕκτον καὶ πέμπτον· τρίτον δὲ φορᾷ ἰέναι, ὡς σφίσι φαίνεσθαι, ἐπανακυκλούμενον τὸν τέταρτον, τέταρτον δὲ τὸν τρίτον καὶ πέμπτον τὸν δεύτερον. Στρέφεσθαι δὲ αὐτὸν ἐν τοῖς τῆς Ἀνάγκης γόνασιν.

* Here the differences between the MSS tradition and the "older version" begin:

ἕκτου,	MSS	:	ἑβδόμου,	older version
τετάρτου,	MSS	:	ὀγδόου,	older version
ὀγδόου,	MSS	:	ἕκτου,	older version
ἑβδόμου,	MSS	:	τετάρτου,	older version
πέμπτου,	MSS	:	τρίτου,	older version
τρίτου,	MSS	:	δευτέρου,	older version
δευτέρου,	MSS	:	πέμπτου,	older version
τρίτον,	Theon	:	τὸν τρίτον,	MSS

1. *Some nonmathematical details.* It is worth noting that certain of the nonmathematical details of this myth can be better shown to be functional if studied in the frame of reference which the notes above have suggested.

For example, the notion that the shaft of light holds the heavens together "like the reinforcing ropes of a trireme," though its exact meaning has been much debated (since we are not clear as to the details of ancient trireme reinforcement), seems not to have been recognized as an extraordinary image for Plato to have chosen. Unless the heavens are being viewed under the aspect of necessity, however, they have no need whatever to be held together or reinforced. No such devices of reinforcement are relevant to the theoretic astronomy of the *Republic,* to the natural theology of the *Laws* and *Epinomis,* to the proof of the existence of a cause of mixture in the *Philebus,* to the mathematical schematism of empirical cosmology in the first part of the *Timaeus,* or to the myths in the *Phaedrus* and the *Statesman.* If Adam is right in his belief that the reinforcing ropes ran around the sides of the hull (and his picture of an ancient trireme model seems fairly conclusive on the point),[102] the image carries out the idea of iron bands of justice holding parts in place by mathematical constraint. This is suggested also by the statement that the shaft and whorl are made of adamant and its alloys.

Further, the over-all analogy to a "spindle," which is in general "like those in common use," also underscores the mechanical nature of the imagery. The world is seen as a machine for spinning the threads of destiny, and this analogy may well be carried over into details of the imagery in ways which have thus far escaped attention.

2. *The Sirens' scale.* Extensive speculation has been devoted to the relation of the described velocities of the hemispheres and the allocation of the notes of the scale to the Sirens. These discussions seem to lose sight of or minimize the intentional echo here of the earlier image of political organization as a harmony of the classes in the state, which, like νήτη, μέση, and ὑπάτη, accord with one another like the fixed notes of a scale.[103] The Sirens must be viewed, in the light of this earlier analogy, as personifying the projection of social harmony and justice into the relation of parts of the cosmos itself, the final resolution of the problem of combining images of harmony with those of cycle. Since each Siren is presented as singing a single note, an echo of the definition of justice as the virtue which keeps parts specialized to their proper function is also projected into the cosmic harmony.

One may wonder at the reason which led Plato here to use Sirens as the mythical personification of the harmony of the spheres. In the tradition which originated the legend, the harmony of the spheres seems to have been thought of as the property of the Muses, and the Muses as patronesses of education have already appeared in a personified form in the *Republic.* It is the Muses, as goddesses of harmony, whose choir the spheres are echoing. If the allegorical significance of the myth set forth above is correct, furthermore, the personification of the celestial harmony is intended to have an educative function which the Muses might more appropriately exercise.

In resolving this problem, the first point to note is that Plato's Muses make their appearance in context as goddesses of intellectual, not sensible, harmony. There is some ground for doubt as to whether the study of "harmony" in the *Republic* has anything to do with musical song except insofar as the beauty of such art is due to the presence in it of an intelligible structure. Song, however, is the perceivable embodiment of the more intellectual mathematical realm of harmonious structure over which the Muses preside.

The substitution of Sirens for Muses is in this detail an exact counterpart of the over-all personification of concepts as visual (and auditory) imagery in the myth, and effects an emphasis on the sensuous vividness of the symbol required to arrest the attention of the inspecting souls and to register pleasurably in their disembodied experience.

The Sirens are related to sensuous satisfaction in heard harmony as the Muses are to the intellectual satisfaction of mathematical insight.

3. *Periods of time.* In general, as analysis of the description of Atlantis has shown, Plato exercised a meticulous sense of fitness in his selection of arithmetical details. The myths he recounts, some of which are explicitly identified by him as of Pythagorean origin, show a preference throughout the choice of detail for the number ten and its multiples and powers. When, therefore, in such a myth, specific numerical details are intruded which are not in this decimal tradition, it is at least worth investigating the possibility that some feeling of the greater appropriateness of his own figures led Plato to deviate from the tradition.

The present myth presents many of the traditional uses of powers of ten; in fact, the prominence of these details has been adduced as an argument for its Pythagorean origin. It also presents many striking deviations which do not have the air of having somehow been borrowed from some alternative traditional use of illustrative number. Among these may be mentioned (1) the prominence of the number 9 in the workings of the astronomical model; (2) the use of 7, 4, and 12 in the description of the soul's journey; (3) the appearance of 20 (2 × 10) as the specific number of one of the lots drawn. (This last is atypical, since it might have been more in line with tradition to make the number 100 or 1,000, and would have been just as probable in the story, unless we find some special relevance in the actual figure chosen.)

The cosmic nines and the lot of Ajax will be discussed later, in connection with their contexts. The key to the periods of travel seems to lie (1) in the calendar, and (2) in the divided line. After starting their progress, the souls journey on the path we have identified as an allegorical progression from becoming to being through *four* days. At the end of the fourth day's journey (the eleventh day), the trans-spatial and temporal region at the center of the heavens is visible.

The training of the soul to perceive being rather than becoming is presented in the earlier allegory of the cave as taking place through *four* stages, given by the image of the divided line. The fact that it has *four* segments is the main identifying characteristic given in the nuptial number of a line which represents levels of knowledge; there is probably a deliberate echo therefore, in the assigning of periods so that the souls have a four-day journey.

We may recall also that the calculation of the happiness of the just man and the tyrant is followed by Socrates' remark that the number is true if days and nights and months and years enter into the calculation in some way as relevant units of the calculation. In other Platonic contexts, two different properties of the number 12 bring it into occasional prominence. The first of these is chronological: the number and its multiples are easily equated with periods of time. The second is computational, and a third is theological. Of these properties, neither the computational nor the theological seems to have any relevance to the choice of 12 days as the period of time occupied by Er's vision in the present passage. In a later discussion of the

12-celled schematism underlying the order of appearance of all but the first and last of the cast of transmigrating souls, a reason will appear which will suggest why the dividing of a vision which progresses through all stages of destiny into 12 units of some kind would have an immediate appropriateness. This subsequent use of the number, however, can hardly be developed as the sole explanation of the detail here, since the assumption has been made that it is *antecedent* context which suggests the appropriateness of such a detail, and no 12-cell matrix appears before this passage. In all of the mathematical imagery in which, once the importance of harmony has been established, an attempt is made to apply harmony to the cycle of temporal process, one dimension or stage or factor of the construction has been devoted to establishing in some degree the relevance of a set of abstract distinctions or types to the actual periods of historical time. In the nuptial number, a mythical factor representing the period of human life in the decimal tradition of Pythagorean myth is included. In the tyrant's number, the third dimension is devoted to temporal factors, and a postscript points out their relevance to the schematic calculation. A large part of the fifth and seventh books deals with the appropriate chronological ages for changes of marital and educational status.

On the basis of these precedents, one may be permitted at least to suggest that the number 12 is intruded here as an allegorical time factor. The ordinary divisions of time are a relevant measure of the stages of development of Er's vision, but with the strange foreshortening which also appears to the view of other disembodied souls from their point of vantage as spectators of all existence and all time. This foreshortening is apparent also in the blurring of the planets in their motion in the model into apparently concentric rings. If this myth allegorically recapitulates the earlier discussion, which in turn, as we are told at its conclusion, recapitulates the lessons learned by extensive human experience (except perhaps for the section on education, which is omitted from the recapitulation at the beginning of the *Timaeus*), the appropriateness of a temporal reference seems clear. Since Plato's myth is about to show that the motions of the planets—measured by years—are the mechanism of human history, this suggestion of a chronological number as an appropriate detail gains in plausibility.

4. *Why the planets are not identified by name.* The various planets are identified here, not by name, but by a description of their appearance in the model. It seems incredible that Plato could not have found for the planets contemporary names which would have been popularly understood before the date of the *Epinomis.* There is an artistry, however, in his refusal to use such names in his myth, which justifies this peculiar circumlocution. To call a planet by its name would have put it in the framework of an intellectual discussion of scientific astronomy. But the time for such discussions is past; the rôle of intelligence has been treated exhaustively in the first moment of this book of the *Republic.* What the model of the heavens presents, with sensuous vividness, to the souls is a direct vision of how the threads of Destiny are spun; cosmic justice is sustained because they are spun as they are. True, in the order of his description, Plato indicates certain mechanical properties of this spindle which give a necessitarian aspect to its mode of spinning, but although these properties are purely ordinal and apparent on inspection, the soul coming to choose another life is not required to analyze this mechanism to gain the intended insight which, if "God is blameless," is necessary for its choice. What is essential is the sight of the twisted continuity of these threads between the eternal present and the receding future, and the rôle of the past in this continuity. The "Fates, robed in white" in Plato's description, are the visual center of attention in the panorama he is describing. It is their songs and their spinning, with the Sirens' "music of the spheres" for contrapuntal background, that activate and explain the revolutions of the "Spindle of Necessity." The colors of the hemispheres are similarly chosen because of their vividness to the spectator of the tableau which is the subject of Er's report. The suggestion of a systematic unity in diversity of concentric cosmic process is made poetic by the vision of the harmonious motion of the spangled, red, white, and yellow hemispheres.

5. *Why the souls see colored rims instead of separate planets.* The reason for assigning the colors of the celestial bodies to their zones in the machine has been explained by citing the Greek belief that the planet is attached to and carried by the motion of its orbit. How this relative activity manages to transfer from the passive planet to the active orbit the color of the former is not made clear, unless we are intended to assume that the planet, being lazy, does not deserve such ornament.

However, the tactics of explaining this device by statements less susceptible to interpretation than the original can be set aside if we bear in mind what the effect of a location on the eminence of being has on the perspective of the spectator of the physical world moving below. The philosopher is not made dizzy by the high altitude from which he is a spectator of all time and all existence; his broadened concepts of space and time are far removed from those of common men, impressed by large estates or ancient lineage; for the size and length of these shrink to points against the standard of the ideas of all space and all time. The souls in this myth have ascended to this vantage point, but, lacking the disciplined intellectual insight of the philosopher, are represented as perceiving only those visible and tangible features of the universe which the senses can convey. The important philosophic concepts which inhabit this eminence reveal themselves in visible personified form when awareness of their presence is required.

The natural effect of this new vision is a foreshortening of time and space to a point at which even the longest celestial revolutions can be intuitively perceived and need not be laboriously calculated with the help of many models, as they must be inferentially established by the terrestrial student of descriptive astronomy. The capacity to "see together" many distant things in space is matched, as it is said to be elsewhere in the description of philosophic insight, by a "seeing in one view" of many things far separate in time. To make the physics and astronomy of the story sufficiently plausible to gain even poetic credibility, we may have to assume that this new vision is actually presented by the Fates as a model of what the souls would see from their vantage point, had sight the power of intelligence to pierce through the outer spheres to the hidden workings of the inner ones. This only underscores the fact that the inspection of the world-machine is one which vastly shrinks its spatial scale and one in which the temporal scale is correspondingly diminished. If the ratios of speed were preserved and if the scale were set so that the ring of Saturn revolved once in a minute, the effect of such temporal shrinkage would require that the fixed stars travel at about 10,000 r.p.m. But, under the impact of its propulsion, the foreshortening of time in this model is evidently carried much further. The blurring of individual planets into what seem to be revolving concentric rings is the natural and apparent effect of the change in temporal perspective which makes possible any direct perception of their relative motion. All that it signifies here is that the souls are seeing a vastly speeded-up model of celestial process, not that there has been any inexplicable transfer of color to orbits from planets, nor, necessarily, that we have a remnant of archaic Pythagorean astronomy.

The extraordinary foreshortening of time-periods in the souls' inter-incarnation careers seems to corroborate this interpretation. The myth opens with pleasures and pains awarded in periods of a thousand years; it closes with a four-day journey and a one-day period of education and choice. The number 10,000 is used in myths to indicate an indefinitely large but denumerable set or period. Consequently, no two periods Plato could have assigned would have been more neatly calculated to set up a contrast between the enduring and the ephemeral. (This shift is paralleled in the change of the earlier description of human life as "a period of one hundred years" to the prophet's opening words, "souls that live but for a day.") The underscored shortening of time periods becomes functional if it is taken as paralleling the more expansive vision of time and the consequent foreshortening of time scale which the souls attain as they move to a "higher" place. It may be assumed that Er, as special observer and messenger, was in an exceptional position with respect to the correspondence between perceived and absolute time. If a year or a hundred years seem merely part of a single day to the soul, we may legitimately allow Er to view events covering several centuries and return to earth in twelve days of elapsed terrestrial time. He must see the whole process, yet he must return to his body with his report. His return takes place in a definite period of time, the length of which suggests that time is somehow important; but, in the absence of any clear anticipations of modern physics, we are not told how the gods who have selected him as messenger accomplish this relativity of time.

Earlier in the dialogue, when his discussion made this becoming-to-being transition, Socrates himself was the subject of an exactly similar time-sense shift, which was indicated in his statement that "the time spent in such discussion seems but a moment compared to the cycle of the soul's transmigration." [104] His contemplation of the forms basic to the discussion of education has so colored his orientation to time that the remark of a member of his audience, who applies a normal time scale, seems to be inappropriate and calls for explicit correction.

6. *Allegorical function of the model.* The concept of equilibrium serves in this account as the observable projection of the principle of perfection; the balancing of parts with one another is the sensible evidence of the presence of an organizing plan. Equilibrium is a property of mass, not of volume or color; but the balancing of the lists of colors and volumes seems to suggest that these are the properties to which the concept is here applied. Mass is itself, however, related to volume and unit density, and unit density can be inferred from color. The lists of colors and volumes are combined to form a balanced list of masses; they are presented separately because only one of the factors—that of density—can have its balanced distribution verified by empirical observation, and the volumes must be inferred from a postulate of balanced masses to which this observable arrangement of colors gives some empirical confirmation. It is the balance of this mechanism which is evidence that the parts have been adjusted to one another by some plan.

A further function of this description is the insight it gives into the apparent motion of the model. The impulsions of the goddesses of future and past are both distributed with complete uniformity through the model, in such a way that some irregularity is created at the point where the transmitted forces are equal.[105] The circle at this balance point must seem, from the mechanics of the model, to have a peculiarly erratic behavior. As if endowed with a free will, it moves against the dominant direction of the total system only to be brought back by the dominating motion. These excursions, while reflecting an irregularity in impulsion, are not sufficiently forceful to alter the entire dynamics of the system. This behavior will be interpreted as visual confirmation of the presence of freedom of will at the moment of confluence between future and past. In this way a present choice is not necessarily in accordance with the natural order, but it cannot succeed in altering nature to its own plan.

The vision of destiny turning the cosmic spindle is thus, as we should expect from its tableau form, an allegorical representation of choices in time by which man creates his own destiny.

7. *Colors of the hemispheres.* The intrusive, detailed description of the colors of the rims of hemispheres of the world-machine in this myth has remained one of the least satisfactorily explained passages in the *Republic.* Unlike the notorious nuptial number, of which each generation of scholars produces a new interpretation, the function of this list of colors has never been given any plausible explanation. A. E. Taylor wrote that while he was sure the list represented some empirical counterpart, and had some principle of order, he could not even imagine what that counterpart and principle might be.[106] Because of his erudition in classical and modern Platonic scholarship Taylor's statement lends credence to the claim that no plausible function has ever been suggested.

The closest approach to such an interpretation was the discovery, late in the nineteenth century, that a principle of balance had been observed in the order of the hemispheres for which colors are mentioned, so that symmetrical pairs add up to a total ordinal value of nine.[107] But what conceivable meaning one is to give this meticulous "balancing" of color is left an open question at least as difficult to resolve as the original problem.[108]

Yet there is an interpretation which seems to explain both the presence and the order of this list of colors and to render the passage functional in context. The key to this interpretation lies in the conviction, which grows with study of Plato's mathematical imagery, that he is too precise a craftsman ever to have intruded a clumsy metaphor treating balance as a property of color.[109] Balance is, as the *Timaeus* makes clear,[110] a property of *weight,* not of volume or of color. The purpose of the present section is to present an original interpretation of the principle of order operative in the listing of colors which (*a*) establishes an empirical reference, (*b*) avoids attribution of this clumsiness of metaphor construction to Plato, and (*c*) gives the list a function which helps justify its sudden, emphatic intrusion in the myth.

First, we should note that these colors are properties of parts of a model of the world-machine, displayed by the goddess Necessity. This goddess seems to be a personification of the concept which, in a "necessary" account of cosmology, represents an analysis of the world under the aspect of mechanism rather than under the aspect of organism.[111] In the natural order, organic functioning as a whole is constructed and mediated by mechanical interaction between separate parts. It is completely in keeping with this identification of the goddess with the concept she personifies as well as with the archaic tone proper to a religious myth, that the cosmic model shown by her to the souls should be an archaic *Ionian* or *Ionian-Pythagorean* device.[112] It stresses celestial mechanics rather than the pure mathematics of astronomical motion typical of the research in the Academy, and more appropriate to a description of the world *qua* organism.[113]

In the necessary account in the *Timaeus,* the key concept to the behavior of the machine as a whole is that of *equilibrium,*[114] a property determined by the relative shapes and volumes of parts, which we observe empirically on a macroscopic scale as differences in hardness and weight.[115] The discovery of a metaphor of balance in the colors and volumes of Plato's lists strongly suggests that their author intended to underscore the presence in his model of some sort of equilibrium proper to the presence of perfection in a world-machine.

If this is true, however, it is not enough for both the volumes and colors of the hemispheres to be balanced; the point intended must be that the masses are balanced.[116] Whether this is in fact the principle of construction underlying the lists of sizes and colors can be investigated from two directions. The first is the consideration of the relation of color and density ("hardness") in the chemistry of the *Timaeus,* as compared to the observed colors in the present list. As we shall find, this comparison, while it does not contradict the hypothesis that apparent color is associated with relative density, is not extensive enough to confirm it definitely. If the list of sizes represents volume, that of colors in some way represents density, and if a balance of masses underlies the construction of the two lists, then a second tactic of confirmation is to construct and examine the new ordinal list which results if we add or multiply together the relative sizes and colors of each hemisphere.[117] The result, which is given in figures 66–73, seems conclusive confirmation. The final list not only shows a balancing of symmetrically paired terms which add up to nine but represents the property in a clearer, more evident form than either of the two sets from which it is constructed. The probability that such a result would come about accidentally, if the two lists so combined had not been planned with just this result in view, can be calculated, and is so slight that this explanation of the balancing of the list of sizes and colors may be dismissed as a practical impossibility.[118]

Plato himself invites the reader to make this association of color and physical composition by his initial remark about the composition of the hemispheres, that "the others [i.e., those other than the outermost circle] had varying amounts of adamant in their composition." [119] He then presents a listing of colors ordered, as we can demonstrate from the *Timaeus* color theory,[120] in terms of the *intensity,* not the hue, of their colors.

The color spectrum of the *Timaeus* theory takes as primary colors black and white,[121] then differentiates (still as primary) brilliant,[122] white, and red,[123] all acting mechanically on the visual stream like white light, but shading in intensity from a maximum to a minimum. The colors of the hemispheres, except the outermost, which has special properties, are listed beginning with *brilliant* and following with *yellow* (a composite color intermediate in intensity between its constituents, brilliant and white),[124] *a pure white, red,* and a *secondary whiteness* (which would be a composite color resulting from the mixture of white and black, a grey lower in intensity than red).[125]

If "the presence of more and less adamant" is revealed by a greater and lesser intensity of color, the principle of order can be functionally explained.

So far as it goes, the chemical theory of the *Timaeus* does not contradict this association. Adamant and gold, the constituents of the outermost hemisphere, as described in the *Timaeus,* are the most dense of materials; [126] bronze, having a yellow color, is less dense than gold; brilliance and reflecting power are the properties of "transparent stone," which, having the most homogeneous elements of all earth compounds, is also the most dense, though the relative density of such stone and bronze is not given. The other colors do not appear as identifying properties of basic types of compound except for the mention of the color white as associated with various saps and juices.

It is not hard to see why Plato complicates his final exposition by combining two separate lists of properties. Of the two properties relevant to a mechanical equilibrium of the world-machine, the relative volumes of zones admit of no empirical confirmation.[127] Through the posited association of density and intensity of apparent color, however, a postulated principle of order in the distribution of celestial masses can be *tested and confirmed empirically*. The empirical fact, that an order is operative in the relation of colors and distances of the planets, is in turn justification for constructing a list of volumes, on the postulate that these also are distributed according to some order, presumably an order showing the principle of mechanical equilibrium in operation.

The justice of the world, which the souls are to learn from their inter-incarnation education, is thus at least in part evident to every man who takes the trouble to inspect the appearance of the heavens. Furthermore, the very colors of the planets constitute an empirical argument for the justice of the world order.

The principal objection that seems likely to be made to this interpretation is that it puts an undue burden of interpretation on the reader. Although it is extraordinarily unlikely that the balance of the composite list is merely a result of chance, it may be objected that this improbability is balanced by an equal unlikelihood that Plato would expect such elaborate mind reading from his intended audience. In reply, I can suggest only that this intended audience would naturally be expected to look for mechanical properties in an *Ionian* astronomical model, and might quite reasonably be expected to take the remark about relative proportions of adamant as a hint as to where these properties were to be sought. Further, the myth is equally valid and effective for the reader if he does not try to interpret these lists of properties *at all,* but reads them simply as detailed evidence that *some* rationale is at work in the cosmic order, a rationale made apparent by the mechanics of this order. As a teller of myth, Plato here needed only to suggest aesthetically (and detailed listings are such a device of aesthetic suggestion) the evidence of a principle at work in the mechanical details of the cosmic order. However, as a conscientious inventor of mythology who believed that the central points stressed in the myth were true, Plato could not honestly have gained his aesthetic effect by inventing a set of descriptive statements about the observable mechanical properties of the world to which the facts did not correspond. His aesthetic device of listings therefore embodies in it, for the reader not willing to take the myth on the level of belief, but insistent on testing it by intellectual inquiry, a basis of empirical fact to which the aesthetic properties can actually be shown to correspond.

8. *Motion of the hemispheres.* Following his account of the structural details of the model, Plato presents a list of the properties of its dynamic behavior. This list displays the same

principle of balance in the velocities of the hemispheres that was the clue to the lists of their sizes and colors; [128] and if we interpret the former lists as determining mass, as was suggested above, this balance of velocities will also establish a balance of momentum.

There is a contradiction in engineering between the two stages of structural and dynamic description which must be recognized, since it prevents the full deduction of the model's dynamic from its structural properties. (Theon of Smyrna actually attempted to make this deduction, with the aid of a reconstruction of the model here described.[129]) Though the operation of the principle of perfection is represented in the same way, as an equilibrium of the ordinal numbers used in the two descriptions, the structural properties do not lend themselves to motions of the sort described. The effect of the relative velocities involves a diffusion of retrograde motion from the innermost circle, where the retrograde impulsion is applied, to the seventh circle; and the strength of this motion decreases in proportion to the distance from this central source.[130] This type of fluid transmission of motion from a center of disturbance was familiar to the Greek scientists, and most often illustrated by the behavior of eddies in water. It is a basic mechanism of the vortex theories with which the formation of the cosmos was explained by the Ionian and later Atomist philosophers.[131] But if the hemispheres are conceived as being rigidly defined by the iron bands of Justice, which hold each to its proper place, this idea of fluid transmission will not work. Consequently, a reader anxious to construct a model of this machine would actually have to build two models, one showing the structure of the machine at rest, the other duplicating its apparent motion. The latter would involve some transmission of impulsion through a fluid medium in such a way that momentum varied with distance, and the forward and retrograde components gave resultant motions to the circles of the sort Plato here describes.[132]

Although this defect in engineering might seriously disturb an Ionian physicist, it constitutes no obstacle to the reader, even to one who interprets the description in detail. The dynamic properties of the model are taken as those given in the list; one can easily visualize them. Having visualized, one can look for the operation of a principle of perfection or order in the interrelations of the motions, and even construct a table of the momentum which hemispheres of the sort described would have if they were moving in the system postulated by the dynamics section of the description. There is nothing difficult about combining a visualization of these sizes, colors, and relative speeds in an imaginative picture, unless one has a singularly limited and literal imagination. We shall see that the detail of these dynamic properties and their relation to the structural account still provide the sense and order that the myth requires, even though some supernatural power is needed to make the structure behave as described.

The reason for this difficulty is of course an interference between the facts of mechanics and the allegorical and poetic demands of aesthetic suitability. In fact, though not in poetry, the celestial spheres are not held to their courses by iron bonds; the adamant walls bring out admirably the presence of cosmic justice, but at the expense of requiring a divinity to make the walled-in spheres responsive to impulsion in the fluid manner that an empirical, atomistic account of transmission of motion describes.[133]

In the motion of this model the momenta and velocities are both balanced into three zones: one predominantly impelled by the forward motion, one by the retrograde, the third (twice in mass to either of the others) intermediately moved by both.

The central axis of all of these symmetrically balanced lists falls at the fourth circle, midway between the two points of forward and retrograde impulsion. The reader with wax tablets, having jotted down and noted the balance of these descriptive details, would naturally have seen that circle as occupying the center of the stage, and wondered whether something dramatic in its behavior might have led to its prominence in the imagery.

In fact, of course, Er singles out for special mention an anachronism in the motion of this circle. The fourth circle moved with "apparent counter-rotations" or "turnings back." The mechanism of balance, however, gives a physical explanation for this aberrant behavior. Since this circle is at the exact point of balance of the forward and retrograde momenta, any change in impulsion will be most evident in its behavior; and the impulsions of the model are constantly changing and mechanically erratic. A backward jump of Mars follows as an effect of a reinforced impulsion of the inner circle. This evidence, which Er has previously mentioned, that the Fates turn erratically can, however, by no means exhaust the significance of the retrogradation if the phenomenon is important enough to deserve the prominence it seems to be given. There should also be, since this is a true myth, an identifiable empirical counterpart, and, since the myth is allegorical and since later context requires it, an identifiable applicability to the phenomenon of human choice, either in this life or in our next selection of incarnation.

The empirical counterpart is, of course, the observed retrogradation of the planet Mars, which the *Timaeus* describes.[134] Cornford's commentary on that passage seems to establish definitely that this is what it is describing, and also that Plato explains this apparently aberrant behavior as an exercise by the planet of its *free-will*.[135] This aspect of the empirical referent of the present passage seems, when we bear in mind that the passage in the *Timaeus* centers in the problem of the relation in human choice of freedom and necessity, to be the key to the allegorical interpretation which Plato may have had in mind when he put the phenomenon into the foreground in his myth.

Before attempting an application of these mythical mechanics to the central contextual problem, it will be necessary to consider the model in terms of another aspect. The machine that the souls are shown is repeatedly qualified in terms that are intended to underscore the fact that a model of celestial motions is also a model of the *nature of time*. More than tradition or caprice underlies the allocation of past, present, and future to the separate jurisdictions of the Daughters of Necessity. The themes of their songs,[136] and the impulses they give the model, alike invite some association of cosmic mechanics, freedom, and temporal flow from the eternal present to the indefinite future.

The song of the goddess of the present is about that which is; the motion she imparts, with her right hand, is the "motion according to the same" of the *Timaeus* account; [137] this motion flows, through a zone of momentum primarily under its domination, to an intermediate zone, balanced between progression and retrogradation, into the contrarily dominated inner zone of the machine, the locus of the indefinite "that which is about to be," as distinct from the locus of the perfectly determinate "that which is." [138] In this passage from same to other, the intermediate zones are balanced much as a moment of present choice exists suspended between present and future. The direction taken by the intermediate zones varies with the differential activity of the goddess of the past, who sometimes reinforces the one motion, sometimes the other.[139]

Considered as a time-machine, the phenomenon of balance takes on new significance in this context when it is equated with the similarly balanced moment of choice which each man confronts many times in his historical career, and is shown reenacting in the present mythical drama. In the *Theaetetus,* human choice is explained by the statement that "each man has many patterns of the sort of life he would like to lead. Of these, he selects one; and his destiny is actually to become the type he has selected." [140] In *Republic* ix, we learn that a good man will contemplate the pattern of the citizen of the best state, a pattern to be found only in heaven, and will live as a citizen of that state, and no other.[141] We also learn, earlier in the *Republic,* that past choices have a predisposition to repeat; a man who imitates something grows into a likeness of the thing imitated.[142] If we now interpret the influence of the motion of the same as representing the influence in any present moment of the forms, always existing and knowable in an eternal present, the motion of the other as a projection of these forms into a temporal medium (where they are subject to distortion and counteraction), and the motion of the past as a persistence of habits influencing choice, the time-machine seems very like a model of the human will.

If we accept this interpretation of the balanced zones of present and future, with a moment of choice between, into which both are alike ingressive (a double ingression which, of course, lends some point to making this intermediate zone exactly twice the momentum of either of the others), the importance of Mars is readily explained. The aberrant behavior of this planet provides direct empirical evidence that freedom is possible within the mechanism of the world-order. The Fates adjust the mechanism of their machine in such a way that its behavior allows expression to a free choice which runs counter to the dominant momentum of the natural order. There is no absolute determinism, because the rigid mechanism adapts itself to implement decisions, just as the parts of an organism adjust mechanically to new conditions which confront that organism as a whole. Once the adjustment is made, it is irrevocable; its mechanical consequences cannot be evaded or recalled.

A close observer of the model would also find that although nothing prevents Mars from acting on its "desire" to move contrary to the natural course of things, it cannot carry out that countermovement long. Its own momentum is so much less than that of the whole set of turning hemispheres that its contrary career is soon overborne and carried along by the inexorable momentum of the total order to which its own desire was opposed. In the same spirit, Plato later argues that the temporary success of injustice is no warrant for disbelieving in the over-all perfection of God's management of the world; the laws of nature are arranged, like the rules of chess, to move each man automatically to the status he deserves.[143] In *Republic* ix, the same thing is suggested. Plato says even that if we consider men's lives as wholes, the span of life is often long enough to show this reward and punishment in operation.[144]

Having seen this model, the soul who has learned from it will be guided in his choice of a pattern of life by the reflection that such a pattern should resemble the world-order in which and with the concurrence of which the chosen rôle will be enacted.

In every moment of choice, the chooser is balanced between the eternal forms, which suggest a right standard, and the impulse of his past interests, which establish a certain inertia and impel him to remain in the same course he has been following. The future lies ahead as the locus of actualization of the career chosen by him as a present possible pattern; whether this actualization will be good or bad depends on the adequacy of the subjective rôle chosen to the proper rôle of man, laid up as an eternal pattern which will be furthered by nature.

On this interpretation, the lessons of human responsibility and human freedom are presented to every soul when it is shown the innermost workings of the world-machine; and the choice of each after this instruction is clearly made on its own responsibility.

This interpretation, in addition to the fact that it explains the functionality of the elaborate details of the mythical machinery, can be partially confirmed by a comparison with the earlier mathematical imagery of the dialogue. If the myth sums up the entire sweep of the dialectic, one might hope that the mathematical image central to that myth would also in some way synthesize and sum up the entire range of the antecedent mathematical images. In this connection, we may note the recurrence of the concept of impulsion and momentum associated in Book iv with the wheellike growth of the state in an inevitable upward spiral of progress.[145] It has already been suggested that this image and the downward spiral of Book viii are combined and balanced in the present imagery of cycle. In a later note, it will be shown that there is also a synthetic inclusion of the tyrant's number computation in the emphasis placed in the present image on the number nine. It has already been remarked that the initial image of harmony is here personified by the singing Sirens whom the souls see and that the length of the soul's journey takes on new significance if read as a cross reference to the four stages in the image of the divided line.

This remarkable synthesis of diverse imagery, which one could not find duplicated in any writer for whom the construction of such images did not hold aesthetic interest and value, is also present in another Platonic passage, the construction of the world-soul, to be interpreted in a later section. The synthesis of antecedent images in the arithmetical and geometric cross references of his final model helps to validate the notion presented earlier that the myth as a whole, read as allegory, performs the same function for the antecedent dialectical discussion.

9. *Rôle of the number nine.* Another fact apparent to the reader with a wax tablet filled with diagrams, but not to the simple observer of the model, is the intrusiveness of the number nine. Although this is the only choice by which one can establish an arithmetical metaphor of balance based upon the ordinal numbers from one to eight, numbers other than one to eight might be chosen if the number nine were not considered to be appropriate in context. The balanced character of the mechanism could certainly be brought out as well by an adumbrated scheme of ratios as by the ordered list; and such an adumbration would not be out of place in the context of the Sirens, whose presence reminds us of the fact that there are internal relations in cosmic connection. Further, the use of nine as a central number breaks Plato's usual pattern of mythical arithmetical analogy; his numbers in myths and semimythical histories are elsewhere presented as tens and powers of ten, presumably as a joint effect of Pythagorean practice and the presence of "myriad" as the largest common number-word in the language.[146] The periods and incidental numbers, such as that of the lot of Ajax in the present myth, preserve this preference for multiples and powers of ten; one may therefore wonder whether there is not some reason why the details of the mythical model do not.

If the mathematical illustrations of the *Republic* are interconnected (and by a sort of poetic economy each is built from the elements of its predecessors), then we may look for an explanation of this use of nine in what would normally be a wholly decimal context by considering the functions of nine in the antecedent mathematical imagery. We are certainly invited to identify this number with the nine-celled matrix of human careers and characters of the tyrant's number and of the nuptial number by substituting nine, in both of those images, as an arithmetical analogue for the nine-part matrix figure whenever a computational metaphor is used to summarize the structure of the diagrams. The representation of each of the three dynamic components of the time-mechanism of the universe as equivalent to the number nine may have been felt to be appropriate because of its connection with the intended moral interpretation of the model's behavior. The symbolic number remains the same for past and future, and in the present is doubled through the copresence of a given present nature and a desired future goal. If this significant relevance to antecedent imagery is actually the explanation of the retention in a mythical context of this particular number, representing moral balance, its significance is clearly the invariance of human character through time. New types of character, and new ideals, are not operative in human choice; rather, the same basic nature and superimposed education are to be found in the present and in the pattern which represents ingredience in the present of an ideal future. Furthermore, the invariance extends to the relation of character as reflected in moral choices. The use of the moral number as a symbol of cosmic balance further underscores the continuity, in human life, of future, present, and past. The irreversibility of the machinery of fate insures that each choice, helping as it does to establish the nature of the chooser, will be rewarded with an appropriate destiny, even though there may seem to be in some given case a temporary retrogradation of cosmic justice, which in human life permits the apparent securing of desired ends by means which are unnatural and inappropriate.

Thus from the model he is shown the soul learns the same law that the prophet proclaims to the assembly: "The choice is yours; God is blameless." [147] Permanent and real, as opposed to transitory and apparent, freedom resides only in the choice in which lessons of past experience motivate the individual to choose on the basis of those eternal truths.

d. Text of the Passage: * Theon's Version

The rim of the largest whorl was spangled; the seventh brightest; the eighth colored by the reflected light of the seventh; the second and fifth like each other and yellower; the fourth was red; the sixth second in whiteness. The Spindle *rolled forward* as a whole with one motion, *that of the [daily revolution of] the cosmos;* but, within the whole as it turned, the seven inner circles *were led around* slowly in the opposite direction; and of these the eighth *itself* moved most swiftly; second and all moving together· *with equal speed,* the seventh, sixth, and fifth; next in speed moved the fourth with that *they said* appeared to be a retrogradation, *more marked than that of the others;* next the third, and slowest of all the second.

Text † of Theon's Version, with Theon's Variants

Κυλίεσθαι δὲ στρεφόμενον τὸν ἄτρακτον ὅλον μὲν τὴν αὐτὴν φορὰν τῷ κόσμῳ, ἐν δὲ ὅλῳ περιφερομένῳ τοὺς ἐντὸς ἑπτὰ κύκλους τὴν ἐναντίαν τῷ ὅλῳ ἠρέμα περιάγεσθαι, αὐτῶν δὲ τούτων τάχιστα μὲν ἰέναι τὸν ὄγδοον, δευτέρους δὲ καὶ ἅμα ἀλλήλοις ἰσοταχῶς τόν τε ἕβδομον καὶ τὸν ἕκτον καὶ τὸν πέμπτον· τρίτον δὲ φορᾷ ἰέναι, ὃν φασι φαίνεσθαι, ἐπανακυκλούμενον μάλιστα τῶν ἄλλων· τὸν τέταρτον, τέταρτον δὲ τρίτον· καὶ πέμπτον τὸν δεύτερον.

Theon's Variants—

Κυκλεῖσθαι,	MSS	:	Κυλίεσθαι,	Theon
φορὰν,	MSS	:	φορὰν τῷ κόσμῳ,	Theon
περιφέρεσθαι,	MSS	:	περιάγεσθαι,	Theon
ἀλλήλοις,	MSS	:	ἀλλήλοις ἰσοταχῶς,	Theon
ἕκτον καὶ πέμπτον,	MSS	:	τὸν ἕκτον καὶ τὸν πέμπτον,	Theon
τὸν τρίτον,	MSS	:	τρίτον,	Theon
ὡς σφίσι,	MSS	:	ὅν φασι,	Theon
ἐπανακυκλούμενον,	MSS	:	ἐπανακυκλούμενον μάλιστα τῶν ἀλλῶν,	Theon
τὸν τρίτον,	MSS	:	τρίτον,	Theon

Though a copyist or scholar might hesitate to tamper with

* *Republic* 616E, trans. Cornford, *Republic,* p. 354. Theon's variants have been supplied in italics. See also the notes to the Greek text, following.

† *Republic* 616E–617C, Chambray, *République.*

the text of a passage he was sure he had not understood, the empirical referents of the present passage must have presented constant temptation to introduce adjustments bringing it into better conformity with later commonplaces of empirical astronomy. That these adjustments did take place is clear from Proclus' "older and better" text of the sizes, already given.[148] The reading is not older, because it disagrees with Theon's text; it is not better, except that a Hellenistic commentator can interpret it more easily. What took place here was an entire rewriting of the list of sizes, apparently on the principle that A. E. Taylor has pointed out, of identifying distance with apparent luminosity. This list is given in Figure 68, following.

One may therefore wonder whether the differences between Theon's text and the manuscript tradition reflect any such intruded accommodations on either side. The variants are given in the text and translation just preceding.

Of these, Burnet proposed to accept the qualification of the retrogradation of Mars as "greater than that of the others" since (1) this is needed to secure accurate correspondence with the account of apparent planetary motions in the *Timaeus*, and (2) the length of the phrase, 15 letters, is exactly what the length of a line in the older manuscript of Plato must have been, suggesting that Theon retained a line lost in later copying.[149] Burnet further suggested that since Theon says he has used the commentary of Dercyllides, he presumably had the latter's text at hand. It is likely that in quoting this passage, on which he spent so much scholarly effort, Theon would have quoted more meticulously than he does elsewhere.

The difference in the descriptions of the motion of the whole could easily result from a transposition in copying, and since Theon's term carries with it a notion of "translation" (linear motion through space) as well as "rotation," it is inappropriate here. There does not seem to be any ground for choice between the other variant pair of verbs describing revolution. In this context, such a verb should be capable of taking on a connotation of some external impulsion or mechanical constraint, as either of these can.

At *b2,** editors have all recognized the superiority of Theon's text to the manuscript version. At *b1,* Theon's addition of two definite articles to differentiate the ordinal designations of the hemispheres from those of their velocities may not be essential, but seems preferable. At *b3,* Theon's omission of the definite article gives less clarity than the manuscript text, and seems inferior.

Theon's addition of "the cosmos" to the reference to the motion of the whole is certainly not an improvement if any separation of cosmos and model is to be preserved in the passage, and the empirical referent is clear without it. Nor, unless it implies some alternative type of measure, does the addition of "equally swift" in Theon's text add anything of importance. If "velocity" here refers to apparent celestial motion, "equal velocities" of different circles would seem to mean "equal periods of revolution," which of course implies different linear velocities. If the ordering principle is linear retrograde velocity, to distinguish the 7th, 6th, and 5th circles in ordinal listing and to carry out the image of "music of the spheres," the qualification that 7, 6, and 5 are "equal in velocity as measured by period" ought to be more explicit.

These problems must be left to future editors of the *Republic,* but some of Theon's variants should be given careful consideration.

Figure 66

LAW OF NINES: SIZES

Order in Series	1	2	3	4	5	6	7	8
Size of Rim	1	8	7	3	6	2	5	4

(19) (17)

J. Cook Wilson's article, "Plato, Republic 616E," *Classical Review,* XVI (1902), 292–3, is the basis of James Adam's interpretation in *The Republic of Plato,* II, pp. 470–79, in which Adam reverses his

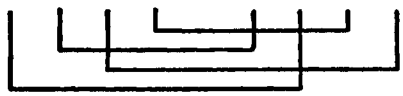
* This symbol and the two following (*b1* and *b3*) are abbreviated references to Stephanus pages; fuller designations would be *617B.2, 617B.1,* and *617B.3,* referring to *b2, b1,* and *b3,* respectively.

own earlier defense of Proclus' alternative text (Adam, "On Plato, Republic 616E," *Classical Review,* XV [1901], 391–93). Cook Wilson's figures ("If the order in which breadths of rim and colors are listed is set down for each hemisphere, the following figures result," *loc. cit.*) are reproduced as figures 66, 67, and 71.

Figure 67

LAW OF NINES: COLORS *

Order of whorls	1	2	3	4	5	6	7	8
Order in which their colors are described	1	4	6	7	5	8	2	3



(18) (18)

* See text comment of Figure 66 for citation.

This is not a very clear-cut instance of balance, and, in its context of balanced lists, suggests that perhaps some error has been made in the reconstruction of the figure, hiding a more spectacular pattern of balance. In the list of sizes the order of hemispheres gave the balanced figure, but the order of mention was the same as the order of whorls. Since this was not the case in the list of color, one might try reconstructing this diagram, taking order of listing rather than order of cosmic location as the ordering principle. The result of this revision is shown in the following figure.

Figure 68

LAW OF NINES: COLORS (ALTERNATIVE VERSION)

No.	1	2	3	4	5	6	7	8
Color	1	7	8	2	5	3	4	6

(18) (18)

This is simply a different form of Cook Wilson's diagram, with the rows transposed; this form will be used in the discussion that follows since it permits certain significant relationships of the two listings to be more clearly indicated than in the original representation. It is preferable also because it presents both lists in the same matrix form of:

ROW 1 : ORDINAL PRINCIPLE 1 2 3 . . . .
ROW 2 : COSMIC LOCATION

For any comparison of the listings, it is of course essential that they be constructed in the same form.

Figure 69

SIZE AND COLOR (VOLUME AND DENSITY)

Width	1	8	7	3	6	2	5	4
Color	1	7	8	2	5	3	4	6
Sums	2	15	15	5	11	5	9	10

Adding the two orders shown to balance in figures 66 and 68 gives this combination of volume and color. The result of translating these sums back into an ordinal list, from 1 to 8, is shown in the following figure.

Figure 70

SIZE AND COLOR (ORDINAL): ANOTHER LAW OF NINES

Order in series	1	2	3	4	5	6	7	8
Rim and color	8	1	2	6	3	7	5	4

This figure results from assigning ordinal numbers to these sums, exhibited in Figure 69, in constructing a list of relative momentum (observing, as is done in the color list, the convention that two equal sums are numbered with the greater to the left). This discovery might have been made by Adam and Cook Wilson if they had realized that their ways of constructing the lists of size and color were disparate and had reconstructed the color diagram in its proper form.

Note that Figure 71 divides the dynamic behavior of the mechanism into three zones of momentum, one dominantly forward, one retrograde, and a third (double in size) balanced between the other two. The starred numbers represent hemispheres with the same velocity; the balance is equally well demonstrated if we assign the average ordinal velocity (6) to each, instead of following the right-left convention in giving them successive ordinal numbers.

Figure 71

LAW OF NINES: BALANCE OF VELOCITIES †

Hemisphere	1	2	3	4	5	6	7	8
Relative Speed	1	8	7*	6*	5*	4	3	2
	(9)			(18)			(9)	

† See text comment of Figure 66 for citation.

Unless Plato's constructive ingenuity gave out at this point, one would expect to find some balanced figure representing the combination of balanced velocity and balanced mass; evidently certain details of the myth call for a recognition of some sort of balance of momentum. (See Fig. 73.)

Figure 72

PROPORTIONALITY OF DISTANCE AND FORWARD VELOCITY

Order of distance	8	7	6	5	4	3	2	1
Order of velocity	1	2	3	4	5	6	7	8
Sums	9	9	9	9	9	9	9	9

The passage in *Laws* x on the reasonableness with which radius and velocity are kept proportionate by nature in revolving planes or solids suggests that this sort of relation of relative distance and speed may have been the root or starting-point of the various demonstrations of "arithmetical balance" of ordinal numbers in the present passage. In an aesthetically ordered system, other properties ought also to have complements which "balance," and as these properties approach more closely the dynamic character of the system as a whole, their symmetrical balancing should become more evident.

Something about this list, or perhaps an initial ordinal listing of another property, suggested to Plato an arithmetical illustration of some aesthetic concept with which a reader of his time would have been familiar. Whatever this concept might have been, it was not recognized by the Hellenistic scholars when they wrote their explanations of this myth.

Figure 73

THE FINAL LAW OF NINES: EXACT BALANCE OF MOMENTUM IN THE SYSTEM

Circles	8	7	6	5	4	3	2	1
Masses *	1	8	7	3	6	2	4	5
Speeds †	1	2	3	4	5	6	7	8
Sums	2	10	10	7	11	8	11	13

Momenta (Ordinal Arrangement of Sums)

8	4	5	7	2	6	3	1

* From Figure 70.
† Linear forward velocity.

This is the law showing the balanced momentum (mass and velocity) of the system. If the series is thought of as cyclical, so that the initial 8 follows the final 1, the principle of location of balancing terms has become the simplest possible in this diagram.

Figure 74

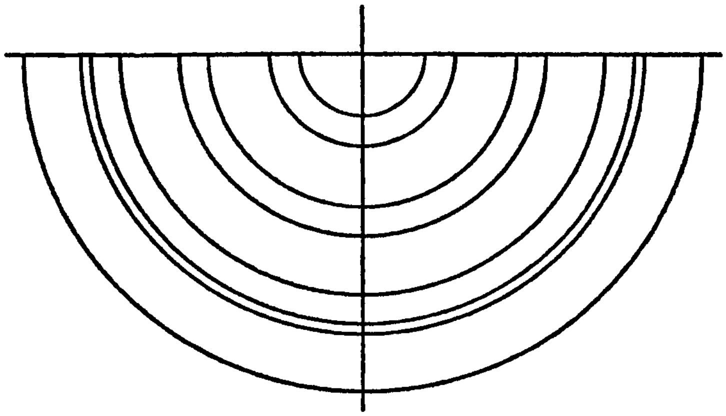
LACK OF BALANCE IN PROCLUS' ALTERNATIVE TEXT OF LIST OF SIZES

Order in series	1	2	3	4	5	6	7	8
Rim	1	7	8	6	4	3	2	5
		(22)	(No balancing of nines)				(14)	
			(No balancing of sides)					

Note that this version shows neither a balancing of nines nor a symmetry of sides; since the lists of colors and velocities do show such balance, the evidence against this text seems conclusive; and the demonstration in the foregoing figures of the balance of the combined lists of colors and volumes based on the MS text confirms the rejection of this "older reading."

Figure 75 shows the spindle cut down the center, to bring out the mechanical nature of the model. If these semicircles are completed on the same scale, the result is a top view of the lips of the hemispheres, which the souls see.

Figure 75



NECESSITY'S SPINDLE IN CROSS SECTION

Scale: Width of smallest circle = 1/16″

IX. The Myth of Er—Transmigration *

It was indeed, said Er, a sight worth seeing, how the souls severally chose their lives—a sight to move pity and laughter and astonishment; for the choice was mostly governed by the habits of their former life. He saw one soul choosing the life of a swan; this had once been the soul of Orpheus, which so hated all womankind because of his death at their hands that it would not consent to be born of woman.[1] And he saw the soul of Thamyras take the life of a nightingale,[2] and a swan choose to be changed into a man, and other musical creatures do the same. The soul which drew the twentieth lot took a lion's life; this had been Ajax, the son of Telamon, who shrank from being born a man, remembering the judgment concerning the arms of Achilles.[3] After him came the soul of Agamemnon, who also hated mankind because of his sufferings and took in exchange the life of an eagle.[4] Atalanta's soul drew a lot about half-way through. She took the life of an athlete, which she could not pass over when she saw the great honours he would win.[5] After her he saw the soul of Epeius, son of Panopeus, passing into the form of a craftswoman;[6] and far off, among the last, the buffoon Thersites' soul

* Trans. Cornford, *Republic,* pp. 357–58. Superior figures refer to Cornford's notes, listed on following page.

clothing itself in the body of an ape. It so happened that the last choice of all fell to the soul of Odysseus, whose ambition was so abated by memory of his former labours that he went about for a long time looking for a life of quiet obscurity.* When at last he found it lying somewhere neglected by all the rest, he chose it gladly, saying that he would have done the same if his lot had come first. Other souls in like manner passed from beasts into men and into one another, the unjust changing into the wild creatures, the just into the tame, in every sort of combination.

Commentators since Proclus have noted that this panorama of choices of transmigration parallels the list of lovers in the *Phaedrus.* This parallel has already been cited in defense of an asserted parallel between the *Phaedrus* list and the tyrant's number in the *Republic.* It has not been explained why the parallel is incomplete; for the relative positions of Epeius and Thersites on the present list transpose the relative positions of mimetic artist and craftsman as these were arranged in the *Phaedrus* scheme. It is reasonable to suppose that the proximity of the metaphysical condemnation of the mimetic artist, at the beginning of *Republic* x, is somehow responsible for this relegation of the imitator to last place, behind the artisan. Had the same metaphysical distinction of artifact and mimetic artifact been observed in the *Phaedrus,* the effect would have been the adding

*Literally, the life Odysseus chose was that of a man who did not engage in politics and who minded his own business. Compare this with the definition of "justice" as "minding one's own business" in *Republic* iv.

Cornford's notes:

1. Orpheus was torn in pieces by the Maenads, the women-worshippers of Dionysus.

2. Another singer, who was deprived of sight and of the gift of song for challenging the Muses to a contest.

3. After Achilles' death a contest between Ajax and Odysseus for his arms ended in the defeat and suicide of Ajax. The first mention is in *Odyssey* xi.543, where the soul of Ajax, summoned from Hades, will not speak to Odysseus.

4. The conqueror of Troy, murdered by his wife Clytemnestra on his return home.

5. Atalanta's suitors had to race with her for her hand and were killed if defeated. Milanion won by dropping three golden apples given him by Aphrodite, which Atalanta paused to pick up.

6. Maker of the wooden horse in which the Greek chieftains entered Troy.

of a fourth level of forgetfulness, making the matrix 12-celled, or 4×3. But just prior to the 9-term list, we are informed that lovers, each following in the train of his patron deity, are 12 in number.[150] I suggest that Plato here had in mind a schematism of the 12 gods analogous to the 3 and 4 of the 3-4-5 triangle in the nuptial number, which, if written out analogously, is as shown in Figure 77. In this diagram, rows again represent dominant parts of the soul of the follower of each god, while the columns represent objects of love: soul, body, artifacts, and mimetic works, respectively. Zeus is the patron of philosophers; Athena, in her character of "lover of wisdom and war," of such men as the good state's auxiliaries; Hephaestus, of craftsmen; Apollo, of poets. Ares becomes the patron of the proud king or warrior chief listed in the 9-celled matrix of the *Phaedrus*. Artemis is patroness of lovers of the chase, who are not concerned with improvement of the body, as the gymnasts and physicians are, but with the capture of external creatures. Hermes, whose name in the *Cratylus* is found appropriate to his attributes of thief and liar, becomes the patron god of Sophists, who imitate intelligent discourse. Aphrodite, the goddess of physical love, becomes the patroness of those who put their faith and center their interest in bodily appearances, as the democratic man typically does in the *Republic*. Hestia, the stay-at-home goddess who never sees any of the forms, is well adapted to be the prototype of the woman who, always preoccupied with her affairs within the house, has no interest in or knowledge of the world outside, and acquires no virtue but only petulance from her preoccupation with domestic minutiae; this is a type of woman whom Plato elsewhere deplores, and intends to abolish in any good state. Poseidon, finally, the Earthshaker, ruler of the stormy sea, driver of horses, founder and patron of the vast and potent state of Atlantis, is, in his unbridled power, presumably the patron deity most akin to the power-seeking tyrant. A sign of this is the lapse of the god's Atlantean descendants into unbridled tyranny. Not only does Hestia never march in the parade, but there would seem to be great doubt as to whether Poseidon's steeds and chariot, designed to be oceangoing, will ascend high enough to show their follower anything but the foam-crested waves of the sea.

Arranging the souls which transmigrate in terms of this list, they seem to represent a section running across the upper row, with the exception of Atalanta, whose juxtaposition with Ajax and Agamemnon helps to indicate their true characters. The list of transmigrating souls now becomes:

ORPHEUS	AGAMEMNON	EPEIUS	THERSITES
THAMYRAS	AJAX		
SWAN AND OTHER MUSICAL CREATURES	ATALANTA		

Here, as in the later *Phaedrus* list, a larger number of instances may tend to indicate the categories of a higher type.[151]

The interrelation of these characters is partially indicated by the comment that the lot of Ajax was the twentieth.[152] There is good reason to suppose, with the ancients, that this piece of information is a functional detail. It appears clear that 20 is chosen because Ajax' category of lover is second in excellence, and viewed in the mythical context of punishment and reward, the ordinal difference is here given its eschatological tenfold magnification.

Nor do the first and last choosers in this panorama fit badly with the present interpretation of the Epeius-Thersites reversal. The first man to choose represents a masculine follower of Hestia. Incuriously accepting life in an orderly state, he appears on the plain with no intellectual vision to guide him in making his choice. As his exact opposite, Odysseus, drawer of the final lot, has seen all the cities of men. He is the very antithesis of the stay-at-home type, and he has been the hated enemy of the god Poseidon. The life of Odysseus' choice is one in which a man has a chance to become a follower of Zeus by cultivating philosophy; and this choice by the hero of widest experience gives a pragmatic confirmation to the doctrine of the relative happiness of the just man developed theoretically in the antecedent argument of the *Republic.*

This analysis raises the question, also posed by the operation of subtraction of segments in the nuptial number diagram, of the reason for the contraction of the larger schematism of 12 lovers to the smaller matrix of 9. It was suggested, in connection with the nuptial number operation, that the four-part division of knowledge, while metaphysically relevant, was not morally so, since normal human experience would by maturity give every man the kind of practical knowledge which is characterized as "opinion." In effect, this collapses the metaphysical distinction between the makers of artifacts and the makers of mimetic artifacts. Psychologically, the poet and craftsman are akin; neither has an art which conduces to his moral excellence or wide experience. Both, therefore, operate on the level of opinion; both may be regarded as lovers who have forgotten everything about the forms once seen except those properties observable in external bodies. The followers of Apollo, Hermes, and Poseidon differ from those of Hephaestus, Artemis, and Hestia neither in forgetfulness nor in vice; therefore the 12 trains of gods are adequately represented by only 9 kinds of lovers. In this reduction, the poet finds himself identified with the lover of beautiful bodies, the follower of Aphrodite, the democratic mentality; and since ignorance is the major principle of organization of the list, the poet is placed a step above the craftsman. The hunter is not included, the Sophist replacing him; the stay-at-home drops out, and is replaced by the ignorant, stay-at-home tyrant.

Figure 76

MATRIX OF CHARACTER WITH FOUR LEVELS OF KNOWLEDGE DISTINGUISHED

INTELLIGENCE	FORMS	BODIES	ARTIFACTS	IMITATIONS
REASON	Philosopher	Auxiliary	Artisan	Poet
SPIRIT	General	Athlete	Sophist	Actor
APPETITE	Merchant	Democrat	Tyrant	Housewife
MOTIVATION				

Though other contexts (e.g., *Republic* 546C) involve imagery suggesting that no adult of normal experience will remain in the state of being of someone whose knowledge is pure εἰκασία, such a life remains a metaphysical, if not a practical, possibility. It would represent a level of ignorance below that of the artisan, whose interests are centered on mimetic artifacts which lack tangible function and reality. The actor, as he projects the poet's insight into physical performance, obviously belongs where he is placed; and this explains why Epeius and Thersites appear in the pageant in that order. The location of the incurious, uninformed housewife, on a level of stupidity equal to that of the mountebank and too out of touch with practice to make possible such social damage as the tyrant effects, is suggested by the fact that the patroness of keepers of the hearth, Hestia, is the only deity who never joins in the parade of the gods, so that her followers patently have no opportunity at all for intellectual insight.

This matrix is not necessarily one that Plato himself had in mind, but is put forward as a schematic way of showing how the contextual stress on the four levels of reality of the divided line could suggest the reversal of actor and artisan.

Figure 77

MATRIX OF CHARACTER APPLIED TO OLYMPIAN GODS WHO ARE PATRONS OF TYPES OF HUMAN LIFE

FORMS	BODIES	ARTIFACTS	IMITATIONS
Zeus	Athena	Hephaestus	Apollo
Ares	Artemis	Hermes	Demeter
Hera	Aphrodite	Poseidon	Hestia

For Plato's myth to be plausible, it is necessary only that the range of interest and temperament of the Olympians be adequate to the range of lives that their followers represent. However, as the present list of patrons shows, the actually recognized personalities of Olympus present a fair one-to-one correspondence with the different categories of type of character derived from a 12-celled reason-motivation matrix. Aphrodite seems a suitable patroness for the democratic man who is a "lover of sights and sounds"; Demeter is identified with the actor, perhaps on tenuous grounds, because she communicates in the mysteries in her honor through enacted, allegorical pageantry; Hestia, as outside the procession, seems to require the corner location in the present scheme.

This figure is intended to suggest that no matter how closely Plato checked the adequacy of his story of the twelve processions and patrons, the correspondence to his purpose at hand would still have seemed adequate.

CHAPTER IV

Algebraic Metaphor

Introductory Comment

PLATO chooses with care and frequently employs mathematical illustrations based on the concept of ratio. The modes of proportionate, qualitative relation may be metaphorically identified with the modes of relation of things, either as parts in wholes or as entities juxtaposed in some sequential order. The metaphor may be projected, in turn, from algebraic to geometric symbolism, in which case proportionate relations are presented as constitutive of a metric scale or net. In this context more than in that of geometrically mathematical imagery, which we usually identify as "mathematical imagery" proper, a later literalness which refuses to accept the concept of algebraic illustration as metaphor has rendered passages opaque to attempted interpretation.

The only way in which the metaphorical significance of proportion can be recaptured is through the device of defining and classifying certain more general types of relational pattern, of which quantitative ratios provide illustration as instances of each type. In this way, some of Plato's less understandable statements about proportion can be exhibited as metaphors of the familiar kind, in which there is "substitution of one species for another within the same genus." But to do this involves a preliminary abstraction and formalization which only a Platonist enamored of the *Parmenides* is likely to enjoy.

In the course of this discussion, several instances will appear of a Platonic metaphor which has later, in a more formal and literal interpretation, provided the basic insight of a branch of mathematical science. Such instances of later development will be cited not with any intention of claiming that Plato anticipated statistics, logarithms, and calculus but as evidence that his choice of algebraic metaphors is not unmathematical, as their contextual vocabulary could lead a modern reader to suspect, and that in some cases the metaphor chosen has been adopted as literal fact by later mathematicians.

In general, four entities, *a*, *b*, *c*, and *d*, are in proportion when the relation of *a* to *b* is identical with that of *b* to *c*. A term *b* is a mean when *a* has to *b* the identical relation that *b* has to *c*, and when the set *a,b* has this same relation to the set *b,c*. Two sets in proportional relation are "similar," though some cases of intransitive similarity probably cannot be construed as literally proportionate. It is, of course, neither necessary nor desirable, as these abstract preliminary definitions indicate, to interpret proportion as an essentially quantitative relation.

The degree and mode of connectedness which a proportion expresses depend on the natures of the two relations (1) between the members of the sets and (2) between the two sets themselves. In effect, the symbol "::" is itself analogical or equivocal, since different types of relation may correlate sets that are analogous. Relations of "::" may be classified into two broad types, the "translative" and "projective." A translative relation is a correlating relation equivalent to a rigid displacement of one set in space or time. If isomorphic and analogous sets of parts are combined in identical relational nets at different dates or positions, the analogy is so highly specific and close that the sets and their elements are called "the same." Thus the organs of one man are "the same" as those of another, since the two men have "the same" structure and form. The ultimate test of translatability is dependent on the identity of *function in context* of any two structures that are the same. We can imagine the one set replaced by the other—a given table by another table, a saw by another saw—and this replacement effects no change in the context or function.

The degree of analogy involved in the class of "projective" relations is less close, and projectively related sets are only "metaphorically called 'the same' " or "similar." As opposed to rigid translation, projection may involve both change of kind and change of scale between the parts related. It is not possible to imagine the one set as simply substitutable for the other, because of a difference in their respective contextual functions and their spatial and temporal scope. Perhaps it would be more accurate to say that the elements of sets in projective relation are correlated on a different principle from that correlating those in translative relation. The elements of translative sets are connected by identical relations, and are isomorphic. With projective sets, this identity no longer holds in the same way; one can say only that any two correlated pairs are connected by relations of the same relational class or type. In organized structures, the relations of elements are always "harmonious." A "harmonious" relation may be described by its stability; it is a relation that preserves and reinforces the identity of each of its relata, both in itself and in its context. All relations not "harmonious" are "hostile." Elements in hostile relation alter both their relational patterns and their identities.

We may now summarize the nature of projection by saying that if the relations of parts of two "similar" sets are correlated, each correlated pair will belong to the same relation-class in respect to its harmony or hostility. Two sets need not be isomorphic to be placed in such a projective correlation; the relations between complex parts of one may be correlated with the relations of incomplex parts of the other. If the one structure is replaced by the other, the result will be invariant in respect to its internal unity, but altered entirely in respect to its function in context. (The nature of the modification of an analogue to the theory of types required to avert paradox when such a broad concept of correlation is introduced would require an intensive study of the formal structure of Platonic logic.)

Some such preliminary abstract discussion as this is needed as a way of explaining and correcting a peculiarity of the interpretation of proportions and analogies that became characteristic of Neo-Platonism. A typical Neo-Platonist would see no sharp difference in kind between the analogical identification of the cosmos with the human organism and that of one individual human organism with another. In the latter case, as in the former, his interpretation of dialectical proportions presupposes a one-to-one correlation of all parts of both sets and an identity of the relational connections between them. Thus, if "man is like the heavens," it would seem legitimate in this mode of discourse to ask where the heart, liver, etc., of the heavens are. If he finds the constellations analogous, astrology becomes a branch of the Neo-Platonist's philosophy. Or he may ask—and this is a key question of Neo-Pythagorean embryology—what seven ages in the life of man correspond exactly to the ratios of the heavenly periods of planetary motion. The assumption that the relations of all members of projectively analogous sets are literally identical leaves room for only one key pattern of proportions, which all subject matters share. Thus a biologist in this tradition can make the assumption a priori that the quantitative ratios of the periods of the planets are identical with the set of key ratios determining organic development, and his only task is that of filling in the blanks in this ready-made pattern with the names of the biological parts with which the planets are in one-to-one correlation. This leads to absurdities ranging from the conviction that all sets with the same number of members are essentially the same (Aristotle [1] satirizes this belief in his criticism of the Pythagorean assertion of the essential analogy of all sets having the cardinal number 7) to the conviction (for example, of Proclus) that each passage in Plato is susceptible of interpretation as a theorem in every known science.[2] Plato's own handling of such matters as the microcosm-macrocosm analogy and the interpretation of other philosophic texts leaves no doubt that such grotesque dialectical practice was not his own.

For Plato, the likenesses of structures which are wholes (i.e., structures adapted to single functions) hinges on the fact that all parts in any such structure must be connected in a way which prevents either the deformation or removal of any functional part, or a transposition of parts which would lead to such deformation or removal. The number and character of "parts" will vary with the function of the "whole"; and the specific relations through which each part, by sustaining the whole, also preserves and sustains the other parts, will also vary. The *invariant* basis of the Platonic projective analogy is that the unity and value of organization demand that the constitutive organizing relations be *the same in kind*. Thus the collection of analogies ceases to be an arbitrary game, because the demand of functionality introduces a restriction to the range of comparable parts, and the demand of strict isomorphism among the equations of the several sciences is replaced by the demand that every science be inclusive and comprehensive, so that each theorem may contribute to and be functional in the system as a whole. A later interpretation of the mathematical images of the *Parmenides* will be concerned with this relation of parts, wholes, functions, and their associated structures.[3]

A second distinction of kinds of relation constituting proportion is a distinction, within each member of the analogy, of the mode of connection of its terms. The connecting relation between terms may be simply an identical juxtaposition, an external relation, or actual identification of one term with another within a connecting constitutive class, which is an internal relation. The Pythagoreans and Plato differentiated multiplication from addition on the ground that a number's factors are constitutive of it, determining its nature in a way that a randomly chosen predecessor in the integer-series, which may be equated to the number by a suitable addition, does not.

Thus if the ratio $a : b$ is a multiplicative relation, a is a factor constitutive of b, and a and b have something in common, since the character of a is reflected in the nature of b. The use of multiplication as a symbol of marriage brings out the basic notion of this interpretation; the children, who are the "product" of father and mother, resemble both in their inherited natures, just as an arithmetical product resembles both of any pair of its factors.

On the other hand, if $a : b$ is an additive relation, only some sort of juxtaposition, and no real connectedness, of a and b is created. The connecting power of a mean term between the extremes a and c can be symbolized by noting that if the mean is geometric, such that $ax = b$ and $bx = c$, then $ax^2 = c$; and that if the mean is arithmetic, $a + 2x = c$. The geometric representation of ratio, in which $a \cdot c$ is a closed rectangular figure and in which $a + c$ is simply two juxtaposed figures, reinforces this notion of the external character of sum as opposed to product.

Two classes may be "bound" or "connected" by the creation of an intermediate class between them, similar to both. Like entities fuse together, unlike separate. Thus if the two unlike and extreme classes, a and c, are brought into contact, they will not hold together. If, on the other hand, a is brought into contact with a like class, b, and b with c, which it is like, the chain a-b-c will hold. Terms in proportionate relation cannot be disconnected by reversal, translation, or any external force causing a transposition, since any such force will have the same effect on all similar entities on which it acts, and will therefore exert the same transposing effect on both sides of the proportion.

The concept of analogy is also fundamental to any definition or theory of measure. In its simplest form, measurement simply establishes a translative relation between an arbitrary "standard" and a "magnitude" homogeneous with it, to which the standard is applied. The relation within members is additive, and the integers are used to count the successive translative applications of the standard to the magnitude. Thus such measurement takes the form of the proportion $l : s :: x : m$.

It is also possible to conceive s and m as related projectively, and the integer-standard as a multiplicative relation rather than an additive one. This leads to measurement of a different type, which Plato calls "evaluation" and opposes to pure "description." This is the sort of measure that a craftsman uses to correct dimensions of parts which are detrimental to the function of the whole. In such evaluation, description may still be involved; indeed, without the accuracy contributed by descriptive technique, the craftsman could not achieve the desired results. The basic relation of such normative measure may be schematized by the proportion: $l : [p : s] :: x : [q : m]$, where p and q are functional parts of the wholes s and m, and the craftsman must solve for x, the descriptive measurement of q.

Two different concepts and employments of metric nets follow from these metaphors of measure. An additive or descriptive metric net is a set the terms of which are connected entirely by repetitions of a translative or additive relation, R. Thus a spatial grid-scheme, with a spatial unit serving as the side of its squares, and with corners as entities connected by the grid, is an example of a descriptive net of this type. The scope of such a net can be increased (that is, entities not falling at intersections can be described) by a further subdivision of the individual net squares. It is significant that Plato nowhere employs a net of this simple descriptive type, although in several passages he introduces variations of it in taking account of mechanical interactions between parts. The appropriateness of this application is perhaps indicated by the fact that any problem for which the operations of solution can be represented in such a descriptive net can be solved by a calculating machine. This remains true if the operations involved include not only R, but also the converse of R, R^{-1}. The operations of an abacus or balance are such descriptive calculi; and their mechanical character is deprecated by Plato when he separates "computation" from "mathematics proper" in *Republic* vii.[4] The "normative" metric net, on the other hand, gives the quantitative analogue of systems of part-whole relations such as are expressed by the artist's proportions.

The simplest example of such a net is the bare enumeration of parts in a hierarchy, where the order reflects a relative importance or scope. The combination-matrix, in which two sets of ordered spatial positions are correlated with two sets of ordering relations, is another such simple projective net. In fact, the use of such symbols to schematize relational order is the fundamental principle of the construction of mathematical imagery. In the *Laws,* Plato suggests that a divine justice would so arrange distribution of wealth that the two series, "relative wealth" and "relative human excellence," would coincide.[5]

A normative system of measurement involves some technique of measuring and comparing degrees of deviation from the standard or norm. Such an "index of aberration" can be derived by comparing the actual dimension of a part with the ideal dimension; the ratio of these two, taken as an index of error, would permit a comparable evaluation of technical works of any type or size. In fact, if we grant the highly un-Platonic axiom that most entities realize the ideal structure, modern applications of the normal distribution curve and the standard deviation are exact counterparts of Plato's metric metaphors of the normative type.

The problem in legislation may seem to be that of making the social privileges of wealth and the power of a citizen deviate from some constitutionally established "normal" by a distance comparable to that by which his actions deviate from the legally established social norm. We should look, then, in the arithmetical details of the *Laws* for some such correlation of degrees of deviation.

One other, more special usage of ratio must be considered. This is the use of geometric progression as a schematism or symbol of organic growth and causal efficacy. The *Laws, Epinomis,* and Aristotle in the *De anima* record such schemata.[6] An incorporeal principle becomes causally effective in space and time first by generating an order, then a two-dimensional plan, and finally an embodying spatio-temporal construction. Plato symbolizes this process by which an idea or insight issues in a physical creation as a 1:2:4:8 progression, from 0 to 3 dimensions. A goal or ideal first manifests itself without any associated plan; more relevant factors are ordered in respect to it, until at last a construction issues in physical action. The organizing power of a principle, as it projects order into structures of increasing complexity, generates a relational net of order correlated in this metaphorical representation with the exponential relation between its terms. The internal relations of organizing, and of being organized by, are used as the framework of the proportions which God marks off on his cosmic scale in the *Timaeus,* and by which he determines the intervals of the circles of the planets.[7] The important aspect of this exponential net in its context seems to be that it carries over the symbolism of the multiplicatively related mean as establishing an internal relation; so that the connection between any sequential entities in the net is internal, and their relation one of "friendship" or "linked connection." The choice of the exponential relation as a symbol of internal connection and harmony results in a construction which the modern mathematician would at once recognize as a *logarithmic* scale. The research into the logarithmic spiral as typical of processes of growth in nature would no doubt have seemed to Plato a sound and illuminating metric metaphor.[8]

In classifying ratios Plato follows the Pythagoreans in setting aside as a special harmonious class those integral ratios which had been found empirically to correspond to the concordant intervals of the musical scale. The peculiarity of these ratios is that they are simple and integral, with integers less than 10. "Concordance" seems to result from a special kind of co-operation or reinforcement between different tones. A concord is not the average pitch of its constituent notes, but a combination in which each constituent reinforces the identities of the others. The peculiar simplicity of the integral foundation of musical theory may have been instrumental in suggesting the doctrine attributed by Aristotle to Plato, that the number series is modular at ten, so that the first ten integers are adequate to describe and establish the whole system of relations of numbers.[9]

A "concordant" relation seems to fall, as a metaphor of connection, somewhere between the internal relatedness established by a geometric mean, and an external relation such as is symbolized by an arithmetical mean. A third type of mean, the "harmonic," symbolizes this intermediate tightness of connection.

The preceding discussion is illustrated by Figure 78, on the following page.

The ordering of kinds of motion in *Laws* x entails a classification of relations similar to that developed in Figure 79. The kinds of change are ordered in terms of the amount by which the identity

Figure 78

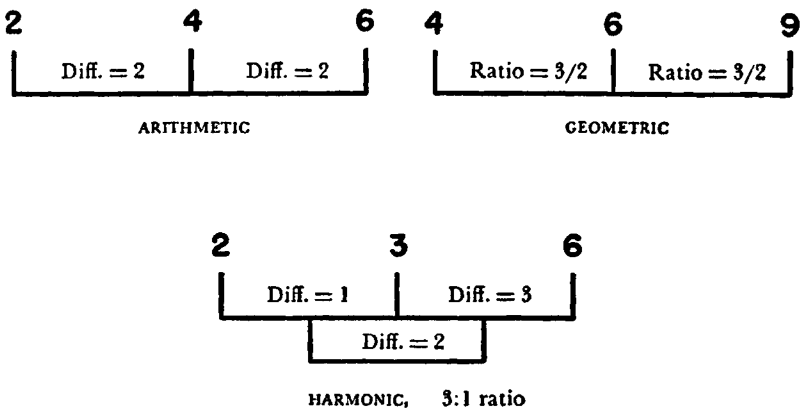
SCHOLION: THE THREE KINDS OF RATIO

(Greene, *Scholia Platonica*, pp. 167–68)

Gorgias 508A

Geometric Equality *

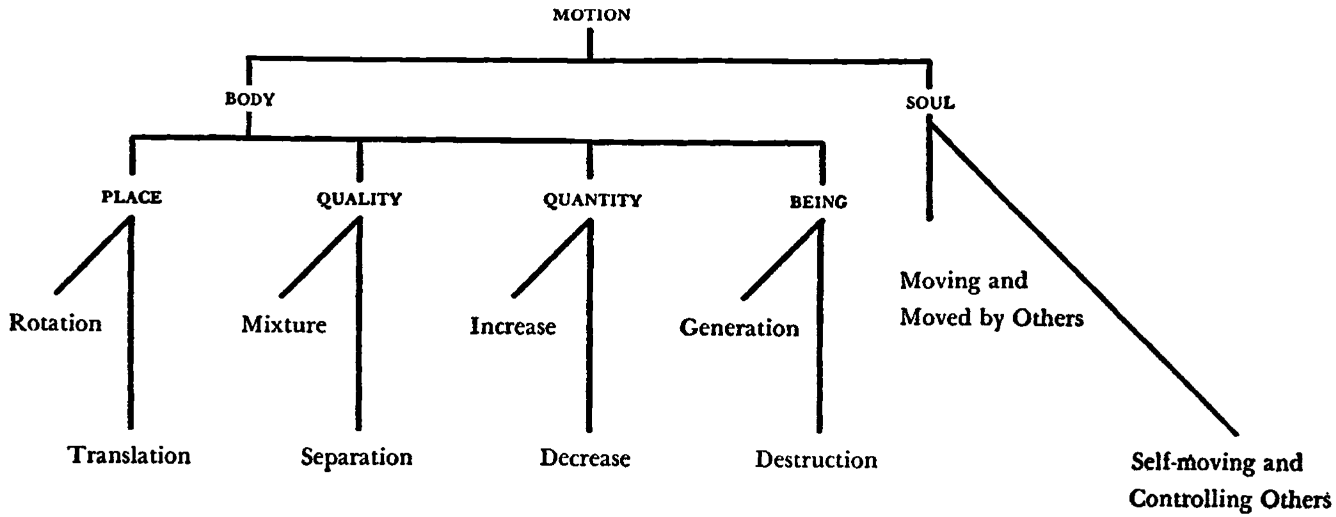
(1) This is justice, for the geometrical proportion is called the order of Zeus in the *Laws,* and through it all things are ordered and determined. (2) A proportion is called "arithmetic," when of three unequal numbers, the mean exceeds and is exceeded equally, e.g., 2:4:6. It is called "geometric" when of three unequal numbers, the ratio of the first to the second is also that of the second to the third, e.g., 4:6:9; it is called "harmonic" when of three unequal numbers, the ratio of the greatest to the least is the same that the larger difference has to the smaller, e.g., 2:3:6.



of the subject is affected, i.e., by the degree of difference between the object in its final changed and original form. Since the classification of tightness of analogy was suggested as based on the kind of relation necessary to transform one analogue into its counterpart, the list should be the same as a list of the relations holding between the initial state of a thing and its form as altered because of change. The list of changes runs: rotation, translation, mixture, separation,

* This is Plato's phrase. It is used by the scholiast to identify the subject of his comment.

Figure 79



SCHOLION: THE TEN KINDS OF MOTION (*Laws* 894B)
(Scholion figure from Greene, *Scholia Platonica*, p. 354)

growth, diminution, generation, destruction, self-motion externally caused, self-motion self-caused. (Of these, we are told that the last two should have been put first, as they are the primary and prior sorts of motion.)

In effect, this gives the following classification of the correlating relations by which analogy can be established:

1. Rotary and translative relations of correlation hold where there is a one-to-one correspondence, and such analogy is so close that it may be stated as an identity.

2. Mixing and separative relations involve a one-to-one correspondence of one analogue with a subset of the other. The two are not interchangeable, and the analogy is less close.

3. Growth and diminution involve, on the one hand, new additions, so that correlation is still only with a part of the changed form; but within this correlated part, the relation is projective; whatever scale is relevant must be stretched or shrunk to show the analogy of the one with the other.

4. Generation and destruction, as relations transforming two sets into each other, can yield only a statement of *contrast,* and no analogy at all.

5. Self-motion is a property of organized dynamic systems, which are analogous in their common possession of some sort of part-whole organization, retained and self-sustaining through changes of the other kinds enumerated. Thus an organism retains its identity through changes of place, context, and growth. Analogy between such organisms rests on the similarity of kinds of relation holding between parts of functional wholes, and on the common resistance of such relational nets to change in spite of other relational changes; the parts remain integrated throughout a motion of the whole. Analogies based on relations of this sort, while more difficult to exhibit as simple correlation, take account of a more basic identity and similarity than those of other kinds.

I. Construction of the World-Soul

Timaeus 35 *

The things of which he (God) composed the (world) soul, and the manner of its composition, were as follows: (1) Between the indivisible Existence that is ever in the same state and the divisible

* Trans. Cornford, *Cosmology,* pp. 59–60, 71, 73.

Existence that becomes in bodies, he compounded a third form of Existence composed of both. (2) Again, in the case of Sameness and in that of Difference, he also on the same principles made a compound intermediate between that kind of them which is indivisible and the kind that is divisible in bodies. (3) Then, taking the three, he blended them all into a unity, forcing the nature of Difference, hard as it was to mingle, into union with Sameness, and mixing them together with Existence.* And having made a unity of the three, again he divided this whole into as many parts as was fitting, each part being a blend of Sameness, Difference and Existence.

And he began the division in this way. First he took one portion (1) from the whole, and next a portion (2) double of this; the third (3) half as much again as the second, and three times the first; the fourth (4) double of the second; the fifth (9) three times the third; the sixth (8) eight times the first; and the seventh (27) twenty-seven times the first. Next, he went on to fill up both the double and triple intervals, cutting off yet more parts from the original mixture and placing them between the terms, so that within each interval there were two means, the one (harmonic) exceeding the one extreme and being exceeded by the other by the same fraction of the extremes, the other (arithmetic) exceeding the one extreme by the same number whereby it was exceeded by the other.

These links gave rise to intervals of $3/2$ and $4/3$ and $9/8$ within the original intervals. And he went on to fill up all the intervals of $4/3$ (i.e. fourths) with the interval $9/8$ (the tone), leaving over in each a fraction. This remaining interval of the fraction had its terms in the numerical proportion of 256 to 243 (semitone).

By this time the mixture from which he was cutting off † these portions was all used up.

In his construction of the world-soul, Plato represents God as a maker of astronomical models. In order that the constituent motions of the system may be specified with greater clarity, the soul is described as constructed of moving concentric bands, like the metal bands representing equator, ecliptic, and plane-

* Cornford retains αὖ περί (see p. 225).
† See notes 11 and 12 to this chapter.

tary orbits in a Greek astronomical model.[10] But before he bends and arranges the strips of the alloy of being, same, and other, God cuts into the bands the marking of a metric scale, to serve as a measure of their relative positions and motions.[11] He subsequently uses the framework of this same scale to adjust the distances and velocities of the subdivisions of the circle of the other.[12]

There is a double problem that must be met in such a cosmic construction. In the first place, same and other, by their natures hard to mix, must be harmonized in such a way that the unity of the world-soul will be preserved. As a corollary, the external relations of part to part in cosmic process, which are relations according to the other, must be given their proper places as the mechanism by which these moving parts are forced into the internally adjusted patterns which relate them according to the same.

The metric scale which God employs, and etches into the substance of the world-soul, is constructed in a manner which, through a meticulous use of mathematical metaphor, symbolizes the structure in which God has blended the other and the same.

There has always been a temptation for later readers to construe these proportions as intended to present an empirical description of astronomical phenomena. There is always also the opposed temptation to treat these proportions as a whimsical elaboration of the poetic concept of the "harmony of the spheres." But neither of these interpretations is quite correct. Plato's metric scale is actually a presentation (as Proclus thought it was) in the form of algebraic metaphor of a schematism of a relation of parts and wholes in adjusted structures, a schematism indicating the dialectical method appropriate to cosmology. In this passage, differences in kind among entities appear as differences in quality between the illustrative numbers and their ratios, and God builds into the world-soul exactly the type of structure which reason will develop as the architectonic of rational cosmology.

The schematism is based upon a pair of different exponential series, of the type previously referred to in the classification of ratios, as symbolizing internal relations. It is important that the key elements be integers, integrally related. Presumably the tradition is quite right which identifies the sequence of even numbers with the element of the other, the odd numbers with that of the same.[13] These two distinct constituents are mixed or harmonized by the insertion between them of arithmetic and harmonic means. Schematically, this is analogous to a gluing or joining of substance between same and other; for the arithmetic mean is a mathematical analogue of the relation holding externally between mechanically connected objects, and the harmonic mean also has this character of external connection.

Although such statements of interpretation as "odd numbers symbolize the same" have a Neo-Platonic sound to them, and therefore are automatically discounted by the modern scholar, this is a case where the Neo-Platonists were entirely right. If it is true, as has been already suggested, that Proclus tried to find in every mathematical image in Plato a schematism of method and relations which the scientist could use as a kind of standard form in which to arrange his data, this particular mathematical image was made to order for his technique. True, Plato presents this as a schematism of the distinctions appropriate to a discussion of cosmology, and there is every reason to think that he would dispute any attempt to apply it to other discussions. It is peculiar to the *Timaeus,* and would be misleading and out of place in the *Republic.* But, in the *Timaeus,* it does just what Proclus expected it to do; and his interpretation, on which the following discussion is based, has never been equalled or surpassed.

To counteract the overtones of this explicit Neo-Platonism, it is only right to state that what is to be developed in the present discussion is not a theory of mystical affinities between numbers and things, but a display of the connections of the terms of a cosmology in an algebraic schematization. Such an algebraic display is thoroughly, in fact surprisingly, modern in spirit. The concern of contemporary symbolic logic is precisely the study and construction of such algebraic schemata. But this modernity of spirit has been effectively masked by an antiquity of form which depends on metaphorical interpretation of a specific instance, rather than on an ambiguous notation, to secure its generality of statement.[14]

The central point in this schematism is its remarkable reflection of the fact that a right connection of separate parts, schematized by the insertion of the arithmetic and harmonic means, can lead to a fusion and emergence of these parts internally connected in wholes. The geometric ratio reappears as a connection between sets in the developed scheme. (The geometric ratio had been identified by Plato with the class of relations that hold together the parts of functional wholes.) The unity, intelligibility, and wholeness of the world-soul are insured by continuing the construction until a harmonious ratio holds between each adjacent set of parts.

The schematization has been developed in stages that successively direct the reader's attention to types of relation which themselves appear successively as central in Plato's own discussion of cosmology. First there is the relation of the pair of constituents, being and becoming, into which the cosmos can be analyzed. Next there is a discussion of the juxtaposition of externally related elements which are the "auxiliary causes" in making the copy resemble its model. Finally, in the *Timaeus*, there emerges a set of organic patterns created from the planned construction of these externally related connections.

The cosmologist, like the creator, founds his discussion on the principles of being and becoming, and same and other, as basic constituents of the cosmos. Within this frame of reference, however, he cannot explain such phenomena as death, disease, and war, which arise from the interference of externally related parts within the whole; a description of creation in terms of its likeness to a perfect model can account only for its total perfection, not for any defections in its parts. A theory of elements must therefore be provided, to explain the mechanism by which being is projected into becoming. The final view, in which the mechanism of elements in the universe is seen to create the perfection of the universe, and of its subordinate constitutive wholes, shows how external relations of elements, built into organisms, can frustrate effective organization, and gives a clearer insight into the way in which reason for the most part persuades a recalcitrant necessity.

This same schematism will therefore apply to a discussion of astronomy, as a study of a special, particularly accurate, aspect of cosmic order. The creator uses the same scale in subdividing the soul of the world that he employed in its original constitution, so the schematism of creation serves as the paradigm of the system of planetary motion as well.

This interpretation of the series of even numbers can hold only if the frequently bracketed αὖ περί is retained in the text, as Proclus retained it.[15] The effect of this phrase is to indicate that the mixture does not involve same and other in their absolutely pure form, but a sameness and otherness already partly blended with substance, so that their natural contrariety is lessened. The relation of otherness, as explained in the *Parmenides,* is solely an external, part-to-part relation.[16] Unless we think of this principle as already sufficiently imbued with some sameness through its mingling with substance, relations of otherness would have to be schematized disjunctively, as an atomic set. However, an otherness manifesting itself as an external relation of physical or psychic entities can be schematized as involving some internal connection. The phrase somewhat complicates Plato's account, but is needed to make the intended point that God is a creator, not a dialectician, and that his creative activity utilizes a substantial raw material as medium, not the dialectician's pure, insubstantial, abstract contrary classes. The result of mixing pure same and other would not be a soul, but an intelligible class, assuming (quite un-Platonically) that God could cause these contrary classes to mix in their pure form. The classification of the raw materials thus follows the same method as the later schematism of cosmological structure: intermediates are inserted to connect the disparate classes of other and same, just as arithmetic and harmonic means will be inserted in the following schematism to connect the disparate arithmetical classes of odd and even.

This is a remarkable synthesis of many complex mathematical metaphors. The initial series from 1 to 27 gives the metaphor of the music of the spheres. The progression to 8 and 27 introduces the metaphor of progression from a principle to its embodiment, through squaring and cubing. The initial contrast of odd and even series schematically reproduces the opposition of same and other, the ingredients blended by God in his creation of the soul. The two "bonds" created by the first insertion of harmonic and arithmetic means reflect the two aspects of cosmic structure central in the dialogue, the arithmetic mean symbolizing the external relations of mechanical causality, the harmonic approaching more closely structures such as those which have a purposive organization and are treated in the "likely account" early in the discussion. The reduction of the whole series to a continuous geometric progression (except for the λεῖμμα, where, since two metaphors conflict, the reduction for some terms is only approximate) insures the organic unity of the cosmos as God has connected its parts. And, if we think of the specific numbers in this net as replaced by variables, the schematism gives an abstract representation of the logical relational structure proper to research in cosmology.

In addition to its synthesis of this imposing set of geometrical and algebraic metaphors, the repetition of the 2:1 ratio of the concordances of the initial octave scale carries with it a cyclic or modular metaphor. (The concept of a modulus, a mathematical metaphor that is natural for reiteration, seems to have been already formulated by Plato's time. The Pythagoreans' concept of the *pythmen*, their identification of 6 and 5 as "circular" numbers, and their insistence upon 10 as the natural base of the number system show this idea emerging in arithmetic.) [17] This matrix of doubles is also extremely attractive as a starting-point for interpretations of the 2:1 ratio as a principle of projection and of the problematic "dyad." [18]

The synthesizing effect of such a mathematical image as this is so like the complex connection of image-classes in lyric poetry that the image may be legitimately classed as an aesthetic construct rather than as an abstract mathematical one. For Plato, in his practice as well as in his theory, the two are closely akin.

A thesis basic to this study, that no peculiarly technical or esoteric reference underlies this imagery, could not well explain a passage which demanded the solution by its reader of thirty-four separate algebraic equations. Such an image would be less a pedagogically useful schematism than a special problem, presupposing a reader who was, for instance, an advanced student in the Academy. Actually, however, this calculation, as presented in the following 8 × 4 matrix, requires only that the reader (1) be familiar with the metaphor of the "harmony of the spheres," (2) know the ratios of the notes of the diatonic scale, and (3) be able to multiply by two. The second condition can be more accurately stated as follows: The reader must know (*a*) that the notes of the octave are in 2:1 ratio, and (*b*) that the insertion of harmonic means in the octave gives musical concords. These items of information are represented as familiar to characters in the dialogues, such as Eryximachus and Glaucon, who know them as parts of their general culture, not as a result of any specialized Platonic or Pythagorean training.

Figures 80–86

Construction of the World-Soul

Figure 80

INITIAL TERMS OF THE SCALE

```
      1
    2   3
  4       9
8          27
```

The lambda-arrangement brings out the distinction of the odd numbers, representing the same, and even numbers, representing the other.

```
SAME        OTHER
         1
       3   2———Linear number
     9       4———Plane number
  27           8———Solid number
```

Figure 81

ARITHMETIC MEANS INSERTED

1 3/2 2 3 4 6 8
1 2 3 6 9 18 27

Figure 82

HARMONIC MEANS INSERTED

1 4/3 2 8/3 4 16/3 8
1 3/2 3 9/2 9 27/2 27

The insertion of harmonic and arithmetic means creates a double bond (according to the other and according to a mixed relation of same and other) between the terms.

Proclus' theorem explains why there need be no specific reference to the geometric progression:

$$X : \text{harmonic mean} :: \text{arithmetic mean} : Y$$

In other words, the insertion of these two means produces a 4-term geometric progression.

Figure 83

BOTH MEANS INSERTED

The inserted links reduce the distinction of the two initial series, so that the lambda-representation can be replaced by a line

1 4/3 3/2 2 8/3 3 9/2 4 16/3 6 8 9 27/2 18 27

The connections are made tighter in Figure 84 by inserting links in 9:8 ratio between the terms which are in the ratio of 4:3, so that the entire series approximates to a continued geometric progression, and reproduces the structure of intervals of a diatonic scale.

Figure 84

MATRIX FORM OF COMPUTATION OF THE SERIES

9/8 9/8 256/243 9/8 9/8 9/8 256/243

1	9/8	81/64	4/3	3/2	27/16	243/128	2
2	9/4	81/32	8/3	3	27/8	243/64	4
4	9/2	81/16	16/3	6	27/4	243/32	8
8	9	81/8	32/3	12	27/2	243/16	16
16	18	81/4	64/3	24	27		

In Figure 84 the relation of doubling between rows of the matrix suggests a way to simplify the intended calculation.

The 2:1 ratio of the octave extends the concordant structure of the initial scale through the compass of four octaves and a sixth necessary to complete the symbolism of extension of the odd series to a term which is a cube.

Figure 85

TRADITIONAL COMPUTATION

To avoid dealing with fractions, commentators traditionally treated each term as multiplied by 384 (= 3 × 128) or 192 (= ½ × 384). The resultant progression is:

1—	384	19—	2304
2—	432	20—	2592
3—	486	21—	2916
4—	512	22—	3072
5—	576	23—	3456
6—	648	24—	3888
7—	729	25—	4096
8—	768	26—	4608
9—	864	27—	5184
10—	972	28—	5832
11—	1024	29—	6144
12—	1152	30—	6912
13—	1296	31—	7776
14—	1458	32—	8192
15—	1536	33—	9216
16—	1728	34—	10368
17—	1944		
18—	2048	Total	105,113

Figure 86

TRADITIONAL COMPUTATION IN MATRIX FORM

The difficulty of computation is considerably reduced if these figures are also constructed in a matrix of octave width, as follows:

384	432	486	512	576	648	729	768
768	864	972	1024	1152	1296	1458	1536
1536	1728	1944	2048	2304	2592	2916	3072
3072	3456	3888	4096	4608	5184	5832	6144
6144	6912	7776	8192	9216	10368		

II. The Theory of Vision

Timaeus 47

Timaeus 46: * There will now be little difficulty in understanding all that concerns the formation of images in mirrors and any smooth reflecting surface. As a result of this combination of the two fires, inside and outside, and again as a consequence of the formation, on each occasion, at the smooth surface, of a single fire which is in various ways changed in form, all such reflections necessarily occur, the fire belonging to the face (seen) coalescing, on the smooth and bright surface, with the fire belonging to the visual ray. Left appears right because reverse parts of the visual current come into contact with reverse parts (of the light from the face seen), contrary to the usual rule of impact. On the contrary, right appears right, and left left, when the visual light changes sides in the act of coalescing with the light with which it does coalesce; and this happens when the smooth surface of the mirror, being curved upward at either side, throws the right part of the visual current to the left, and the left to the right. The same curvature turned lengthwise to the face makes the whole appear upside down, throwing the lower part of the ray towards the top and the upper part towards the bottom.

Timaeus 43: * And so at the moment we speak of [when the soul, newly joined to the body, was upset by the violent, communicated motions, these sensations] causing for the time being a strong and widespread commotion and joining with that perpetually streaming current in stirring and violently shaking the circuits of the soul, they completely hampered the revolution of the Same by flowing counter to it and stopped it from going on its way and governing; and they dislocated the revolution of the Different. Accordingly, the intervals . . . were twisted in all manner of ways, and all possible infractions and deformations of the circles were caused; so that they barely held together, and, though they moved, their motion was unregulated, now reversed, now side-long, now inverted. It was as when a man stands on his head, resting it on the earth, and holds his feet aloft by thrusting them against something; in such a case right and left both of the man

* Trans. Cornford, *Cosmology*, pp. 148, 154, 155–56. Discussion of *Timaeus* 47 is preceded by Cornford's translation of *Timaeus* 46, 43, because the latter supply material indispensable for the interpretation of the theory of vision in *Timaeus* 47.

> and of the spectators appear reversed to the other party. The same and similar effects are produced with great intensity in the soul's revolutions; and when they meet with something outside that falls under the Same or the Different, they speak of it as the same as this or different from that contrary to the true facts, and show themselves mistaken and foolish. . . .

Timaeus' account of the genesis of knowledge[19] gives the impression that true insight is inevitable. For the observation of the cosmic environment stabilizes in the soul those revolutions to which the movements of the cosmos correspond. Two factors intervene which prevent this automatic acquisition of knowledge, and provide a physiological mechanics of error. The first of these is the shock of external relations caused by union with the body, which interrupts the natural progress of the soul.[20] The second is the unreliability of the senses, which makes possible misinterpretation of our observations, so that they seem to confirm an incorrect mode of psychic revolution.[21] In the theory of optics, here cited to explain the mechanism of vision, Plato is providing a theory of error as well. For the mind, whether it be warped or straight, will seem to find a reflection of itself in visible phenomena. In a tradition which has always used analogies of vision and illumination (and the history of which is usually traced by optical analogies, with the study of optics often central in that history), this epistemological extension of the laws of sight does not seem unnatural. The dialogues abound in examples of lawgivers, statesmen, and philosophers who seem to be the victims of some consistent distortion of intellectual vision. These errors in empirical intelligence parallel the analogous distortions of physical sight.

The interaction of the eye of an observer and a plane mirror typically produces an image which is reversed, with left and right transposed.[22] The distinction of same and other finds its natural projection, in space and motion, in the differentiation of left and right. Thus the circle of the other always revolves to the left. Reversed mental vision would therefore transpose the columns of matrices, resulting in a reversed apprehension of same and other.[23] This is distinct from inversion, which trans-poses the rows of matrices and results in a reversed apprehension of being and becoming.[24]

Moreover, if the mirror is distorted in such a way as to be concave from top to bottom, the image is inverted. History is full of references to "inversions" of Platonism, in which the flux of becoming is taken as the ultimate reality, and the forms which emerge in that flux are merely its transitory modifications, appearing for the moment from an unreal, or less real, shadowy realm of possibility. Plato refers in *Epistle* vii to the ease with which anyone who chooses can invert the *logos* of the man trying to describe forms, by centering on the inadequacies of his symbols of description, which are themselves only images.[25] Thus the warped mind of an inverted Platonist must seem to perceive reality as appearance, and appearance as reality. How the world looks from this point of view is well brought out in the "account according to necessity," which follows the present passage in the *Timaeus*.[26]

If the mirror is concave from left to right, however, the object is not inverted, and the relative location of left and right in the object and its image correspond.[27] This perception is therefore adequate and suggests that the perceiving mind does more than simply "hold a mirror up to nature"; the trained mind must be a corrective mirror in which reflection corrects the discrepancies between objects and purely sensuous images. More specifically, there must be no inversion; reality must be known as permanence, not change. Further, there must be no reversal; the mind must not allow the vividness of sensory accompaniment to distract it from the fact that things are known only by insight into their intelligible structures. Mathematical training is particularly suitable for correcting the undue vividness of appearances by accustoming the mind to look for structures themselves.

A further distortion of objects which mirrors produce in their images occurs when the mirror is slanted or oblique.[28] In this case, the real and apparent proportions have no metric analogy; some dimensions are elongated by the glass, others foreshortened. A philosophic system seen thus obliquely loses its coherence and symmetry and much of its plausibility. It is to such oblique mental vision that Aristotle's paraphrases of the Platonic position would probably have been attributed by Plato, as doubtless would many more recent oblique interpretations of the Platonic philosophy.

To grasp forms, the mind must be a blank substrate. But the mind, jostled by the intrusive body to which it is attached, lacks the complete absence of quality that characterizes pure space.[29] And as perceptions are stamped out through the senses on the soul, the quality of the things perceived will be identified with characteristics which are really contributions of the perceiving mind.

This passage on optics, like the introductions of proportion in the *Republic* and *Gorgias,* provides an insight into Plato's analogical method of discussion, and an analogy for examining the deviations of men who, on the basis of the same experience as the Platonist's, have followed other philosophies.

Figures 87–93

Theory of Vision: Distorting Effects of Mirrors

A modern reader can disregard the mechanics and physics of Plato's visual ray theory of perception, in dealing with mirror distortions, since the effects observed are entirely independent of the postulated mechanism which produces them.

In the following figures, terms in parentheses are not parts of the matrix reflected, but indicate the spatial orientation of the observer to the thing seen, hence the designations and interpretations that he attributes to its spatial relations. In figures 87–93 the mirror images are used to schematize other philosophies as transformations of Platonism.

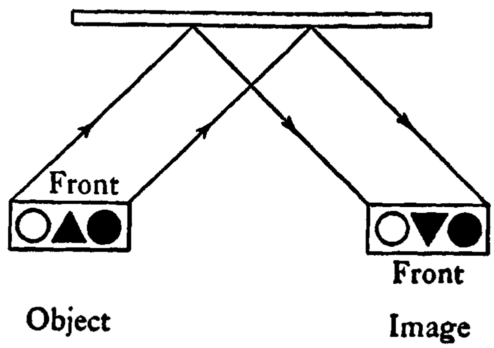
If we substitute the *Sophist* matrix for the letters in Figure 87, the effect of this simple reversal is (since the observer's orientation, determining the co-ordinates, remains the same) to give an idealistic emphasis by suggesting that human artifacts and imitations have a primary reality, while natural objects and images have a derivative one.

Figure 87

REVERSAL IN A PLANE MIRROR

MATRIX SEEN DIRECTLY

	(*Same*)	(*Other*)
(*Being*)	A	B
(*Becoming*)	C	D



MATRIX AS SEEN IN MIRROR

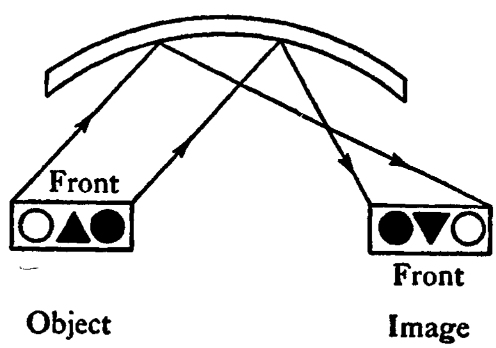
Matrix

	(*Same*)	(*Other*)
(*Being*)	A	B
(*Becoming*)	C	D

Image

	(*Same*)	(*Other*)
(*Being*)	B	A
(*Becoming*)	D	C

Figure 88



RECTIFICATION OF MIRROR
TO PREVENT REVERSAL

Probably one could extend the analogy of physical sight and intellectual vision to take account of Plato's mention of the rectifying mirror in this passage by pointing out that education in the same way "rectifies" the mind's orientation toward objects in experience when we have stabilized and corrected the revolutions in the soul.

Figure 89

EFFECT OF REVERSED INTELLECTUAL VISION

NORMAL INTERPRETATION

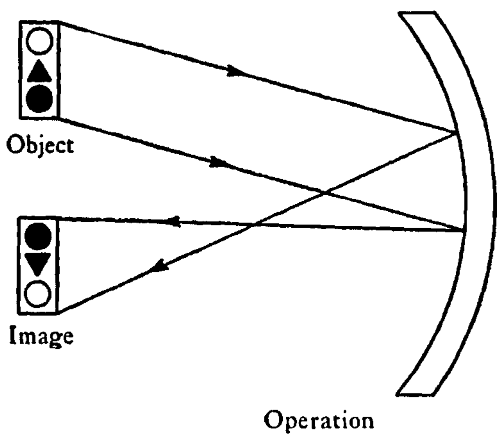
	(Creations [Reality])	*(Constructions [Appearance])*
(Objects)	Natural Objects	Man-made Artifacts
(Images)	Images of Such Objects	Human Fine Arts

REVERSED INTERPRETATION OF EXPERIENCE

	(Creations [Reality])	*(Imitations [Appearance])*
(Objects)	Man-made Artifacts	Natural Objects
(Images)	Human Fine Arts	Images of Such Objects

Note that the transposition of columns suggests, by its reorientation of terms, that natural objects may properly be explained as a kind of secondary artifact, a sense of "object" later to become central in Kant's philosophy.

Figure 90



INVERSION IN MIRROR

Figure 91

EFFECT OF INVERTED INTELLECTUAL VISION

A. LINEAR IMAGES

Normal Interpretation	*Inverted Interpretation*
(*Being*)	(*Being*)
eternal forms	representations
abstract hypotheses	opinions
opinions	abstract hypotheses
representations	eternal forms
(*Becoming*)	(*Becoming*)

B. PLANE IMAGES: REVERSAL AND INVERSION

NORMAL INTERPRETATION

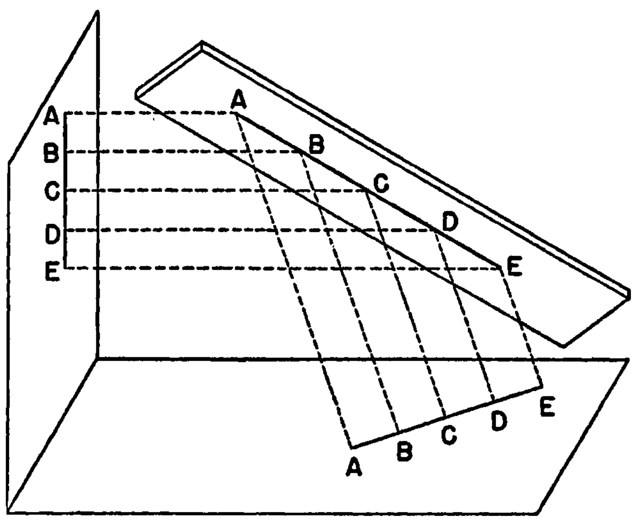
Nature	Artifacts
Shadows	Mimetic Works

INVERTED AND REVERSED

Mimetic Works	Shadows
Artifacts	Nature

Note that in Figure 91 the ontological status of the inverted segments of the divided line here corresponds exactly with the philos-

Figure 92



EFFECT OF MIRROR AT OBLIQUE ANGLE

ophy of Bergson, in which process is put forward as the primary nature of reality.

In Figure 92 an image in which the continuity of the parts depends on the metaphor of height is viewed as it appears turned on its side, in a position where this continuity is no longer operative. The result of such obliquity is to introduce a *separation* of entities, terms, and disciplines which in the original diagram were meant to be shown as synoptically connected. Again, the divided line is a peculiarly suitable illustration of this distortion.

Figure 93

EFFECT OF OBLIQUE INTELLECTUAL VISION

NORMAL INTERPRETATION

	(Being)
Sciences	philosophy
	mathematics
Arts	technology
	poetry
	(Becoming)

OBLIQUE INTERPRETATION

(Being)

Sciences: philosophy mathematics **Arts:** productive fine

(Becoming)

Note that the being-becoming axis ceases to interconnect the terms. Compare Aristotle's classification of knowledge:

Arts		Sciences		
Fine	Productive	Mathematics	Physics	Metaphysics

A part of the total Aristotelian scheme is here presented; it is intended to be that part which shows characteristically differentiated disciplines replacing the Platonic scale, hence as a typically oblique vision of Platonism. Aquinas and Kant have classifications of the sciences that lend themselves to this same interpretation.

III. The Theory of Geometrical Elements

Timaeus 51

Timaeus 53-56: * Such being their (the elements') nature at the time when the ordering of the universe was taken in hand, the god then began by giving them a distinct configuration by means of shapes and numbers. That the god framed them with the greatest possible perfection, which they had not before, must be taken, above all, as a principle we constantly assert; what I must now attempt to explain to you is the distinct formation of each and their origin. The account will be unfamiliar; but you are schooled in those branches of learning which my explanations require, and so will follow me. In the first place, then, it is of course obvious to anyone that fire, earth, water, and air are bodies; and all body has depth. Depth, moreover, must be bounded by surface; and every surface that is rectilinear is composed of triangles. Now all triangles are derived from two, each having one right angle and the other angles acute. Of these triangles, one has on either side the half of a right angle, the division of which is determined by equal sides (the right-angled isosceles); the other has unequal parts of a right angle allotted to unequal sides (the right-angled scalene). This we assume as the first beginning of fire and of the other bodies, following the account which combines likelihood with necessity; the principles that are yet more remote than these are known to Heaven and to such men as Heaven favours. Now, the question to be determined is this: What are the most perfect bodies that can be constructed, four in number, unlike one another, but such that some can be generated out of one another by resolution? If we can hit upon the answer to this, we have the truth concerning the generation of earth and fire and of the bodies which stand as proportionals between them. For we shall concede to no one that there are visible bodies more perfect than these, each corresponding to a single type. We must do our best, then, to construct the four types of body that are most perfect and declare that we have grasped the constitution of these things sufficiently for our purpose.

* Trans. Cornford, *Cosmology*, pp. 198, 212, 213–14, 215–16, 217–18. Consideration of *Timaeus* 51 is preceded by Cornford's translation of *Timaeus* 53–56 because the latter has an important bearing on the theory of geometrical elements.

Now, of the two triangles, the isosceles is of one type only; the scalene, of an endless number. Of this unlimited multitude we must choose the best, if we are to make a beginning on our own principles. Accordingly, if anyone can tell us a better kind that he has chosen for the construction of these bodies, his will be the victory, not of an enemy, but of a friend. For ourselves, however, we postulate as the best of these many triangles one kind, passing over all the rest; that, namely, a pair of which compose the equilateral triangle. The reason is too long a story; but if anyone should put the matter to a test and discover that it is not so, the prize is his with all good will. So much, then, for the choice of the two triangles, of which the bodies of fire and of the rest have been wrought; the one isosceles (the half-square), the other having the greater side triple in square of the lesser (the half-equilateral). We must now be more precise upon a point that was not clearly enough stated earlier. It appeared as though all the four kinds could pass through one another into one another; but this appearance is delusive; for the triangles we selected give rise to four types, and whereas three are constructed out of the triangle with unequal sides, the fourth alone is constructed out of the isosceles. Hence it is not possible for all of them to pass into one another by resolution, many of the small forming a few of the greater and vice versa. But three of them can do this: for these are all composed of one triangle, and when the larger bodies are broken up, several small ones will be formed of the same triangles, taking on their proper figures; and again when several of the smaller bodies are dispersed into their triangles, the total number made up by them will produce a single new figure of larger size, belonging to a single body. So much for their passing into one another. The next thing to explain is, what sort of figure each body has, and the numbers that combine to compose it.

First will come the construction of the simplest and smallest figure (the pyramid). Its element is the triangle whose hypotenuse is double of the shorter side in length. If a pair of such triangles are put together by the diagonal, and this is done three times, the diagonals and the shorter sides resting on the same point as a centre, in this way a single equilateral triangle is formed of triangles six in number. If four equilateral triangles are put together, their plane angles meeting in groups of three make a single solid angle, namely the one (180 degrees) that comes next after the most obtuse of plane angles. When four such angles are produced, the simplest solid figure is formed, whose property is to divide the whole circumference into equal and similar parts.

A second body (the octahedron) is composed of the same (elementary) triangles when they are combined in a set of eight equilateral triangles, and yield a solid angle formed by four plane angles. With the production of six such solid angles the second body is complete.

The third body (the icosahedron) is composed of one hundred and twenty of the elementary triangles fitted together, and of twelve solid angles, each contained by five equilateral triangular planes; and it has twenty faces which are equilateral triangles. Here one of the two elements, having generated these bodies, had done its part. But the isosceles triangle went on to generate the fourth body, being put together in sets of four, with their right angles meeting at the centre, thus forming a single equilateral quadrangle.

Six such quadrangles, joined together, produced eight solid angles, each composed of a set of three plane right angles. The shape of the resulting body was cubical, having six quadrangular equilateral planes as its faces. There still remained one construction, the fifth (the dodecahedron); and the god used it for the whole, making a pattern of animal figures thereon.

At the outset, it is essential to note that Plato is not here drawing his illustrations from a pure stereometry of the sort described in *Republic* vii.[80] Such a theoretical science of solid geometry must properly proceed from different principles peculiarly appropriate to three-dimensional mathematical objects. Any reduction of solids to the simpler elements of plane geometry or arithmetic is merely an operational aid to show the natures of the solids themselves, as geometers engage in absurd talk about plane figures when they talk of squaring and applying, as though they were studying combinatorial technique, and trying to build something rather than to understand it. In a proper stereometry, "volume" must be a central concept, and "division" must be construed as a division of volumes.

The present passage, on the contrary, treats the subject with a technique that is throughout (except when God's motives are specified) combinatorial, and the relevant stereometrical theorems proper must, for the most part, be read in from outside the given text. The elements of plane geometry are used as the principles of solids, in apparent opposition to the dictum in the *Republic* that each mathematical science postulates its own contrary principles.[31] In fact, in the references to the "numbers" of which solids are composed, the analytic reduction has been carried a step further, and the constituent plane elements are themselves viewed as though they were the units of arithmetic.[32]

This strange treatment of volumes as planes is a mathematical analogue of the treatment of things as radically disconnected by the removal of all bonds of proportion, as they would be "in the absence of God." [33] The principle of a physics of such unintegrated parts must be shock or contact, as it is in the later physics of Descartes.[34] This is the simplest external relation that can hold between bodies which are in no way harmonized as parts of a common organized whole. When applied to mathematics, the analytical technique by which Plato represents organisms dissolved into constitutive material elements represents solids as dissolved into mere abstract "sets of points of contact" —the volume is shrunk to a bounding plane, the plane to an infinitesimal unit of area.

Even this identification of elementary entities externally related by contact requires, in the flow of process, a postulate of definiteness and order. Pure flow, of an Anaxagorean sort, is radically unintelligible. The intelligibility of the outcome of mechanical process, however, requires the postulate that there be something definite in the flux. Plato indicated this by saying that "God has marked out" the basic units, separate from each other, but each retentive of its own spatial identity.[35]

At this point, there is an obvious objection to the claim made that this account of Plato's involves an atypical solid geometry. The synthesis of these planes into solids results in four of the five regular solids and presupposes the proper advance of stereometry through the classification of solids as regular and irregular. This seems, in fact, the most advanced point which the science had attained in Plato's time, and such a development as might have served as a model for the directed research advocated in the *Republic*.[36]

Though from God's point of view the results of the fitting together of these solids do exemplify an application of pure geometrical science, and possess unusual intellectual and aesthetic merit, the process of this synthesis seems not to be developed as a proper geometrical construction should. Another reason for the appearance of the "regular" solids here is related to mechanics rather than to stereometry. The geometer of solids sees why these solids are best; but a Milesian engineer would be the man to see why the exclusive fusion of planes into *regular* solids is necessary.

If the basic concept of this physics is contact or shock of corpuscles in a plenum, the basic concepts used to describe this process for the system as a whole are equilibrium and disequilibrium. It is relevant at this point to recall that the mechanical equilibrium of volumes and densities is made the ordering principle of the model displayed by Necessity in her personification as goddess, in the Myth of Er.[37] The formation of solids must respect this same principle. Since the series of corpuscular shocks is incessant from all sides, only those figures capable of being inscribed in a sphere will have the equilibrium necessary to attain stability and escape disintegration. Any irregular projection or asymmetry in a combination of planes under these conditions would, when hit, cause the planes to separate again. The idea of the atomists, that atoms are of all shapes, and equally cohesive regardless of shape, would have seemed to Plato to defy mechanics and geometry. One must go to the geometer of solids to find out how many and what sort are the solids satisfying this condition of equilibrium, but it is a mechanically, not geometrically, imposed condition.

A further confirmation of the fact that this account is mathematically atypical is found in the use of the dodecahedron.[38] God again enters explicitly to insure that the universe will not be lacking in perfection, but, as it included all living creatures in the earlier account of creation, it will now comprehend all of the "fairest" solid figures. God includes this dodecahedron by using it for the ornamentation of the heavens. Only in an account "according to necessity" could Plato recognize this sort of decoration as a legitimate mode of "inclusion." In the *Republic,* the work of the scene painter is criticized because it is not "included" in nature as real objects are.[39] But immediately after that criticism, Plato goes on to a myth in which the pictorial qualities of a cosmic model give the observer his clue to its mechanism, and in this myth omits the names of the planets presumably because their use would give an intellectual emphasis not in keeping with the pictorial context. All of the regular solids are "included" in the cosmos as they might be included in a box of mathematical models. The sculptured relief in the heavens is as much a "part" of the world as are the molded figures of the atoms. Neither is, after all, explanatory of the important organic aspects of cosmic structure.

The analysis by which volumes are reduced to elementary plane triangles is dependent on the prior reduction of relations and causality to external superficial contact. As a first step, the solid approaches its component capable of such contacts, and is reduced to a surface. Through bisection of the angles of plane figures the surface can be progressively analyzed into triangles until at last repetition of the division will only go on producing smaller triangles of the same type. Such self-reproducing triangles are the end-points of division; their properties remain homogeneous, however often the division is repeated. The unit of contact can be described as having certain geometric properties and ratios even if it is defined as infinitesimal, and hence can never be actually constructed.[40]

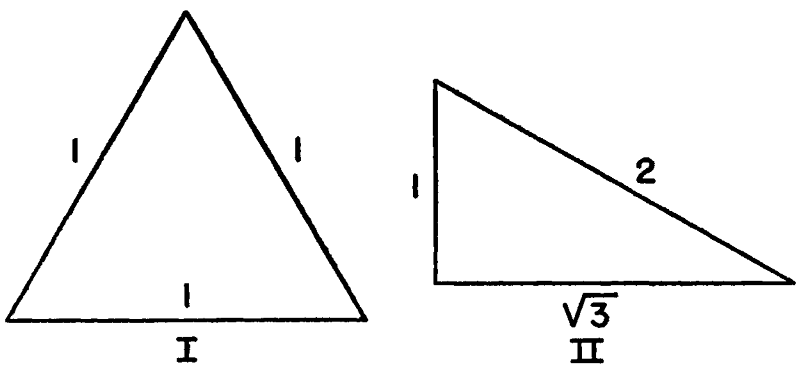
Since the shapes of these smallest triangles cannot be used as the surfaces of the stable solid figures in their uncombined form, one can explain the existence of stable solids (which the objects in our experience attest to be a fact) only by reversing the process of analysis through which these elements were derived. The ultimate point of contact will not serve as the ultimate element of physics; presumably, then, some simple aggregate or complex of these smallest elements does so.

This casts some light, perhaps, on a puzzling part of Plato's procedure. For in the course of the demonstration that the elementary triangles can aggregate to form the bounding surfaces of regular solids, the isosceles plane is reconstituted from six (rather than two) of these smallest scalene elements.[41] One can see from a diagram that this synthesis exactly reverses the antecedent analytic construction; the lines of junction in this figure are the lines of division resulting from analysis.

From this point on, there are really two tables of elements of which chemical and physical theory must take account. On the one hand, there are the elementary plane surfaces bounding the regular solids; on the other, there are the infinitesimal plane triangles of contact, the existence of which represents the least postulate needed to give an intelligible account of flux. Though the former of these elements are most closely connected with observed differences in phenomena, the latter are prior in theory. The dodecahedron is relegated to a decorative design in the heavens just because there is no simple way to show its bounding planes as aggregates of the primary elements.

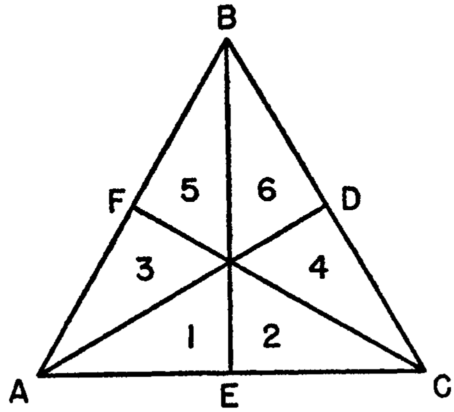
The significant point, which these detailed observations are intended to show more specifically, is that in this passage the method of Plato's geometry mirrors the method of the contextual dialectic, and just as the emphasis of the dialectic on analysis into elements is "uncustomary," [42] so is that of the geometry.

Figure 94



THE ATOMIC TRIANGLES

Figure 95

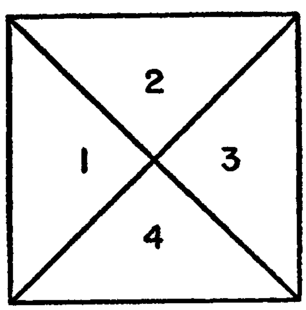


SYNTHESIS OF THE EQUILATERAL MOLECULAR TRIANGLES

Pairs of elementary scalene triangles are here synthesized by joining them "diagonally," with the diagonal repeated three times.

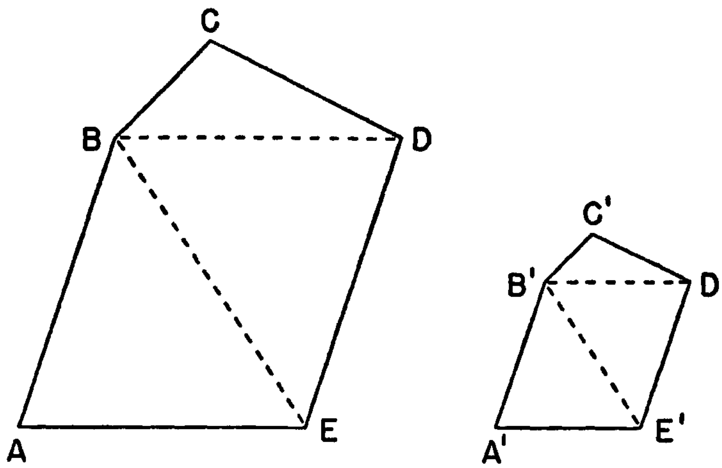
The question raised by this construction is why triangles 5 and 6, for example, are not simply joined along *BF, BD* to form a molecular isosceles triangle.

Figure 96



SYNTHESIS OF THE SQUARE MOLECULAR PLANES

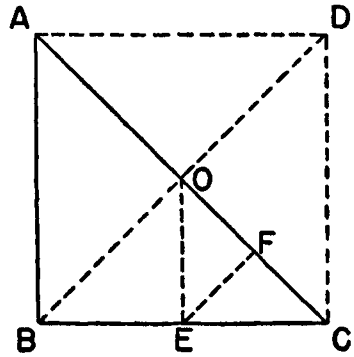
Figure 97



DECOMPOSITION OF POLYGONS INTO TRIANGLES (Euclid vi. 20; see I. L. Heiberg, ed., *Euclidis Elementa,* II [Leipzig, 1884], figure p. 133, text of Corollary, p. 138, and compare Todhunter's figure in *Euclid's Elements,* p. 200.)

In the figure, the two polygons are similar; each triangular subdivision of the larger has two similar sides and the same angle as the corresponding subdivision of the smaller. The theorem is that "similar polygons have to one another double the ratios of their corresponding sides"; since this has already been proven for similar triangles, the construction showing that similar polygons can be treated as sets of such triangles proves the more general theorem. Euclid adds, as a corollary: "The same theorem can be demonstrated for similar quadrilaterals, that their ratio is twice that of their similar sides. And this has been demonstrated for triangles: and so it holds universally that similar rectilinear figures are to one another in double the ratio of their corresponding sides." This suggests that Euclid thought of the division of his five-sided figure as giving a quadrilateral plus a triangle; so that, for a figure of any number of sides, the same type of construction will show it equal to a sum of a quadrilateral and added triangles. Further subdivision of triangles is not needed in this proof, and is not carried out. Since Euclid vi. is Pythagorean, it must be some construction like this one that Plato has in mind when he says that "all plane figures can be analyzed into triangles."

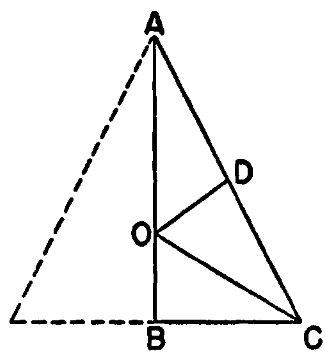
Figure 98



HOMOEOMEREITY OF THE FIRST ATOMIC TRIANGLE (Cornford, *Plato's Cosmology*, p. 233)

"The half-square *ABC* is divisible into two smaller half-squares, *ABO, BOC;* and Plato does in fact so divide it when he constructs the whole square face, *ABCD,* of the cube." (See next figure.)

Figure 99



HOMOEOMEREITY OF THE SECOND ATOMIC TRIANGLE (Cornford, *Plato's Cosmology*, p. 234, and note)

"In the same way, the half-equilateral, *ABC*, can be subdivided into smaller half-equilaterals by bisecting the angle at *C* and dropping the perpendicular *OD*. It is actually so subdivided into three elementary scalenes in Plato's figure [Figure 95]; and the subdivision can be carried on *ad infinitum*. . . .

"Since the triangles, not the solids, are Plato's 'elements,' this meets Aristotle's objection that not every part of a pyramid or cube is a pyramid or cube. 'Homoeomereity" was first clearly defined by Anaxagoras, but Empedocles had no doubt already assumed that every part of fire was fire."

Figure 100

TRANSMUTATION RATIOS OF THE ELEMENTARY SOLIDS

	EARTH *	WATER	AIR	FIRE
EARTH	1:1	—	—	—
WATER	—	1:1	1:2½	1:5
AIR	—	2½:1	1:1	1:2
FIRE	—	5:1	2:1	1:1

* Earth does not transmute.

It would be most natural for a physicist or even for a practitioner of purely theoretical solid geometry to derive these ratios from considerations of relative volume and planes of fracture, or topological deformation. But these ratios cannot be explained by such a derivation. As the correspondence of this figure with the next shows, these transmutation ratios are computed on the basis of disintegration and recombination of the elemental, triangular plane boundaries of the solid corpuscules.

Figure 101

RATIOS OF THE NUMBERS OF BOUNDING PLANES OF THE ELEMENTARY SOLIDS

EARTH	EARTH † (6)	WATER (20)	AIR (8)	FIRE (4)
WATER	—	20:20	20:8	20:4
AIR	—	8:20	8:8	8:4
FIRE	—	4:20	4:8	4:4

† Ratios of earth are not given, since its atomic bounding planes differ from those of the other elements.

CHAPTER V

Mathematical Jokes: the Limits of Mathematical Metaphor

Republic vii. 434D *

"But surely," said I, "if you should ever nurture in fact your children whom you are now nurturing and educating in word, you would not suffer them, I presume, to hold rule in the state, and determine the greatest matters, being themselves as irrational as the lines so called in geometry."

Republic ix. 580D †

"Very good," said I; "this, then, would be one of our proofs, but examine this second one and see if there is anything in it." "What is it?" "Since," said I, "corresponding to the three types in the city, the soul also is tripartite, . . . [the calculative part can provide ‡], I think, another demonstration also."

Symposium, 190B §

ARISTOPHANES: Now the sexes were three, and such as I have described them; because the sun, moon, and earth are three; and the man was originally the child of the sun, the woman of the earth, and the man-woman of the moon, which is made up of sun and earth.

Statesman 266 ||

STRANGER: Every tame and herding animal has now been split up, with the exception of two species, for I hardly think

* Trans. Shorey, *Republic,* II, 209.

† Trans. Shorey, *ibid.,* p. 371.

‡ The text of this passage, including the version translated in brackets, is given in note 8 to this chapter.

§ Trans. Jowett, *Dialogues,* I, 316.

|| Trans. Jowett, *Dialogues,* IV, 293.

that dogs should be reckoned among gregarious animals.

YOUNG SOCRATES: Certainly not; but how shall we divide the two remaining species?

STR.: There is a measure of difference which may be appropriately employed by you and Theaetetus, who are students of geometry.

Y. SOC.: What is that?

STR.: The diameter; and again, the diameter of a diameter.

Y. SOC.: What do you mean?

STR.: How does a man walk, but as a diameter whose power is two feet? [*AC*, Fig. 102]

Y. SOC.: Just so.

STR.: And the power of the remaining kind, being the power of twice two feet, may be said to be the diameter of our diameter. [*CF*, Fig. 102]

Y. SOC.: Certainly, and now I think that I pretty nearly understand you.

STR.: In these divisions, Socrates, I descry what would make another famous jest.

Y. SOC.: What is it?

STR.: Human beings have come out in the same class with the freest and airiest of creation, and have been running a race with them.

Though a concluding discussion of mathematical jokes may seem on first view a capricious digression from a discussion of mathematical imagery, it is in fact the only dialectically appropriate conclusion for such a study. The mathematical joke typically involves correction of an overpretentiousness in the use of the mathematical metaphor and imagery, of the expectation of more from the image than it should be expected to give. Plato's mathematical humor serves as a corrective to various types of misinterpretation that result if his images are taken too seriously. In the dialogues themselves they occur in locations where such safeguard is needed against overenthusiasm in the pursuit of mathematical metaphor; and to treat them as the conclusion of a serious study of the use of mathematical imagery is therefore one way of displaying the limits of such imagery, and of bringing out the absurdity of overstepping those limits in interpretation. In each case, the point of the humor is to be found in the practice of treating a metaphor overliterally; this takes the various forms of using geometrical differentiae in biology, where they are inappropriate, of using a geometrical vocabulary in politics, where it does not belong, and of postulating a causal connection on the basis of an arithmetical resemblance which is far from justifying such a postulate. The humor of these passages is lost on the modern reader because he does not recognize the contemporary counterparts of the literal-minded Pythagorean lecturer on ethics of Plato's time against whom the shafts of wit are directed. (In a twentieth-century transposition they would be aimed at the pedantic statistician.)

Nothing has been so diversely or implausibly interpreted by scholars as Plato's sense of humor. A legend of a "freakish" sense of humor that expresses itself in deliberate unintelligibility is a godsend to readers lacking the patience or insight to interpret difficult dialogues and passages constructively. The entire *Parmenides* and the mathematical images of the *Republic* and the *Menexenus* have all been called specimens of Platonic jest; and the *Republic* as a whole has been set aside as a piece of satire. The present study has already tried to show that such hilarity is not the intended tone of the *Republic* imagery.

In the dialogues Plato actually displays a very normal and well-developed sense of humor. The best way of beginning a discussion of the alleged grand jokes, such as those in the *Parmenides,* would have been in connection with the various passages in the dialogues which show this humor at work. There is nowhere in them any evidence of a humor which takes its point from some sudden obscurity, jargon, or unintelligibility, as jokes often do in, say, Aristophanes and Lucian. The humor is usually ironic, and aptly illustrative of Bergson's observation that a joke "calls attention to the physical when it is the spiritual [or intellectual] that should be in question." The hiccough of Aristophanes in the *Symposium* and his entire subsequent speech are excellent illustrations of this principle at work. So is Socrates' indictment of the Athenian statesmen, that "they have filled the city so full of harbors and public buildings that they have left no room for temperance and justice." [1] Another humorous device, the failure of a professional man to adapt normally, because in a novel situation he absent-mindedly, or overliterally, applies the categories of his profession where they do not belong, is evident in Plato's portrait of the vain and pompous blunders of Prodicus as he ineptly tells Socrates the etymologies of words instead of their meanings.[2]

It seems quite reasonable to believe that only a strangely incoherent mind could display such a normal sense of humor in the use of these devices and then deviate to a different principle of humor that is applied with what seems to the modern reader (who has been told that the *Parmenides* is a great joke) abnormal, tedious, and monstrous exaggeration.

Plato's intended pleasantries involving the vocabulary of mathematics are abnormal for the modern reader in still another way; they seem insipid, dull, and not funny. It is therefore hard to see how an author capable of a brilliant parody of Aristophanes could have amused himself by flat punning. The interpretation of Plato's use of mathematical metaphor, however, casts some new light on the principle of this mathematical humor. The thesis will be defended that in Plato's own time there were professional men who, given a mathematical term in a discussion, insisted on transferring the conversation to mathematical subjects. Some of Plato's own passages, in praise of geometry, for example, might be misinterpreted as advocacy of a second-rate Pythagorean pedagogy in which mathematical pedantries were substituted for dialectical inquiry. By an aside showing that he himself would recognize such a literal assimilation of dialectic to mathematics as humorous, as the sort of humor we recognize in the absent-minded and constant recurrence to his own preoccupation of the enthusiastic specialist, Plato detaches himself from any misinterpretation in this direction. Such asides function as safeguards to perspective; mathematical analogies do not always apply, and beyond a certain point must not be taken literally. The humor of such asides lies in their being exactly the remarks that an absent-minded professional man (in this case, a Pythagorean-trained, not overly intelligent, lecturer on ethics) might advance in the discussion as his serious, though inept, contribution. Our own Pythagoreans are statisticians, not geometers; the change of vocabulary that this entails prevents us from seeing, in Plato's illicit identification of the mathematical and nonmathematical senses of a term in one of his asides, any commentary on the statements of the misplaced professionalism of a group of experts in our own culture. (Perhaps the final conclusion of an educational statistical study that I have seen, "between men and women there is a significant difference," is a close approximation to the inadvertent punning of the not-very-bright Pythagorean in Plato's time.)

Four such passages in Plato will be discussed: the use of a geometrical pun in *Republic* vii to describe young men untrained in mathematics, the use of another such pun in *Republic* ix to differentiate logic from logistic, the explanation given by Aristophanes in the *Symposium* of the number of sexes, and the geometrical differentiation in the *Statesman* of bipeds and quadrupeds.

The "normal" use of mathematical metaphor, in the Platonic and Pythagorean traditions, depends on the conviction that physical, moral, and mathematical objects have many of the same properties. The etymologies of the mathematician's vocabulary, deriving technical terms from biology, ethics, and technology, record such recognized similarities between quantities and objects of other types. This conviction underlies Plato's deliberately ambiguous use of mathematical terms in such a way that their literal, technical meaning refers to a theorem or construction, and their broader, metaphorical sense specifies the intended interpretation of that construction.[8] There is another side, however, to this use of a common name as the sign of an observed common nature. In some cases a discrepancy of nature holds between quantities and other entities designated by a single term. This happens whenever the similarity that the common name reflects is fortuitous and nonessential, but the discrepancy between the classes named is not. One safeguard against the Sophistical assumption that in finding a common name one has discovered a common nature is the right use of humor as a corrective of conceit.[4] An overliteral faith in verbal technical devices is corrected by displaying a case in which the result of such literal-mindedness is a patent absurdity.

As with other sorts of Platonic imagery, the interpreter who finds considerable abstract aptness in the choice and location of mathematical wit in the text should not commit himself as to the extent to which these reflect the author's clear-cut and conscious plan. The author presumably has a feeling that this aside fits well here; the interpreter can recover only objective, abstract, intellectual relationships that are relevant, but not quite equivalent, to this intuitive feeling of aptness. Since an interpretation can restate felt imaginative affinities only as abstract intellectual connections, the reader who expects an intuitive and direct presentation from the interpreter is always disappointed and incredulous; such an abstract analysis is not what we ourselves experience when we are writing creatively, nor is it plausible to suppose that in this respect Plato differs much from ourselves. In the case of humor, this difference between intuitive appreciation and analytical interpretation is peculiarly apparent.

The boys who grow up "as irrational as the lines so called in geometry," in *Republic* vii, have close affinity to the antecedent discussion of the mathematical curriculum.[5] The pun on the irrationality of a man and a line invites an illicit identification of education with geometrical construction. The context of the geometrical line is purely quantitative; its construction as operation is external and mechanical. But the reasoning capacity of the human mind cannot be elicited by the analogous external application of courses in geometry. The pun points out that education cannot be conceived as a kind of engineering, nor geometry as magic; it does this by projecting the absent-minded professionalism of the pedantic geometer to the point at which we find him describing human nature and training with the concepts and operations of geometry.[6] By this aside, Socrates dissociates his own intention in the proposed mathematical curriculum from that of a misinterpreter who might think its purpose were to show all life, not under the aspect of reason, but as composed of the elements and operations of geometry. (In Book vii, Socrates himself uses a similar analogy of reason to the rational line, not to suggest a literal identification [7] but to bring out a relevant resemblance between them.)

In *Republic* ix, there is a humorous effect in Glaucon's bewilderment by Socrates' lightning calculation which helps provide a clue to the intention of the earlier mild pun on logic and logistic.[8] The relevant aspect of Socrates' "marvellous computation" is the summarizing diagram he envisions, which retains a clear relevance to his intended interpretation but which seems to lose any such evident relevance when it is replaced by a single arithmetical number representing the product of its dimensions.[9] The humor here gains in point if we recognize that the number, rather than the illuminating diagram from which it was computed, would have been fastened on and presented as a literal solution of the problem by the hypothetical Pythagorean-trained lecturer on ethics of Plato's time—just as our modern, statistically trained moralist might insist, in all seriousness, that the typical American family had $1\frac{2}{3}$ children.

This gives a clue to the humor Plato presumably saw in the earlier mild pleasantry in the same book, cited here. Socrates has remarked, that "since there are three parts of the soul, [the calculative] can show, by another demonstration . . ." that the tyrant's life is not a happy one. The humor arises from the suggested dependence of rational demonstration on the apparently fortuitous presence of the definite number, three, in the subject matter. Here, in anticipation, Socrates is dissociating himself from any assumption that the logical faculty is identical with the logistical.[10] In any use of mathematical imagery, there is always latent the danger of this confusion. Confronted with a complex situation, of which we have grasped several externally related parts, reason, through human inertia, is tempted to operate with those parts abstractly, to indulge in computation with these externally related units, rather than to try the more difficult task of seeing the integration of these parts in some larger, more intelligible whole. The logistical approach loses sight of the continuity and interdependence of the "parts," which it treats as the "units" of its computation.

Plato's aside underscores an ambiguity in the faculties and techniques of "calculation." The humor lies in the implied identity of counting and reasoning, of enumerating and understanding. The similarity of names leads to a silly result if we take it as showing an identity of things.

There is also latent in this pun an element of irony: experience does not respect the preferences of the logician by keeping itself free from confusion. The neatness and ease of manipulation of the abstract schematism does not correspond to an equal neatness and ease of control of immediate, experienced fact. If it did so, thinking and calculating might come much closer to being really, not just nominally, the same.

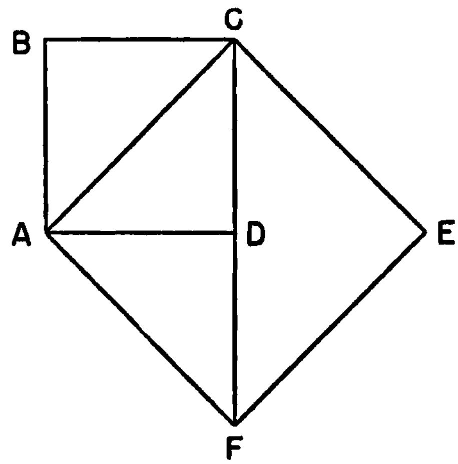
There are several reasons for restoring τὸ λογιστικὸν to the text. Apelt's arguments (see *Platons Staat,* p. 525) on philological grounds seem sound. Further, it is hard to see how the phrase could have come in as a corruption. (One can understand the corrupted version found in one manuscript, which lists all three of the parts of the soul at this point.) There is also a dialectical reason, already given, that seems equally cogent. This aside, or one like it, is needed to anticipate and protect against a misinterpretation of Socrates' later calculations (a type of misinterpretation into which the editors who have bracketed this aside have almost uniformly fallen). Further, this remark helps to make intelligible the cause of Glaucon's later bewilderment.

Aristophanes' clownish theory that the sexes are three because the sun, moon, and earth are three depends for its humor on the familiar Aristophanic comic device of *non sequitur.* The passage is important as an instance of mathematical humor both because the error it represents is basically mathematical, and because the sort of mathematical reasoning that Plato finds so Aristophanic tends to be attributed to Plato himself by his later expounders and admirers.[11] The notion that sets have the same cardinal numbers because of some affinity in essential nature becomes a principle of the microcosm-macrocosm analogy underlying alchemy, astrology, homoeopathic medicine, numerology, and their allied branches of speculation. In all of these studies, passages of Plato, particularly the mathematical ones, are cited as authority. For Plato, however, though similar things may have similar arithmetical properties, enumeration is no guide to demonstrating essential similarity. Even the substitution of a number for a geometric diagram is accompanied by a caution to the reader, since the important aspect of the diagram does not carry over to its number.

The joke about the difference between a biped and a quadruped in *Statesman* 266A (that the former equals the diagonal of a diagonal of the latter) is really a joke. Its humor lies in the unnecessary injection of higher mathematics into a biological discussion to accomplish a simple differentiation of species. This is completely compatible with Plato's view, expressed elsewhere, that mathematical imagery tends to give such clear-cut schematic pictures of a confused reality that there is something ironic in the juxtaposition. It is also an excellent illustration of what I have called the principle of aesthetic economy in Plato's construction of mathematical imagery: the ingredients of this joke are derived from the example of definition provided by Young Socrates and Theaetetus in the first dialogue of this trilogy.[12] The phrasing hinges on just such a geometrical representation of numbers as Theaetetus described, with the incommensurable given its full prominence. The intrusion of this example gives some clue to Glaucon's conviction, in *Republic* vii, that "only a few of the mathematicians he has known are good reasoners," [13] and constitutes an oblique criticism by the Stranger of the poor showing made earlier by Theaetetus in his conversation with Socrates.

It is interesting to note here the similarity of the diagram illustrating the present passage (see Fig. 102) to the one used in the discussion with the slave boy in the *Meno*.[14] This similarity seems to confirm the assertion that in his choice of theorems, Socrates was trying to find a problem which, to the dilettante Meno, would have the appearance of higher mathematics.

Figure 102

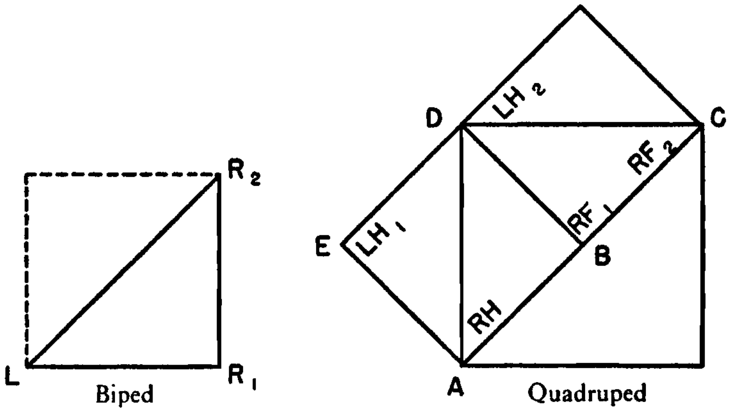


"DIAGONALS AND THEIR DIAGONALS"
(*Statesman*)

ABCD is a unit square; its diagonal, $AC =$ the square root of two The square on this diagonal, *ACEF,* has a diagonal equal to the square root of $2AC^2 = 2$. *CF* is thus the diagonal of the (square of the) diagonal *AC*. The next figure will show that in the present context, as applied to the locomotion of animals, this interpretation of the "diagonal of a diagonal" makes sense.

Aristotle's whole analysis of locomotion (*De incessu animalium*) applies what, from Plato's allusion here, must have been a technique devised in the Academy. The relation of the legs and the ground is treated geometrically, in analogy to properties of right and isosceles triangles. In Figure 103 the relative lengths of bases of two such triangles are shown, one representing a one-step advance by a biped of the right foot from R_1 to R_2 (where the diagonal resulting from the advance of the right foot lengthens the base to the diagonal of a unit square), the other a one-step advance by a quadruped, where the diagonal advance of the right fore and left hind feet (right fore from RF_1 to RF_2, left hind from LH_1 to LH_2) lengthens the base to the diagonal of two squares with sides equal to square root of two (= line *AC*). This figure and the Platonic passage should be compared with Aristotle's analysis.

Figure 103



THE LOCOMOTION OF ANIMALS

APPENDIX A

Effect of Context on Three Platonic Images of Cycle

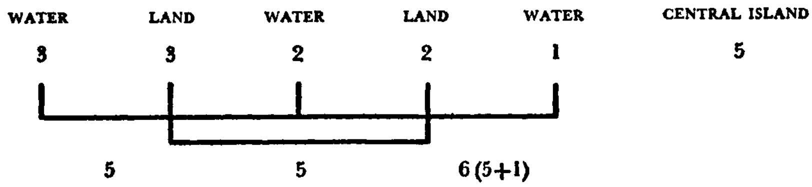
IT HAS been suggested that the significance of the peculiar engineering of Poseidon in planning Atlantis is deliberately introduced as a parody of the other images of concentric systems, one of which is prominent in each of the two preceding dialogues of the *Republic-Timaeus-Critias* trilogy. A comparison of these three sets of circles seems to show that the construction of Poseidon lacks the perfection proper either to an ensouled cosmos bound by bonds of friendship and proportion, or to a world-machine in which bonds of adamant alloy constrain the parts in such a way that by their balance cosmic morality and justice are preserved. The sole perfection of Poseidon's construction is its mechanical regularity; being a god, he describes his circular canals with ease.[1] It is also appropriate that a system of canals, rather than of celestial motions, be the product of the deity who is the patron god of the sea.

If this contrast were deliberate, we should expect some violation, in this circular system, both of the perfection of proportion evident in the world-soul, and the perfection of equilibrium shown in the world-machine. We should expect also a deliberate avoidance of the ratios elsewhere described as "concordant," which are used to describe the structure of planned, organic, internal relations.

In fact, the dimensions of the canals flatly violate both principles, and avoid any aspect of proportion. Their progressive sizes are ordinal, and although each pair of circles of land and water is equal, they form no systematic balance in combination.

Figure 104

LACK OF BALANCE IN POSEIDON'S ENGINEERING



These numbers show no such symmetrical balance as those of the astronomical model; for example, the ratio of circles of land to those of water is 5:6; total land to water is 10:6; island to circles is 5:11.

The dominant ratio of the system is 6:5, whether we compare the outer circles to the central island, or the circles of land and water with one another.

The accentuation of fives and sixes throughout the arithmetical details of the context directs attention to these more outstanding aspects of imbalance, away from the fact that a purely ordinal, logistical technique has actually introduced an adumbrated harmonic proportion among the paired sets of circles. The pairing of circles is significant only to the balanced mechanical structure, where each paired set is equal to the others; and the proportions are significant in giving the structure of the system with each circle treated separately. However, the system of circles here, treated in the way which brings out harmony elsewhere, yields the series 5,1,1,2,2,3,3; while equating the pairs, which gives an effect of balance elsewhere, cannot be carried out (because of the central island of radius 5), and, so far as it can be constructed, gives the sets 2,4,6, as mentioned above.[2]

A program of engineering based on power without enough insight into the combination of harmony and cycle which is the basic metaphor of social construction elsewhere in the text, results in precisely the lack of rationale and efficiency which the institutions of Atlantis on a large scale exemplify.[3]

The remarkable property of Platonic mathematical imagery in its tight integration with dialectical context completely defies either purely mathematical or symbolist interpretation. The relevant properties of a mathematical illustration cannot be determined simply by an examination of the purely mathematical properties of the image, for the relevance of these properties to the function of the illustration as a whole is differential and variable. Nor can an interpretation by a dictionary of symbolism in which each figure is correlated with a single associated concept ever do justice to the variability of Plato's actual choice and use of these figures.

Figure 105

PROPERTIES OF THE IMAGE OF CONCENTRIC CIRCLES IN THREE TYPES OF CONTEXT

PROPERTY	REPUBLIC	TIMAEUS	CRITIAS
Creator	Ananke	God	Poseidon
Structure (evidence of rationale)	Balance	Proportion	None
Material	Adamant	World-soul	Earth & water
Source of account	Religious vision	Calculation & science	Recorded history
Purpose	Education	Creation	Self-defense
Context	Myth, in pure dialectic	Empirical science	Historical legend

A convincing proof of this point is the use made of the same basic geometric figure—that of the set of concentric circles—in each dialogue of the *Republic-Timaeus-Critias* trilogy. In each case, we can say that the image symbolizes the efficacy of some cause, introducing a spatial or temporal order. The type of order introduced and the nature of the cause are not, however, specified by the symbolism or the geometry of concentric circularity. Neither is there any single varying mathematical property which accounts for variations in the significance of these images. In one case, ordinal sums and products of the widths of the individual circles are the mathematical property central to interpretation; in another, the ratios are the crucial property; in a third case, merely the size of the sum of widths is centrally relevant. As in the description in the *Statesman* of the rôle of descriptive measure in the arts, so in the art of interpreting Plato's dialogues recognition of a total function must provide the norm in terms of which we select those quantitative properties which are functionally relevant as the ones to be descriptively presented and measured.

APPENDIX B

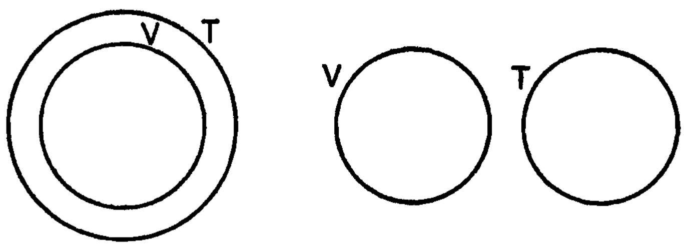
Symbolism: the Significance of the Specific Figures Chosen as Illustrations from Pure Mathematics

THOUGH IT has been suggested that in illustrating a method, any literal theorem from pure mathematics which involves an analogous process of reasoning will serve the desired function, so that the peculiar interdependence of imagery and context conspicuous in other cases is at a minimum, there is a tendency for certain figures to reappear as the content of such pure illus-

Figure 106

LATER DIAGRAMS OF CLASS-INCLUSION: EULER, LEIBNIZ

Euler



"ALL VIRTUE IS TEACHABLE" "NO VIRTUE IS TEACHABLE"

Leibniz

"TEACHABLE VIRTUE (EXISTS)" "NOT-(TEACHABLE VIRTUE EXISTS)"

In the usual current schematizing of class-relations, we follow Euler in representing all classes by the same plane figure, the circle, then schematize class-relations by the inscription or non-inscription of circles in each other.

Leibniz, however, whose appreciation of the value of schemata for logic led him to propose a philosophic language in which the propositions could be analyzed from the schematic relations of constituent primary ideographic symbols, did not introduce Euler's restriction to similar plane figures. For example, in his "Lettre sur la charactéristique" (Couturat, *Opuscules et fragments inédits de Leibniz,* Paris, 1903, pp. 29 ff.), Leibniz denotes moral qualities by distinctive geometric figures, their inclusion or coexistence by inscription or superimposition. His sign for God in this language is a triangle inscribed in a square, inscribed in a circle.

Thus the Platonic inscription figure has recurred independently in the history of logic as a diagram signifying either class-relations or coexistence of attributes in a single individual or class, and the suggestion that this may explain Plato's selection of it here cannot be dismissed as a merely fanciful positing of a resemblance which no one else would recognize.

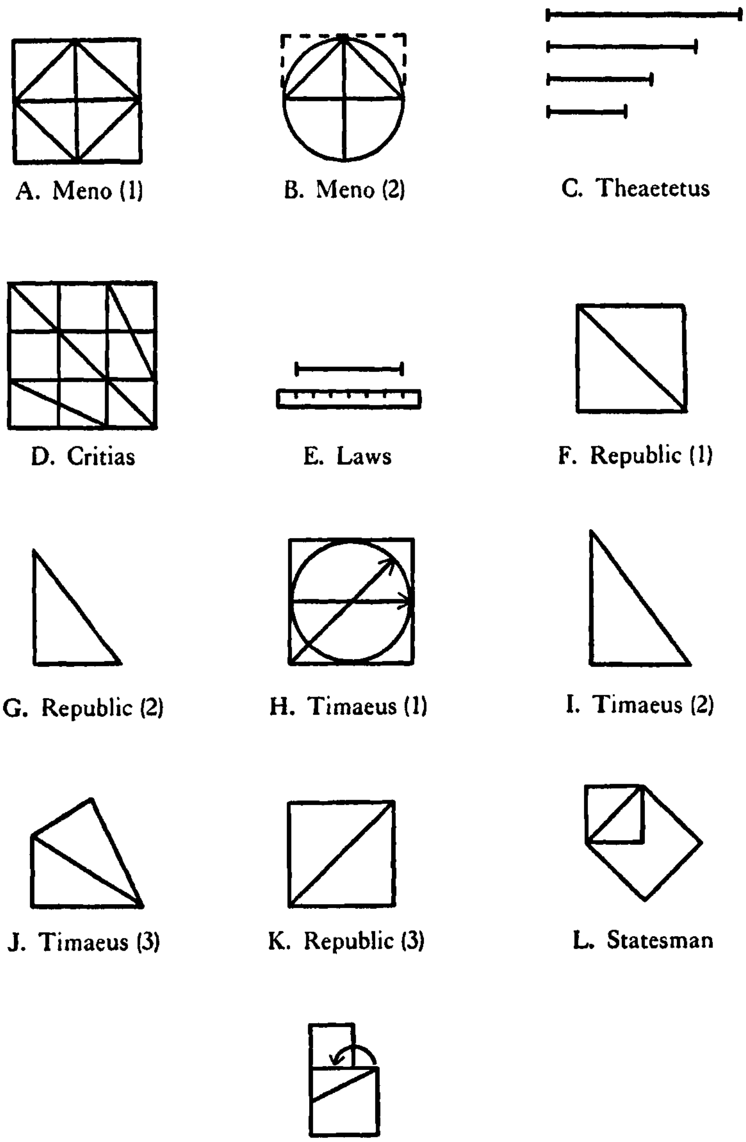
tration. This suggests that the choice of these figures for repetition was not purely coincidental.[1]

Some comment has already been made on the aptness with which a diagram of inscription gives intuitive illustration to a more general methodological problem of class-inclusion. In fact, as was noted, this aptness is so great that analogous problems of class-relation are studied at the present time with the aid of imagery of inscription.

One may therefore speculate on whether some equal intuitive aptness does not underlie the fact that a diagram based on the square of a diagonal appears several times independently in Plato's text, and is referred to several times more. In the *Meno,* this figure is used to demonstrate that knowledge is innate and can be elicited by questioning even from a young boy with no mathematical education, even when the knowledge in question lies on the advanced frontier of contemporary science. In the *Theaetetus,* this diagram underlies the construction by which

Figure 107

SYMBOLISM: FIGURES INVOLVING INCOMMENSURABILITY IN PLATO'S MATHEMATICAL IMAGERY



A. Meno (1) — B. Meno (2) — C. Theaetetus

D. Critias — E. Laws — F. Republic (1)

G. Republic (2) — H. Timaeus (1) — I. Timaeus (2)

J. Timaeus (3) — K. Republic (3) — L. Statesman

M. Republic (4)

Figure 107 is an attempt to bring together the various references to the side and diagonal and to the incommensurable, to show their frequency, and also to show in certain cases (for instance *A, B,* and *L*) their similarity of geometric form.

The reader will recognize *A* as the final construction in Socrates' discussion with the slave boy in the *Meno* (Fig. 9, Chap. I, Sec. 2). *B* is one of the figures suggested to illustrate the example of the method of hypothesis from the *Meno* (Fig. 14, Chap. I, Sec. 3). *C* is the diagram from Euclid x.9, the theorem of Theaetetus (Fig. 17, Chap. I, Sec. 4). *D* once more represents the Atlantean canal net; in this context, one notices that the communication canals superimposed on the irrigation grid produce regions embodying all sorts of incommensurability in the relations of their boundary canals (Fig. 19, Chap. II, Sec. 1). *E* is a pictogram to illustrate the study of measure and incommensurability in the educational program of the *Laws* (Chap. II, Sec. 2). In *F,* the square and its diagonal are the examples given of the kinds of objects mathematicians study in *Republic* vi. *G* is the Pythagorean genetic symbol basic to the Nuptial Number in *Republic* viii; it will be recalled that Plato's construction of this involved the squaring of an irrational (Fig. 53, Chap. III, Sec. 6). *H* illustrates the description of the motion of the other in the *Timaeus* as "along the diagonal" (that is, the diagonal of the rectangle formed by extensions of the tropics and vertical tangents to the equator). *I* is the atomic triangle from the *Timaeus* element theory, the longer side of which equals square root of three (Fig. 94, I, Chap. IV, Sec. 3). *J* represents the synthesis of molecular triangles in progress, with two of the atomic triangles joined "diagonally," that is, along their hypotenuses (Fig. 95, Chap. IV, Sec. 3). *K* represents the allusion in *Republic* vii to young men who grow up incommensurate with their responsibilities, the allusion to side and diagonal balancing that of *Republic* vi (see Chap. V). *L* represents the joke in the *Statesman* about the locomotion of bipeds and quadrupeds (Fig. 102, Chap. V). *M* shows the construction of the Divided Line by section in mean and extreme ratio, as in Euclid ii. 11, showing stages of the construction.

The final diagram is Figure 43, in Chapter III, Section 4 of this study; the construction is shown in Figure 108, of this Appendix.

roots and lengths have been defined and classified, in the manner in which Theaetetus wants to divide and classify opinion and knowledge. In the *Statesman,* the diagram is made the basis of a joke, the point of which is an illicit substitution of a mathematical for a biological differentia. In its context this probably has some critical reference to the way in which Theaetetus and Young Socrates have responded in the previous conversations recorded in the *Sophist* and *Theaetetus.*

In each of these passages, the context involves a discussion of the relation of two kinds of knowledge. The *Meno* passage brings out the difference between recollection and experience as sources of insight; the *Theaetetus* throughout proposes equivalences of opinion and knowledge which crucial cases, usually of mathematical knowledge, show to be indefensible; the *Statesman* joke involves substituting an easily understood mathematical property for the more relevant property based on right observation.

The consensus of these contextual passages is that knowledge and opinion are, in some puzzling sense, mutually "incommensurable." A mere addition of experiences is so far from an adequate account of knowledge that the hypothesis of a previous existence in which learning took place seems more plausible. On the other hand, the assumption that biology, an empirical science, is to be best formulated in the theorems and concepts of pure geometry (an assumption which some of the Pythagorean contemporaries of Plato no doubt seriously made) is so ridiculous as to be a "joke."

The suggestion of the diagram is that an alternative unit or technique of measure would permit an exact description of this baffling relationship, which would resolve or circumvent the impasse analogous to that of trying to find a linear unit of measure which would resolve the apparent incommensurability of the diagonal and side of a square. The later stress on the study of two- and three-dimensional problems in incommensurables is directed toward the better citizenship of men who can see the interconnection and relevance of social institutions and personal choices, as an uneducated man could not. True, the immediate applications of this education are to be to techniques of social implementation (in commerce, public works, etc.); but their importance is based on the assumption that their influence will be far more pervasive than this limitation would imply.

An alternative image expressing the relation of knowledge and opinion is the "divided line" in the *Republic,* in which the various segments (if established by a series of sections in extreme and mean ratio)[2] are proportionate to one another, though linearly incommensurable. Here the impasse reflected by the alternative image of incommensurable diagonal and side appears to have reached some contextual resolution, comparable to the construction by which the incompatible lines of the other figures are discovered to be commensurably and proportionately related in square.

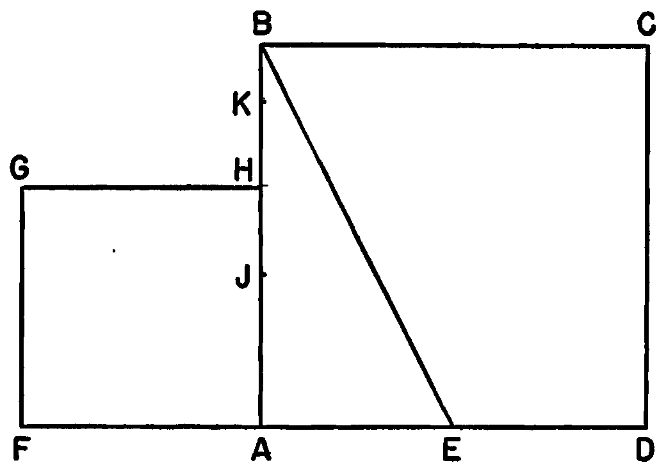
I suggest that it is the presence or the absence of an agreement about the theory of ideas which differentiates the contexts in which the relation of disparate levels of knowledge is presented as a simple proportion, and those in which it is presented as a geometrical impasse. In the absence of this theory, which provides the organizing insight of the image in the *Republic,* Socrates can explain knowledge to Meno only by the postulate of recollection; Theaetetus can in no way explain knowledge to Socrates by following the method previously used in defining roots and magnitudes; and the Stranger in the *Statesman* suggests that one could not find any principle for discriminating relevant and irrelevant subdivisions of a class.

If the objects of experience and opinion receive their character by participation in the forms, which are the proper objects of knowledge, this discrepancy in nature may be recognized without destroying the possibility of our making valid statements about the mutual relevance and connection of the two types of objects. This appears to be the dialectical equivalent of the geometrical construction showing that lines linearly incommensurable may be compared and have their exact relationship defined when they are combined as side and diagonal of a square.

These considerations suggest that Plato's repeated choice of this specific figure may have been guided by a felt appropriateness of its intuitive structure to symbolize the dialectical impasse and its resolution attendant on the demonstration of the "incommensurability" of knowledge and opinion.

What at first sight appears to run counter to this thesis is that the imagery of diagonal and side actually appears in Book vi and Book vii of the *Republic,* where the rule just suggested would seem to exclude it. As if by open and close brackets, the divided line is prefaced by an explanation that geometers study "the side and diagonal *as such,*" and is followed by the joke about boys and irrational lines (of which side and diagonal of a unit square are the standard symbolic illustration). But, on closer study, it will be noted that the geometer is not acquiring knowledge in its highest form (in fact, Glaucon remarks later

Figure 108



CONSTRUCTION OF THE DIVIDED LINE BY DIVISIONS IN MEAN AND EXTREME RATIO

(Method of Euclid II. 11)

AB is the given line. The square *ABCD* is constructed, and *AD* is bisected at *E* by a line drawn from *B*. *AD* is extended, and *EF* is marked off equal to *EB*. The square *AFGH* is completed, with

H located on *AB* so that *AB* : *AH* :: *AH* : *HB*. *AJ*, equal to *HB*, is laid off on *AB;* and *HK*, equal to *JH*, on *HB*. In this way *AH* and *HB* are divided in the same ratio as *AB*.

Here the incommensurables, instead of presenting a problem, have been added to produce a rational result; with the theory of ideas in the background, the dissimilarity of knowledge and opinion (symbolized as an incommensurability) is presented in a context which recognizes the possibility of presenting their exact relation in an analogical statement. The problem arises when the attempt is made to find some element or unit which can serve as the least common measure.

that their studies have not generally made the mathematicians he has known better philosophers), and that the risk of having boys grow up "incommensurate with their responsibilities" (as Cornford translates the passage) is run only if we neglect their philosophic education.

If we then modify the statement already made about the theory of forms to include the case in which there is a tension between the relation of mathematics and dialectic (their relation is, of course, exactly analogous to that of knowledge and opinion), we can also explain this use of side and diagonal imagery on either side of the diagram which sets up a proportionate relation among the irrational lines that represent knowledge of various kinds. Further, the idea of this symbol as marking a delimitation between an exact theory of knowledge and contextual discussion of inquiry and instruction, processes in which that theory may be misapplied, adds something to our understanding of the *Republic.*

Notes

CHAPTER I

1. *Euthyphro* 12.
2. *Meno* 82, 89.
3. *Phaedo* 104.
4. *Theaetetus* 147.
5. *Charmides* 166.
6. *Statesman* 259 ff.
7. *Euthydemus* 290
8. *Gorgias* 454.
9. *Protagoras* 356.
10. *Theaetetus* 196.
11. *Symposium* 190C.
12. *Meno* 74.
13. *Timaeus* 17A.
14. *Phaedo* 104 ff.
15. *Euthyphro* 7D.
16. The existence of the multiplication table, as it must have been set up with alphabetic number notation, would be no mean demonstration of the possible extent, synthetic power, and pedagogical value of a matrix. Inevitably, the difficulties of operating this notation would lead to an interest in computational shortcuts; and in the Pythagorean *pythmen* and the reported Platonic use of ten as modulus, there is evidence of an examination of number theory along lines which would make such simplification possible. The simplest aid to computation would of course be a mechanical abacus, with some principle of positional pebble arrangement. A vase painting shows such an abacus in use, and a ruled table discovered by archaeologists has been thought to be an actual abacus (though opinion is divided, some scholars explaining it as a game-board). The allusions in Plato which compare computation to the game of draughts seem to presuppose familiarity with some such computer's aid, which would in fact resemble the arrangement of lines and counters, and the regulated moves, of a 5 times 5 game-board. From the frequent association of priests and temples with banking and treasury functions, one may plausibly infer that the abacus would be a device with which a religious Greek was familiar. At any rate, the special tables of the money-changers were a familiar feature of the market place. See *Apology* 17C. 8,

and Burnet's note, *Plato's "Euthyphro," "Apology of Socrates," and "Crito,"* ed. with notes by John Burnet (Oxford, 1924), p. 71. Sir Thomas L. Heath, *A History of Greek Mathematics,* 2 vols. (Oxford, 1921), gives citations (I, 46–52) which show that the use of the abacus in calculation was common in Plato's time. See Fig. 1.

17. *The Dialogues of Plato,* trans. B. Jowett (3rd ed.; London, 1892), I, 298.

18. This is clear from the description of its eleven lines, each of which is marked with the initial letter of a unit of currency.

19. *Phaedrus* 275.

20. Figure numbers and letters in parentheses refer to the scholia figures given just after the text. These figures and references have been added to Jowett's translation; Jowett's original figure is given in Fig. 13.

21. *Meno* 85D.

22. Professor Greene has written concerning this proposed emendation of the final scholion figure "You are at liberty if you wish to quote me as saying that the printed figure and letters [in *Scholia Platonica*] correctly represent the tradition of the scholion in question, but that I concur with your emendation. . . ." Mr. Daniel M. Dribin, of Arlington, Va., a student of the history of mathematics, also concurs: "Your emendations of the figure used by the Platonic scholiast are quite correct. . . . The figure itself is Pythagorean, if Greek at all, and is typical of empirical proofs of [Euclid] I. 47, which could easily be suggested by geometric designs in tile decorations, panelling, etc. . . ." A further discussion of proofs of Euclid I. 47 of this type is given in George J. Allman, *Greek Geometry from Thales to Euclid* (Dublin and London, 1889), pp. 29–31. This material is of particular interest in connection with Plato's "combinatorial" approach to stereometry in the theory of elements in the *Timaeus,* discussed in Chap. IV, Sec. 3, following.

23. A. Benecke, *Ueber die geometrische Hypothesis in Platons "Menon"* (Elbing, 1867); J. Gow, *A Short History of Greek Mathematics* (Cambridge, 1884), p. 175, notes 2 and 3.

24. Heath, *History,* I, 300.

25. Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements,* 3 vols. (2nd ed.; Cambridge, 1926), III, 29–31.

26. In connection with this reference to figures already drawn, see in the text of Fig. 14 the suggestion that Socrates may have drawn a circle earlier in the discussion.

27. C. Demme, *Die Hypothesis in Platons "Menon"* (Dresden, 1888); A. E. Taylor: *Plato: the Man and His Work,* new ed. (New York, 1946), p. 138, n. 3.

28. See I. Todhunter, *The "Elements" of Euclid, Books i–vi and xi and xii* (London: Everyman's Library, 1933), figure p. 108.

29. See Appendix B.

30. Eva Sachs, *De Theaeteto Atheniensi Mathematico,* diss., Berlin, 1914; Heath, *History,* I, 209–12 ("Theaetetus"), 322–35 ("Eudoxus").

31. Aristotle, *Prior Analytics,* 41a. 26–27; Heath, *History,* I, 91.

32. Heath, *History,* I, 202–9 ("Theodorus"). The present discussion follows Heath's presentation and evaluation of the three proposed reconstructions of Theodorus' demonstration.

33. This construction is discussed in most of the interpretations of *Republic* 546A, cited below; it is very well described in F. Hultsch, "Die Pythagoreischen Reihen der Seiten und Diagonalen von Quadraten und ihre Umbildung zu einer Doppelreihe ganzer Zahlen," *Bibliotheca Mathematica,* 3rd ser., I (1900), 8–12.

34. Heath gives such a generalized form, *History,* I, 204–5.

35. H. G. Zeuthen, "Sur la constitution des livres arithmétiques des *Eléments* d'Euclide et leur rapport à la question de l'irrationale," Oversigt over d. kgl. Dansk videnskabens Selskabs Forhandlinger, 1915, pp. 422 ff., as cited in Heath, *History,* I, 206–11.

36. Heath, *Euclid,* III, 29–31; see Fig. 17.

CHAPTER II

1. *Critias* 112D.

2. *Ibid.,* 119D. 3. As Mr. Rosenmeyer has pointed out (*Classical Philology* XLIV [1949], 117), my translation of this as "every five and every six years" (*Classical Philology* XLIII [1948], 40) is not so literal as "every fifth and every sixth year." But since the ordinal numbers are being assigned to distances, not to points separated by them, I am convinced that "every fifth year" here does mean "every five years." Compare Plato's similar use of ordinals assigned to intervals discussed in Chap. III, sections 6 and 7, and Chap. IV, Introductory Comments. The point is particularly clear in III, 7, where a list from first to ninth is said to have its terminal items separated by a distance of nine units. The difference between the number of intervals and the number of points they separate is recognized clearly in *Republic* 546A, and whenever there is a choice open in constructing mathematical images, Plato seems to assign ordinal numbers to the intervals.

3. *Critias,* 113E.

4. *Ibid.,* 116D.

5. *Ibid.,* 113D.

6. *Ibid.,* 116A.

7. *Ibid.,* 115D–E.

8. *Republic* 616C–617D: J. Cook Wilson, "Plato, *Republic,* 616E," *Class. Rev.,* XVI (1902), 292–93; J. Adam's notes in his edition of the *Republic* (*The Republic of Plato,* 2 vols. [Cambridge, 1902]), in which he rejects his earlier interpretation, presented in his note "On Plato, *Republic* X 616E," *Class. Rev.,* XV 1901), 391–93. See Chap. III, Sec. 8c, and Appendix A.

9. *Critias* 113E.

10. *Ibid.,* 118C. 5.

11. *Ibid.,* 118A.

12. *Ibid.,* 118D. These canals actually represent an arrangement too haphazard to fall under the rule of 6's and 5's; Plato says that there were transverse canals every 100 stades (a total of 31 canals, counting the outer ditch), with connecting canals which had been cut between them. The image of a geometrical maze is used here with the arithmetical metric frame to reinforce the notion of confusion. See Fig. 19, following.

13. *Ibid.,* 118C. 7–D. 2.

14. The use of "myriads" has a special function, discussed below; but the length is stated in a way which does emphasize its determination by the sides; hence its derivation in this context from the summation of 2's and 3's, which elsewhere is presented by 5 as a symbol of the confusion of odd and even.

15. *Ibid.,* 118E–119B. See Fig. 22.

16. *Critias* 120D.

17. *Ibid.,* 116D. 1: "σταδίου μὲν μῆκος, εὖρος δὲ τρισὶ πλέθροις. . . ."

18. *Ibid.,* 116E. 1.

19. Jay Hambidge, *The Parthenon and other Greek Temples* (New Haven, 1924).

20. The discrepancy between exact accuracy in construction and apparent exactitude presents an aesthetic problem which the Greek architects had resolved, but which Plato does not develop or explain; in fact, he seems to avoid analogies to architecture.

21. *Republic* 546 ff. See Chap. III, Sec. 6*f* (12 and 14).

22. Hambidge, *The Parthenon,* Appendix.

23. *Laws* 701D, 757A.

24. *Ibid.,* 757.

25. *Ibid.,* 719C.

26. *Ibid.,* 817A, 746E.

27. See Fig. 24, following.

28. The sharpest distinction between logistic and arithmetic is made in *Gorgias* 451B–C. Arithmetic deals with odd and even numbers, and the quantities of each (i.e., classifications and factors), whereas logistic deals with the same numbers, but considered as sets, not as classes (as "having *plethos*"). The type of problem considered "logistical" is clarified by the

scholion to *Charmides* 165E (Greene, *Scholia,* p. 115; translation and comment in Heath, *History,* I, 14–15).

29. See Fig. 23, following (factors of 5,040).

30. A mathematician today can verify Plato's calculation easily by the theorem that if $p^a \cdot q^b$ are the prime factors of a number, N, then the total number of divisors of N is $(a + 1)(b + 1)$. Since 5,040 equals $2^4 \cdot 3^2 \cdot 5 \cdot 7$, the number of factors is $5 \cdot 3 \cdot 2 \cdot 2$, or 60.

31. A. E. Taylor, *Plato,* p. 477, n. 1.

32. *Laws* 746E.

33. *Ibid.,* 771B, 746E. Similar proposals for a duodecimal number system, based upon its superiority for calculation, continue to be put forward. See, for example, E. Ullrich, *Das Rechnen mit Duodezimahlzahlen* (Beiträge zum Programm der Realschule zu Heidelberg, 1891 [1117]), Heidelberg, 1891; and F. Emerson Andrews, "Excursions in Numbers," *Atlantic Monthly,* CLIV (1934), 459–66.

34. *Laws* 817C.

35. *Ibid.,* 697C; *Euthydemus* 305C–D.

36. Cf. *Philebus* 24, 52B, 64E, 65D; *Statesman* 283–85; *Laws* 746E; *Parmenides* 149B, 151B, 164–65.

37. Cf. Appendix A.

38. A. E. Taylor, in "A Note on Plato's Astronomy," *Classical Review,* XLIX (1935), 53–56, gives a clear, brief statement of the problem that results if we take the doctrine of "one path" as representing a theorem of theoretic astronomy. L. A. Post's review of J. B. Skemp's *The Theory of Motion in Plato's Later Dialogues, American Journal of Philology,* LXV (1944), 298–301, suggests an interpretation of the discussion of circular motion in *Laws* 893 which would be relevant to the present passage. On Post's interpretation, it should follow that the reference here to one path is to some single closed curve generated as a combination of circular or circular and linear movements. These conditions, however, are met completely by the *Timaeus,* where the helix generated by a point on a rotating circle which at the same time oscillates through an angle of 46 degrees is used to describe planetary motion. One can accept both the present interpretation of "one path" and Post's rendering of the later passage.

As for the common mistake of "calling the swiftest runner the slowest," this is the direct result of failure to recognize that planetary motion has two components. The retrograde component is the proper motion of each planet, since the forward component is transmitted from the outer heaven. To the untutored observer, it looks as though the planet with the slowest period were falling least far behind the fixed stars in a forward race. No elaborate system of negative numbers is needed to recognize that the slower runner falls further behind the swifter than one less slow. See also n. 130, Chap. III, following.

39. *Laws* 819C.
40. See Chap. III, Sec. 8c, following.
41. See Figs. 66–74, following.
42. *Laws* 819D.
43. *Republic* ii.
44. *Statesman* 283–88.
45. *Laws* 894A; see Chap. III, Sec. 4, following.
46. Aristotle, *De anima*, 404a.
47. *Laws* 945.
48. *Ibid.*, 948.
49. *Ibid.*, 934C.
50. *Ibid.*, 959.
51. *Ibid.*, 774A.
52. *Ibid.*, 956C.
53. *Ibid.*, 914B.
54. *Ibid.*, 774A.
55. See Chap. III, Sec. 8a and Sec. 8b, following.
56. *Republic* 615A.
57. *Timaeus* 23E, *Critias* 108E.
58. *Phaedrus* 249A.
59. *Phaedrus* 257.
60. *Critias* 119B ff.

CHAPTER III

1. Scott Buchanan, *Symbolic Distance in Relation to Analogy and Fiction*, Psyche Miniatures, General Series, No. 39 (London, 1932), pp. 101 ff.
2. See the scholion figure (Fig. 33), following.
3. *Cratylus* 432C.
4. See Figs. 26–28, following.
5. See Fig. 27, following.
6. *Gorgias* 508A.
7. See Chap. IV, Sec. 2, following.
8. It is easy to calculate the equivalence of any given Platonic ordinal number in a schematized matrix, and the equivalent row and column subscripts. Since these are verbal, not numerical, matrices, the ordinal numbers in the Platonic schematism do not respond in the same way as the subscripts to matrix operation.
9. *Republic* 400B ff.
10. *Ibid.*, 443D.
11. *Ibid.*, 424B.

12. *Ibid.*, 546A ff.
13. *Ibid.*, 514C–543.
14. *Ibid.*, 509D–511.
15. *Ibid.*, 587D.
16. *Ibid.*, 616 ff.
17. For a further discussion of the analogy of the musical scale, see discussion of the world-soul, Chap. IV, Sec. 1; also J. F. Mountford, "The Musical Scales of Plato's *Republic*," *Classical Quarterly*, XVII (1923), pp. 125 ff.; and Aristoxenus, *Harmonics*, edited with a translation, by Henry S. Macran (Oxford, 1902), pp. 11 ff.
18. Jowett, *Dialogues*, III, 686.
19. See Fig. 41, Chap. III, Sec. 3.
20. Adam, *Republic*, I, 214–15.
21. P. Shorey, "The Meaning of *kuklos* in Plato's *Republic*, 424A," *Classical Philology*, V (1910), 505–7.
22. *Republic* 616 ff.; see Chap. III, Sec. 8, and Figs. 66–74.
23. The discrepancy between the suggestion that the lengths of these segments somehow symbolize relative clarity of knowledge, and the analogies said to hold between them, is vividly presented by Warner Fite, *The Platonic Legend* (New York and London, 1934), p. 251.

To a reader of his time familiar with geometry, Plato's directions for construction involving the division and subdivision of segments "according to the same ratio" would very likely suggest a mean and extreme ratio section. One of the most spectacular geometrical properties of this construction is precisely the way in which the shorter segment, laid off along the longer, cuts the latter again in mean and extreme proportion. The construction once made can thus be applied indefinitely for redivision of segments in a constant ratio. As has been suggested, this property of recurring ratio seems to be the line of research developed by Theodorus in his investigation of the irrationals. A diagram developed by this construction will have segments of lengths which exactly satisfy the set of proportions attributed to the figure in Socrates' summary of its properties. This mean and extreme ratio construction of the diagram is shown in Appendix B, Fig. 108.

24. The other uses of unequal segments referred to are the figure in *Laws* x, describing the progress of causality from point to solid, and the Platonic schematism reported by Aristotle in *De anima* 404a, representing the faculties of the soul.
25. Note the use of higher and lower in the *Sophist* matrix, already discussed, and the frequent use throughout the discussion in the *Republic* of metaphors indicating that we have reached a higher place from which more can be seen.

One must protest against the urge of typographical convenience, which has led scholars to represent this image by a line that is *horizontal.* The vertical distinction of "higher" and "lower" is clearly part of the immediate intuitive effect of the diagram, and to ignore it shows complete lack of feeling for those properties of space which make possible significant diagram and matrix representations. This point is dealt with in detail in Chap. IV, Introductory Comments, preceding the section dealing with optical images in the *Timaeus.*

26. See the positions of "one" and "many" in the Pythagorean table of contraries, Fig. 27, preceding.

27. As in Fig. 44, following.

28. Compare Figs. 31–34 and the differentiation of temperance and justice indicated in Fig. 35. In the present case, inequality : justice :: analogy : temperance. It is important not to let one kind of knowledge do the work that should belong to another; it is also important to see the resemblances which make possible transition from one level to another.

29. The third segment of the divided line has usually puzzled interpreters who take it in conjunction with Aristotle's attribution of an intermediate realm of mathematical objects to Plato. There is nothing intrinsically puzzling about it. The objects treated by hypothetical techniques are those of the real sciences; the treatment differs only in that there has been no dialectical investigation of the principles of these sciences. The result is that the test of a hypothetical science is its internal consistency; the symmetry of the system determines its acceptability as a hypothetical account. The test of a true science, on the other hand, is its correspondence with forms, which are grasped in an immediate act of noetic insight. Aristotle repeats an analogous distinction in his own differentiation of "dialectic" and "demonstration." The "sciences," therefore, are strictly a realm of truth, tested by correspondence; whereas the systems of mathematics represent a realm of symmetry tested by consistency; the imitation of these structures in the natural order is a realm of beauty. This introduction of the touchstones of the good from the *Philebus* to explain the theory of truth in the *Republic* is mediated and justified by Socrates' account in the *Phaedo* of his own technique of hypothesis.

This Aristotelian attribution of a realm of intermediates is discussed in *Aristotle's "Metaphysics,"* a revised text with introduction and commentary by W. D. Ross, 2 vols. (Oxford, 1924), I, xxxiii–lxxvi; it is examined in more detail by L. Robin, in *La Théorie platonicienne des idées et des nombres d'après Aristote,* Paris, 1908.

Aristotle, in treating Platonism as an anticipation of the use of formal causes in his own philosophy, seems to introduce his own distinction between "abstraction" and "intuition" into his paraphrase of Platonic statements. Aristotle does this in spite of the fact (which he elsewhere points out) that Plato himself did not sharply separate the two faculties and their objects in the way Aristotle has done. But since Aristotle's purpose is to test the truth of Platonism as a philosophy (and Aristotle is convinced that any true philosophy must respect this distinction), he can paraphrase Platonism with the distinction presupposed. The result of this is a hypostatized realm of intermediate mathematical entities in the paraphrase of Plato. Robin's study brings out this point very clearly.

30. *Republic* 537C ff.

31. *Statesman* 284–86.

32. *Republic* 530D–531E; see Heath, *History*, I, 286.

33. *Republic* 531A.

34. Fig. 49, following.

35. Most of the following mathematical terms are taken from topology. All seem metaphorical, and in two cases statements about their coinage establish them as intentional metaphor. In their technical use, however, these terms are given formal definitions which restrict them to "mathematics" proper, and preclude any confusion or wrong identification of, for example, topological and physiological "nerves." On this point, Professor Dan E. Christie, of Bowdoin College, has written:

> . . . nerve of a covering and kernel of a homomorphism are certainly *used* metaphorically. Whether they were *coined* as metaphors, I do not know. . . . As for references, my dissertation, "Net Homotopy for Compacta," *Trans. Am. Math. Soc.*, LVI (1944), 275–308, uses *nerve* as a tool but hardly goes into the subject. [My original note had cited pp. 277, 280 of Christie's dissertation as authority for the usage of "nerve." The letter here quoted is a correction inspired by equal unwillingness to deprive a friend of a footnote, and to be misrepresented in it.] Better references are either P. Alexandroff and H. Hopf, *Topologie* (Berlin, 1935), p. 152, where "Nerv eines Mengensystems" is discussed, or P. Alexandroff, "Untersuchung über Gestalt und Lage Abgeschlossener Mengen," *Annals of Mathematics*, 2nd series, XXX (1929), 104: "Nun definiert aber jedes endliche Mengensystem $\beta = F1, F2 \ldots Fs$ folgendermassen einen (abstrakt zu denken) Komplex $N(\beta)$, den ich den Nerv des Mengensystems nenne. . ."
>
> As for kernel, on p. 557 of the aforesaid *Topologie* you will find a definition of "Kern eines Homomorphismus. . . ." It is assuredly true that these terms have precise mathematical definitions which rule out confusion of topological nerves and algebraic kernels with physiological [or botanical] things.

The references cited will enable the interested reader to see how such technical mathematical definitions operate in the cases of "nerve" and "kernel," which seem typical of such coinage. Christie continues:

> Here is another item for your file: part of a footnote from G. Joos, *Theoretical Physics*, trans. by Ira M. Freeman (Hafner, N. Y.), p. 562:

". . . but that at lower temperatures an ankylosis of degrees of freedom takes place. . . ." (Poincaré borrowed this term [ankylosis] from pathology to denote the "freezing up of a degree of freedom. . . .").

This example is therefore a case of documented metaphorical coinage, as well as use.

In conversation, Professor J. W. T. Youngs, of Indiana University, has suggested as other examples: "skeleton," "braid," "hair," "shaving the hairs off [a surface]," "trees," "diaphragm," "dendrite," "cactoid," "to cap [a hole]," "osculating planes." He also told me that he was thinking of a suitable name for a new topological concept, of an element of a given kind, capable of a given development, and that "germ" seemed to him a suitable metaphorical term. This is a second case where coinage as well as use can be documented as definitely metaphorical.

The reader must remember that reference to "new" mathematical terms in the text means terms that are "new" in contrast to ancient Greek coinages, though I believe the examples given in this note have not had their novelty and feeling of being metaphor worn off through long circulation.

36. Compare James Joyce's *Finnegan's Wake,* in which a principle of polyphony or systematic ambiguity is exploited to create lines and phrases with several simultaneous meanings, often in different languages.

37. Cornford has done this in his English, Cousin in his French, translation of the *Republic.*

38. Cf. Chap. V, following.

39. *Timaeus* 43.

40. See discussions, preceding, of *Meno* 82, *Sophist* 266.

41. *Timaeus* 43 ff.

42. Cf. in *Republic* 587 Glaucon's ability to interpret the tyrant's number in its arithmetic and geometric forms. See also Chap. III, Sec. 7, following.

43. Aristotle, *Politics,* 1313a2–b28. Aristotle, criticizing the theory of revolutions and the notion of a cycle of constitutions put forward in the *Republic,* quotes the phrase "of which the 4:3 base joined to the pempad produces two harmonies when thrice augmented," as the *principle* of Plato's theory. He explains the "thrice augmented" by adding "that is, when the number of the figure has become solid." From the passage, we can infer (1) that Aristotle takes this passage seriously and apparently understands it, (2) that the phrase he quotes is in his opinion that part of the passage which serves as "principle" for the rest, (3) that the theory is illustrated with a diagram, which, in the later development that Aristotle does not feel the need to quote directly, represents a modification of the figure described in the phrase quoted such that three factors are needed for arithmetical statement of the relations of that final figure.

The chief objections Aristotle urges are that the schematism is too remote from empirical fact to have any value as a contribution to a genuine Aristotelian science of politics; he notes that Plato himself was constrained by the facts of the matter to leave the cycle incomplete, and not to carry out this basic cyclic metaphor by postulating a revolution from tyranny to aristocracy. For the most part, Aristotle is interested in the later developments of *Republic* viii–ix, not in this passage; but he does recognize the passage as a schematic explanation of the principle organizing the later discussion.

44. *Timaeus* 51 ff.; discussed in Chap. IV, Sec. 3, following.

45. The basic idea underlying the Pythagorean use of this triangle as a symbol of marriage seems, from its use here, and from the general character of Greek genetics, to be correctly represented by later tradition as stated by Heath. The point is discussed as follows by Heath (*Euclid,* 2nd ed., I, 417, "Popular Names for Euclidean Propositions. I.47."):

1. *The Theorem of the Bride* (*theorema tes nymphes*).

This name is found in a MS. of Georgeus Pachymeres (1242–1310) in the Bibliothèque Nationale at Paris; there is a note to this effect by Tannery (*La Géométrie grecque,* p. 105); but, as he says nothing more, it is probable that the passage gives the mere name without any explanation of it. We have, however, much earlier evidence of the supposed connexion of the proposition with marriage. Plutarch (born about 46 A.D.) says (*De Iside et Osiride,* p. 373F) "We may imagine the Egyptians (thinking of) the most beautiful of triangles (and) likening the nature of the All to this triangle most particularly, for it is this same triangle which Plato is thought to have employed in the *Republic,* when he put together the Nuptial Figure [*gamelion diagramma—diagramma,* though literally meaning "diagram" or "figure," was commonly used in the sense of "proposition"] . . . and in that triangle the perpendicular side is 3, the base 4, and the hypotenuse, the square on which is equal to the sum of the squares on the sides containing (the right angle), 5. We must, then, liken the perpendicular to the male, the base to the female and the hypotenuse to the offspring of both. . . . For 3 is the first odd number and is perfect, 4 is the square on an even side, 2, while the 5 partly resembles the father and partly the mother, being the sum of 3 and 2."

Plato used the three numbers, 3, 4, 5, of the Pythagorean triangle in the formation of his famous Geometrical Number; but Plato himself does not call the triangle the Nuptial Triangle nor the number the Nuptial Number. It is later writers, Plutarch, Nicomachus, and Iamblichus, who connect the passage about the Geometrical Number with marriage; Nicomachus (*Introd. Ar.* II, 24, 11) merely alludes to "the passage in the *Republic* connected with the so-called Marriage," while

Iamblichus (*In Nicom.*, p. 82, 20, Pistelli) only speaks of "the Nuptial Number in the *Republic*."

It would appear, then, that the name "Nuptial Figure" or "Theorem of the Bride" was originally used of one particular right-angled triangle, namely (3,4,5). A late Arabian writer, Beha-ad-din (1547–1622), seems to have applied the term "Figure of the Bride" to the same triangle; the Arabs therefore seemingly followed the Greeks. The idea underlying the use of the term, first for the triangle (3,4,5), is that of the two parties to a marriage becoming one, just as the two squares on the sides containing the right angle become the one square on the hypotenuse in the said theorem.

46. See Adam, *Republic*, II, 264 ff.

47. In general, in the texts of passages involving any sort of mathematical reference or matrix construction, the manuscript tradition seems more reliable than most editors have believed it to be.

48. The myth of the gold, silver, and iron varieties told in *Republic* 415B insists (1) that the differences are not in species, hence will not prevent crossbreeding; (2) that the varieties will tend to breed true, but will not necessarily do so. For example, our citizens are told, two iron parents may produce a silver child, two silver parents an iron child, "and so for all the other combinations." The three-part difference in these varieties is a difference in capacity, with environment held constant, to rule wisely; hence the axis representing intellectual capacity in the present combination diagram is anticipated. The explicit mention of all the other combinations after the examples given suggests that behind this story lies some scientific explanation of the genetic phenomenon, which would be based on a combination diagram of the possible kinds of parental ability.

49. The suggestions made to citizens, that marriages between couples of opposite temperament are likely to produce the best children and to be best for the state (*Laws* 773A), would probably have been made to the artisan class in the *Republic.*

In a forthcoming article on Plato's genetic theory in the *Journal of Heredity*, I have summarized evidence from the history of genetics in the pre-Platonic period and from Plato's work which indicates the following conclusions: (1) The concept of "normative measure," involving two directions of deviation from a norm, gives a genetic matrix which explains the practical marriage regulations described in the *Statesman*, where in fact these regulations are presented as instances of "normative measure." (2) This same formulation makes the Myth of Metals in the *Republic* a very precise piece of "popularized science"; the analogy of metals and alloys is a fiction bypassing complex physiological phenomena, but the expectation of variation between proper social functioning of parents and children of every kind is exactly what the theory predicts. The provision for equality of opportunity of all children, whatever their parentage, in a just state, is therefore not a mere fabrication contradicting known scientific fact. (3) If the view of the relation of the *Republic* and the main divisions of the *Statesman* suggested in my "Plato Studies as Contemporary Philosophy," *Review of Metaphysics*, VI, (1952), is correct, the eugenic program that Plato would advocate for an actual human society would be about that of the *Statesman*. The basic problem Plato was treating is given a twentieth-century statement in Lee R. Dice, "Heredity and Population Betterment," *The Scientific Monthly*, LXXV (1952), 273–79.

50. See text of Fig. 43.

51. Compare the scholion to *Phaedrus* 244A (Greene, *Scholia*, p. 79).

52. J. Dupuis, "Le Nombre géométrique de Platon; Mémoire définitif," in Theon Smyrnaeus, *Exposition des connaissances mathématiques utiles pour la lecture de Platon*, ed. with a French translation by J. Dupuis, Paris, 1892. Bibliographical citations in notes 53–64, following, are, except for material in brackets, from Dupuis.

53. Fr. Barocius. Franciscus Barocius, Iacobi filii, patritii Veneti, commentarius in locum Platonis obscurrisimum, et hactenus a nemine recte expositum in principio Dialogi octavi de Rep. ubi sermo habetur de numero Geometrico, de quo proverbium est, quod numero Platonis nihil obscurius, Bologne, 1566.

54. Bodin, J. Les dix livres de la République, par J. Bodin, Angevin, 1583. Ensemble une Apologie de René Herpin, Paris, 1581.

55. Peucer, G. Les divins ou Commentaire des principales sortes de devinations, distingué en quinze livres, lesquels les ruses et impostures de Satan sont descouvertes, solidement refutées et séparées d'avec les sainctes Prophéties et d'avec les prédictions naturelles, tr. par Simon Goullart, Anvers, 1854. Tome IX, Chap. viii.

56. Mersenne, P. Traité de l'harmonie universelle par le sieur de Sermes, Paris, 1627, t. II, théorème xiii, p. 430.

57. [Thomas Taylor, *The Theoretic Arithmetic of the Pythagoreans*, with an introductory essay by Manly Hall, Los Angeles, 1934, pp. 148–57. The interpretation here corrects some details of Taylor's earlier presentation, in his translation of the *Republic*, cited by Dupuis; but Dupuis' summary applies equally well to the conclusions of this later version.]

58. Le Clerc, J.-V. Pensées de Platon sur la religion, la morale, la politique, Paris, 1819, p. 310.

59. Schneider, C. E. Chr. De numero Platonis commentationes duae. Quarum prior novam jus explicationem continet, posterior aliorum de eo opiniones recenset. Breslau, 1821.

60. Vincent, A.-J.-H. Notice sur trois manuscrits grecs relatifs à la musique. Supplément à la note L, sur le nombre nuptial. (Notices et extraits des mss. . . . Vol. XVI, pt. 2, pp. 184–194.)

61. Martin, Th.-H. Histoire de l'Arithmétique, le nombre nuptial et le nombre parfait de Platon (Extrait de la révue archéologique, 13[e] année.) Paris, 1857.

62. Mynas, C. M. Diagramme de la création du monde de Platon découvert et expliqué en grec ancien et en français après 2250 ans. Paris, 1848.

63. [Eduard Zeller, *Plato and the Older Academy,* trans. by S. F. Alleyne and A. Goodwin, new ed. (London, 1888), note 110, pp. 423–28.]

64. Weber, Otto. Gymnasium zu Cassel, Programm vom Schuljahre, 1861–2. Inhalt: De numero Platonis scripsit Dr. Otto Weber, Cassel, 1862.

65. P. Tannery, "Y a-t-il un nombre géométrique de Platon?" *Rev. des Ét. Grecques,* LXX (1903), 173–79.

66. Ivor Thomas, ed., *Greek Mathematical Works, I: From Thales to Euclid* (London: Loeb Classical Library, 1939), pp. 399–401, n. *c,* summarizing Laird's and Adam's interpretations. Of interest in connection with the latter is G. Kafka, "Zu J. Adams Erklärung der Platonischen Zahl," *Philologus,* LXXXIII (1914), 109–21.

The reader must remember, as a counterweight to Thomas and Kafka, that no number of the size proposed makes any sense if we try to interpret it as a period of a political cycle. Social changes do not take 36,000 years, particularly in societies as small and turbulent as the Greek city-states. From the postulates in context in the *Republic,* the first step of the downward cycle must occur in less than 160 years, and the full progression in less than 800. Any realistic study of history would show that changes in social forms do not take anything like Zeller's more modest 7,500-year period. Neither does the recession from an ancient "Golden Age" approach these large figures; the time between the existence of Plato's Athens and a mythical "Ancient Athens" which was a living embodiment of *Republic* i–v was 9,000 years, and the great cycle is 10,000; numbers that, even translated into days, are much too small for Adam or Thomas Taylor (not to mention Sosigenes, cited by Proclus as making the number something larger than 33 trillion). Nor is there any warrant for treating this as the life-cycle of the human race. Men have survived many catastrophes, and presumably always will, because the perfection of the cosmos requires that there be human beings in it. So even if this were what the Platonic text *said* (and certain specific comments already made have given some of my reasons for thinking it is not), we should be very far from seeing what it could possibly *mean.*

67. Since the list is usually visualized as written out in a line:

(1) aristocrat
(2) timocrat
(3) oligarch
(4) democrat
(5) tyrant

In this arrangement the interval is readily seen to be 1:5, not 1:9.

68. Since 729 equals $364\frac{1}{2} \times 2$, it equals the number of days and nights in a year. Cf. Adam, *Republic,* II, 358–59.

69. *Sophist* 266. See Chap. III, Sec. 1, preceding.

70. *Republic* 581C. 3.

71. *Ibid.,* 586E–587C.

72. If a total matrix is

	a	b	c
A	Aa	Ab	Ac
B	Ba	Bb	Bc
C	Ca	Cb	Cc

a gnomon-section is Aa-Ba-Ca-Cb-Cc.

73. The auxiliary differs from a timocrat, and the artisan from an oligarch, because both artisan and auxiliary are temperate, i.e., motivated by reason, not by ambition or avarice.

74. *Phaedrus* 248.

75. Thus there are three characterizations given of the best kind of lover, two each of the second through the eighth kinds, and only one of the ninth. The same principle repeats in the listing of choices of life in *Republic* X (see Chap. III, Sec. 9). This suggests that the emendation <ἢ> at *Phaedrus* 248D. 6 (*Platonis Opera,* ed. J. Burnet, II [Oxford, 1910]), which spoils the principle, is not desirable.

76. *Republic* 557C. To lend credence to the parallel between *Republic* ix and *Phaedrus* 248 here suggested, it may be noted that in *Republic* x, the souls changing into animals (Orpheus, Thamyras, Ajax, Agamemnon, Atalanta, and Epeius and Thersites) represent, as Proclus and Adam have noted, the musician-ruler-athlete-artisan-mimetic artist of the *Phaedrus* list, and they are presented in almost the same order. See Chap. III, Sec. 9.

77. *Republic* 587C.

78. Adam, *Republic,* II, 360–61.

79. *Republic* 582A. (See Fig. 65.)

80. *Ibid.,* 582D.

81. *Ibid.,* 580D ff.

82. See Fig. 64, following.

83. *Republic* 577C–580B.

84. *Ibid.,* 571C–572B, 575A.

85. *Ibid.,* 612C–614A.

86. A. G. Laird, "Note on Plato, *Republic,* 587C–E," *Classical Philology,* XI (1916), 465–68.

87. *Republic* 534A.

88. *Ibid.,* 611C.

89. *Ibid.,* 614B to end.

90. *Theaetetus* 176E.

91. *Republic* 617D.

92. See below, Section (7), (8).

93. The mechanism by which reason modifies appetite through the presentation of images of its concepts, which in themselves have no appetitive appeal, is presented in the account of the function of the liver, *Timaeus* 71B–E. The strange notions of physiology in this passage should not blind the reader to the importance of the basic psychological insight into the relation of thought, imagination, and desire.

94. Some of the factors preventing uniform coincidence of a priori expectation and actual perception are discussed in *Timaeus* 43D–44D; see also the discussion of optics in the *Timaeus,* Chap. IV, Sec. 2, following.

95. *Republic,* 400–1.

96. The suggestion that what the souls see is really a *model* of the universe, which Necessity holds in her lap, seems to have been made first by J. A. Stewart, in *The Myths of Plato.* This suggestion has two arguments in its favor. In the first place, it reconciles the position of the souls with what they see; from the place of choice, which is on a trans-celestial plain, the girders of the world can be seen stretching out into space, but a bird's-eye cross-sectional vision of the cosmic hemispheres would not be possible. In the second place, it accords with Plato's technique in the *Timaeus,* where God's creation of the world is presented by a description of the techniques that a maker of astronomical models would use in constructing a machine to show the motions of stars and planets. Further, only in a model could the celestial spheres be seen in cross section, so that their relative structure and the mechanism would not be hidden by the all-enclosing outermost sphere. Of course, there must be complete correspondence between the model and the world it represents, or the model ceases to have value as a guide to the choice of life which the souls who are shown it are about to make.

The present interpretation is based on the conviction that what the souls see is in fact such a model as Stewart suggests.

97. Cf. *Timaeus,* mechanics of perception, 44D–47C; rôle of space in thought, 52B. Since both perception and imagination are limited by space, neither structures nor principles can be directly perceived or imagined, but must be recognized by reason.

98. Cf. the basis of the differentiation of reason and opinion, *Timaeus* 51D.

99. *Timaeus* 57E–58C.

100. Cf. Heraclitus, Frag. 24 in J. Burnet, *Early Greek Philosophy* (4th ed.; London, 1938), p. 135. See also the definition of justice in *Republic* iv.

101. *Phaedrus* 248E.

102. Adam, *Republic,* II, 443, Fig. ii; 445–7. Since this study was written, however, I have decided that the evidence is by no means "conclusive," as I had thought. In discussion with Mr. Lawrence Hall, of Reed's Cove Boatyard, Orr's Island, Maine, and several of his friends, I found that experts in small sailing craft agree that any functional reinforcing cable or strap must pass from side to side beneath the hull, not from stem to stern. To quote one boatwright, the cable in Adam's model "may be a fender, or something of that sort, but it couldn't be a reinforcing cable; boats don't break that way." Mr. Hall is convinced that this would necessarily be true for a Greek trireme. As a Platonist, one should probably defer to the judgment of experts in a matter of their own art, and reject Adam's interpretation of the *hypozomata.*

103. Cf. Chap. III, Sec. 1, preceding.

104. *Republic* 498D.

105. *Ibid.,* 617B.

106. A. E. Taylor, *A Commentary on Plato's "Timaeus"* (Oxford, 1928), pp. 161–62, n. 2: "The MSS text [of this list of colors] should therefore be accepted as the only authenticated one. I confess I am entirely at a loss to understand what the order it gives is meant to represent. It is hard to acquiesce in Adam's view that it is a mere *jeu d'esprit;* it ought to stand for something which can be detected in the 'appearances,' but for what?"

107. See Figs. 66–74.

108. As Taylor remarks in the note just quoted, Adam has no definite function to suggest for this phenomenon of balancing.

109. The interpretations of Chapters II–IV of this study show that Plato's images are meticulously constructed. See Figs. 66–75, which, as calculation will show, would be very unlikely to balance as they do if they had not been constructed with this result in view.

110. E.g., *Timaeus* 52D–E.

111. *Timaeus* 68E is a clear, brief statement of this distinction.

112. See Adam, *loc. cit.;* John Burnet, *op. cit.,* pp. 188, 190–91, 304, n. 1.

113. Sir Thomas Heath, *History,* I, 335, summarizes his conclusion, developed in his studies of Greek astronomy, that Aristotle reintroduced the Ionian notion of astronomy as celestial mechanics in his notion of "counteracting spheres." The mathematical description of the world *qua* organism is given by Plato in the proportions used in the construction of the world-soul, *Timaeus* 35A–36E.

114. *Timaeus* 52D–53C, 57E–58C.

115. *Ibid.,* 58D ff.

116. Mass equals volume and density. The term is not used in its technical modern sense, but as representing the concept that heavier bodies are harder to lift or move, or, once put in motion, to stop, a concept

which Plato uses, for example, in *Republic* iv. to describe the inevitability of the growth of a state, once it has started gaining such "momentum."

117. See Figs. 69, 70.

118. I would like to correct here a statement made in my article, "Colors of the Hemispheres in Plato's Myth of Er (*Republic* 616E)," *Classical Philology* XLVI (1951), 173–76. On p. 174 I say that the probability of the properties of symmetry and balance appearing by chance in lists of digits from 1 to 8 "can easily be calculated." In footnote 10 (*ibid.*, 276), an approximation of this probability is calculated. But the approximation assumes the independence of certain factors which are not in fact independent, and the correct calculation is "easy" only for a professional calculator.

The conditions stated in the footnote cited are too far from those of this particular problem to give the right result. Actually, the correct figure is arrived at as follows. Given any list of digits from 1 to 8, take the first one and its complement (9 minus this first digit). How many other pairs of positions adding to nine would we describe as "symmetrical"? Taking the digit which lies in one position of a second "symmetrical" pair, what is the chance that the other position of that pair will contain its complement? Similarly for third pairs, given the first two. (If the first three pairs meet these specifications for symmetry, the fourth pair must also meet them.) Of these symmetrical arrangements, which will always have the sum of the first four digits equal to those of the last four? In what proportion of the patterns will only half the symmetrical arrangements approximate this balance?

The answers to these questions must be worked out separately for each location of the complement of the first digit in the list. The result is that the chance of having both symmetry and balance in a single such list, constructed at random, is less than 1/10 (actually, 9/98). To have four independent lists show these properties if they are random in construction has a probability of 1/10.[4] (The renumbering in order of size of the sums of two such lists interferes with both properties sufficiently to justify treating the fourth balanced figure [of masses] as independent.) The symmetry of the fifth list, also independent, has a likelihood of 1/7. The probability that these properties are only coincidence is therefore 1/70,000. Ordinarily, we assume that in a given single case an event with a probability of 1/10,000 or less does not in fact happen. (Ordinary statistical procedures make a much less rigorous assumption.) The fact that the principle of symmetry appears in a clearer form in the fifth list (if the reader will grant that balance of adjacent pairs is the most clear and natural way of representing mutual adaptation) is, since we admit the first and last digits as "adjacent," 3/105. But, having already included the condition that this fifth list is symmetrical, we must use here only the probability that the favored type of symmetry will appear in the fifth of five *symmetrical* lists, which is 1/7. The final figure, therefore, expressing the probability that the properties present in these lists would appear in such a set of lists constructed by chance, is 1/490,000. Though this is considerably less than the figure that would result if the properties were treated as independent for each pair in each list (the calculation in my footnote would apply exactly to certain techniques of list construction, but presumably not to those that Plato used), the revision makes no practical difference. It is still mathematically shown that the appearance of these properties is so unlikely to be coincidental that we can disregard that explanation.

The reader is asked to verify the fact that in the present calculation the problem has been stated in a way which evaluates the chance of an interpreter's being able to find these properties in lists when he is looking for them. In other words, conditions are used which operate against finding a high improbability; a very liberal notion of "symmetry" is employed, and the interpreter is assumed to look in each list for any pattern that can be called "symmetrical." By assuming slightly different conditions (for instance, that fewer arrangements are admitted as having the property of "symmetry," or that we are dealing with the construction of lists by drawing digits and position numbers from a hat), one can set up conditions that seem to apply to this problem, but, not being exactly enough applicable, give too high an improbability.

119. *Republic* 616C. 6–8.

120. *Timaeus* 67C. 5–68E. 9.

121. *Ibid.*, 68A.

122. *Ibid.*, 68B. 1–5.

123. *Republic,* 617A. 2–3.

124. *Ibid.*, 617A. 4.

125. *Timaeus* 68C. 2–4. In connection with this entire passage on color perception, and with the device of arranging colors by intensity in the Myth of Er, the descriptions of color mixture and the emphasis on intensity correspond closely to the first-hand color experience of a man at least partially red-green color blind. (This is especially true of the description, in the *Timaeus* passage just cited, of bright green as the result of mixing grey and yellow.) It is therefore possible that many of the peculiarities of the color description do not reflect general Greek practice, but idiosyncrasies of the color vision of the author; and that, in turn, may help to explain why Plato feels this whole field is controversial, and difficult to secure agreement about.

126. *Timaeus* 59B–60D.

127. Unless, as Taylor suggests in the quotation cited in n. 106, the "older version" of the text given by Proclus (discussed by Adam) was an

attempt to judge this property from apparent luminosity, a very unscientific undertaking.

128. Cf. Figs. 66–75.

129. Theon says that he himself built a model of the astronomical machine described in the Myth of Er. Presumably this was one of the water-driven orreries which Hellenistic mechanics had devised; see Heath, *History,* II, 428–29.

Though the variant κυλίεσθαι in Theon's text would describe a planet moving within its zone in a cycloid, nothing in Plato's description of the model corresponds to the planet as distinct from its zone or orbit, and of course no motion compounded of rotation and translation fits the mechanics of concentric hemispheres in the machine.

The difference here between Theon and the manuscript tradition suggests that both may be alternative restorations of a corrupt original. In that case, it is possible that Plato originally gave some indication here of how the transmission of momentum in the model came about.

130. This diffusion of velocities is somewhat obscured by the fact that whereas the swiftest circle has a forward velocity, the velocities of the others are measured by length of period and are compounded of forward and retrograde motions. The diffusion of the retrograde impulsion applied to the inner circle is shown by the proportionate lengthening of period as distance from the center increases. Taking the retrograde component as the proper motion of each planet (since the celestial revolution is given it from outside), we can see why, in the *Laws* treatment of astronomy discussed in Chap. II, preceding, ordinary common sense is blamed for "praising the slowest runner as the fastest"; this follows if we think of planetary motion as having only a forward component, and the planet with the longest period as coming closest to "keeping up with" the motion of the stars in a celestial race. See n. 38, Chap. II.

131. The place of the vortex concept in Ionian cosmology is well developed in Burnet, *Early Greek Philosophy.*

132. Evidently the Hellenistic notion of fluid transmission is the most natural one to apply in designing a model that will duplicate the motions Plato describes and that will respect the diffusion of impulsions applied to the outside and center of the system. At any rate, Professor Wallis Hamilton, of Northwestern University, in reply to an inquiry as to how a model could most easily be made that would duplicate the dynamics of Plato's world-machine as I described them, has suggested a model with cardboard rings floating on water and an inverted can in the center setting up the retrograde motion by its rotation.

133. *Timaeus* 57E–58C.

134. *Ibid.,* 38D.

135. See also F. M. Cornford, *Plato's Cosmology: The "Timaeus" of Plato*

translated with a running commentary (New York, 1937), pp. 74–115.

136. *Republic* 617C. 5–6.

137. *Timaeus* 36C.

138. *Ibid.*, 37E–38B.

139. *Republic* 617C. 6.

140. *Theaetetus* 176E–177A.

141. *Republic* 592A–B.

142. *Ibid.*, 395.

143. *Laws* 903E.

144. *Republic* 612B–613E.

145. If we further assume an intentional or unintentional connection between the various images of cycle in the *Republic*, the image of the momentum of a repeatedly impelled wheel, introduced in Book iv, may be cited as an indication that the inexorability of this machine is the result of its tremendous momentum; the massiveness of the cosmic hemispheres makes it impossible for any human agency to stop their motion. This compares very accurately with the "necessary" physics, in which process is treated in terms of the velocities and densities.

However, the fact that we are inclined to look for correspondences among the cyclic images introduced previously and in the present image reinforces the initial assumption that the nature of the model does not preclude the possibility of free choice. For both the cycles of evolution and degradation are presented as spirals. If a state at any time directs its attention to educational reform, it will improve just as inevitably as it will decline when, neglecting dialectic and impelled only by the motives inherited from its immediate past history, it does not. The political situation, as presented in these parallel images, seems to be one in which the state at each moment may either blindly follow its trajectory to an ultimate tyranny, or through new insights initiate a reform leading it back to an aristocracy. Certainly we should not expect a model of cosmic process to render what has been presented as political fact a physical impossibility.

146. The eschatological details, however, retain the emphasis on ten characteristic of the Orphic-Pythagorean tradition. For the importance of ten in Plato's myths generally, see Chap. II, Sec. 3, preceding.

147. *Republic* 617E.

148. J. Adam, "On Plato, *Republic* 616E," *Classical Review*, XV (1901), 391–93 (which must be read with the fact in mind that Adam later rejects Proclus' readings), and A. E. Taylor, *Commentary*, p. 161, n. 2, suggests the interpretation which seems to have inspired this text. See Fig. 69, following, for a demonstration that this text does not follow the balanced pattern of the other lists of details in the passage.

149. In his *Early Greek Philosophy*, p. 304, n. 1, Burnet defends his decision to recognize Theon's qualification of Mars's retrogradation. See

A. C. Clark, *The Descent of Manuscripts* (Oxford, 1918), pp. 383–418. I have accepted Burnet's hypothesis in this section, though most contemporary scholars would question it. Certainly, his argument needs qualification; but one is tempted to think that something of the sort is the case, particularly in view of the occurrence of five consecutive variants at intervals of 15 letters, between Theon's text and the manuscript version.

150. *Phaedrus* 252C.

151. See Sec. 7, note 75, in this chapter.

152. *Republic* 620B.

CHAPTER IV

1. Aristotle, *Metaphysics* 1092a.

2. For this principle of multiple interpretation at work, see *The Timaeus and Critias or Atlanticus*, trans. Thomas Taylor, reprint ed. (The Bollingen Series; New York, 1944), Introduction; and *The Commentaries of Proclus on the "Timaeus" of Plato*, trans. Thomas Taylor, 2 vols. (London, 1820).

3. An inquiry into the use of mathematics in connection with metaphysics, as in the *Parmenides*, was originally planned as part of the present study, but has proven so complex and extensive that it will require a separate work. I discuss the dialectical context of this problem in Chap. III of *The Rôle of Mathematics in Plato's Dialectic* (Chicago, 1942).

4. See the discussion of logistic and mathematics preceding, particularly Chap. II, Sec. 2b.

5. *Laws* 757B.

6. *Laws* 894A; *Epinomis* 990D–991C; Aristotle, *De anima* 404a.

7. See subsequent discussion in Chap. IV.

8. E.g., D'Arcy W. Thompson, *On Growth and Form* (Cambridge, 1917).

9. Aristotle, *Physics* 206b32.

10. *Timaeus* 36C. The frontispiece in Cornford, *Plato's Cosmology*, is a photograph of such a metal-band astronomical model.

11. Still following the analogy between God's creation of the cosmos and an artisan building a metal astronomical model, Plato seems here to envision the "cutting off of strips" as marking off and cutting into the metal band the intervals of a metric scale. Such a scale is evidently an important part of a model of this kind. To believe, as most translators and editors have, that the big band is exhausted by being "cut up" into smaller pieces, leaves no material from which the outermost circles of the cosmos are made; and such interpretations render the text at 37C, where the strip is split lengthwise after this "cutting," quite unintelligible. By

what may be a happy arithmetical coincidence, if we take a wide metal band incised with a scale from 1 to 27 and split it into three, the third strip can be cut up to give exactly the lengths needed for the rings within the model, representing planetary orbits, if the circumferences of these correspond to the basic scale (since $1 + 2 + 3 + 4 + 8 + 9 = 27$).

12. *Timaeus* 36C–D.

13. The best English source for an idea of Proclus' full interpretation is the translation cited in n. 2 of his commentary on the *Timaeus.*

14. A good deal of Pythagorean "number-symbolism" seems to have resulted from the Pythagoreans' having had no simple notation for a mathematical or logical variable. Consequently, the first (i.e., smallest) integers having a given property are used with the convention that these represent *any* term or number that has the same relevant property. In dealing with opposed pairs of properties, such as the "same" and "other" in the *Timaeus,* specific *odd* and *even* integers may be used, the former standing for terms in scientific analogies representing the "same," the latter for those representing the "other." (This suggestion answers Aristotle's question as to why the Pythagoreans identify a given property with the *first* integer that has it, and makes possible a sensible interpretation of such Pythagorean maxims as "justice is four.")

15. Cornford, *op. cit.,* pp. 59–66; A. E. Taylor, *Commentary,* pp. 106–8.

16. *Parmenides* 164C–166.

17. Heath, *History,* I, 115–17: "The *pythmen* and the rule of nine or seven."

18. Cf. L. Robin, *Théorie platonicienne* (Paris, 1908).

19. *Timaeus* 47.

20. *Ibid.,* 44A, 64D.

21. *Ibid.,* 51E ff.

22. *Ibid.,* 46B, see Fig. 87, following.

23. *Ibid.,* see Fig. 89, following.

24. *Ibid.,* see Fig. 91, following.

25. *Epistle* VII, 343C ff.

26. *Timaeus* 47A ff.

27. *Ibid.,* 46B. 7–13.

28. *Ibid.,* 43B, 90D, 92A.

29. Cf. *Timaeus* 50D ff.

30. *Republic* vii, 528A.

31. See the table of principles, Figure 49, preceding.

32. *Timaeus* 54D.

33. *Ibid.,* 53B.

34. Compare the "shock" physics of Descartes, "The World," in *Extracts from Descartes' Writings,* trans. H. A. P. Torrey (New York, 1892).

35. *Timaeus* 53B–C.

36. Eva Sachs, *Die fünf platonischen Körper* (Berlin, 1917); Heath, *Euclid,* III, notes on Book xiii; Euclid, *Elements,* xiii.

37. See Chap. III, Sec. 8a and Fig. 66, preceding; the rôle of equilibrium is presented in *Timaeus* 52D–53C, 57E–58C.

38. *Timaeus* 55C. If we read the earlier postulate of God's "marking out triangles" as equivalent to the assumption that an integration of *places* produces a *space* (treating a place as the locus of some homogeneous property, the space as a field within which there is constant, continuous interaction and qualitative flow), this embroidery takes on more point. The use of points of light to mark outlines of animal figures in the sky is technologically very like the mathematician's postulate of infinitesimal "places" which provide the groundwork necessary for a description of objects in space.

The concept of a limit and the problem of defining it make their appearance with the Pythagorean attempt to build magnitudes from nonextended spatial elements; Zeno's paradoxes demonstrated some of the defects in logic of their formulation. By Anaxagoras' time, it was evidently felt that some justification must be given for reasoning from the properties of finite things to the properties of their infinitesimal parts, and Anaxagoras' concept of homoeomereity is his justification for such inference. Cornford's demonstration of an analogous property in Plato's elementary triangles seems a valid and needed one. If space is continuous, the process of decomposing plane figures can go on indefinitely; but, once the process of analysis yields figures similar to the ones analyzed, the inference from finite to infinitesimal is a reasonable one.

Particularly in the light of A. N. Whitehead's recent analyses of space, place, and location, Plato's development of this geometrical element theory is a philosophically suggestive one. Plato's statement that "we assume God marks out the elementary triangles" translates into more contemporary terms as the postulate that "space equals an integration of places," taking "place" as a limit. That postulate connects mathematical and empirical space, since a region (a nexus of places) can be observed and described, but a pure empty space cannot. Pure space can be "known" only by the psychological act of emptying consciousness of all particular perception and thought; space remains. This is exactly the process recommended by Bergson for apprehending pure "duration," and the similarity suggests that Platonic "space" has dynamic properties.

The construction of volumes from bounding "places" is a neat mathematical formalization of the psychological construction (explained in nonmathematical terms in Aristotle, *De anima,* Book iii) by which we associate notions of touch and motion with perception of plane visual images, and perceive volumes and solids.

A comparison of this entire passage in the *Timaeus* to the relation of the monad and the infinitesimal in Leibniz' philosophy is extremely interesting.

39. *Republic* x, 601D ff.

40. Cornford, *Cosmology*, pp. 231–39, where this concept of the homoeomereity of the elemental triangles is developed. See Figs. 98–99, following.

41. *Timaeus* 54E; see Fig. 95.

42. *Ibid.*, 53C: ἀήθει λόγῳ.

CHAPTER V

1. *Gorgias* 518E–519A.
2. Prodicus in *Protagoras* 340C–341C.
3. See "Ambiguity of metaphor," Chap. III, Sec. 6a, preceding.
4. *Euthydemus*, throughout; also the *Protagoras* passage already cited, n. 2.
5. *Republic* vii. 534D.
6. A. E. Taylor (in *Plato*) reads many of the earlier dialogues as refutations of the Sophistic idea that virtue can be, as it were, externally and mechanically applied.
7. See above, Chap. III, Sec. 6e and 6f (14).
8. The text is:

'Επειδή, ὥσπερ πόλις, ἦν δ' ἐγώ, διῄρηται κατὰ τρία εἴδη, οὕτω καὶ ψυχὴ ἑνὸς ἑκάστου τριχῇ, δέξεται*, ὡς ἐμοὶ δοκεῖ, καὶ ἑτέραν ἀπόδειξιν.

* δέξεται	W (Chambray), Chambray, Adam, Shorey
τὸ λογιστικὸν δέξεται	A (Chambray)
λογιστικὴν δέξεται	q^1 (Adam)
λογιστικὴ δέξεται	q^2 (Adam)
λογιστικὸν δέξεται	A^2, F, D, M (Shorey), Apelt
λογιστικὸν ἐπιθυμητικὸν θυμικὸν	–Par. 1642 (Shorey)

Chambray = Chambray, *République;* Adam = Adam, *Republic,* II; Shorey = Shorey, *Republic,* II; A (Chambray) = cod. Parisinus 1807; q^1 (Adam) = cod. Monacensis 237; q^2 (Adam) = later corrections of and additions to q^1; Apelt = *Platons Staat,* trans. and ed. O. Apelt (*Der Philosophischen Bibliothek,* Bd. 80, Leipzig, 1920); A^2 (Shorey) = later corrections of and additions to cod. Parisinus 1807; F (Shorey) = cod. Vindobonensis 55; D (Shorey) = cod. Venetus 185; M (Shorey) = cod. Malatestianus XXVIII, 4; Par. 1642 (Shorey) = cod. Parisinus 1642; W (Chambray) = cod. Vindobonensis 54. The translation given in my text follows Apelt and Shorey's A^2, F, D, H.

9. See discussion of the bewilderment of Glaucon, Chap. III, Sec. 7, preceding.
10. Compare the differentiation of "logic" (dialectic) and "logistic" throughout *Republic* vii.

11. See the analysis of characteristic Neo-Platonic interpretation of analogy, Chap. IV, Introductory Comments.

12. *Theaetetus* 146, discussed in Chap. I, Sec. 4, preceding, and in Appendix B.

13. *Republic* vii. 531E.

14. *Meno* 83 ff.; Chap. I, Sec. 2, preceding; and Appendix B.

APPENDIX A

1. Here, as was suggested in connection with the *Phaedrus,* the power of Poseidon is his most striking attribute; and the mixture of a divine nature inherited from him with a mortal nature produces descendants characterized by their desire for domination and power.

2. See Chap. III, Sec. 8, preceding.

3. See Chap. II, Sec. 1, preceding, for discussion of the confusion of same and other represented by this use of 5 and 6; and compare with the use of same and other as orienting co-ordinates of matrices, e.g., *Sophist* 266A, discussed in Chap. III, Sec. 1, preceding.

APPENDIX B

1. The figures involving imagery of quadrilaterals and their diagonals are presented together for comparison in Fig. 107A–M, following.

2. See Chap. III, Sec. 4, preceding; and Fig. 108.

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