Space and Counterspace

An introduction to modern geometry

Louis Locher-Ernst

AWSNA

Published by:
The Association of Waldorf Schools of North America
3911 Bannister Road
Fair Oaks, CA 95628

Title: *Space and Counterspace*
Author: Louis Locher-Ernst

ISBN # 1-888365-36-6
German Title: *Raum und Gegenraum*, second edition
Original ISBN: 3-7725-0266-0
Printed by: Philosophisch-Anthroposophisher Verlag am Goetheanum
English edition published with permission of Georg Unger secured by the translator
Translator: Paul Courtney
Editor: David Mitchell
Proofreaders: Swain Pratt, Ann Erwin
Cover: Hallie Wootan

Curriculum Series

The Publications Committee of AWSNA is pleased to bring forward this publication as part of its Curriculum Series. The thoughts and ideas represented herein are solely those of the author and do not necessarily represent any implied criteria set by AWSNA. It is our intention to stimulate as much writing and thinking as possible about our curriculum, including diverse views. Please contact us with feedback on this publication as well as requests for future work.

David S. Mitchell
For the Publications Committee
AWSNA

Contents

PART THREE: THEORY (FIRST ORDER)

PART FOUR: REFERENCES

Foreword

After the appearance of his books *Urphänomene der Geometrie* and *Projektive Geometrie*, Louis Locher-Ernst allowed a long interval of time to pass before writing another textbook in the field of projective geometry. This, *Space and Counterspace*, is an introduction to modern geometry based on a spatial/counterspatial conception of space. The latter Locher had developed, more or less simultaneously with but independently from George Adams, following certain indications of Rudolf Steiner. In *Space and Counterspace* the fruits of a decades-long concern with counterspace, and also with issues of teaching, had taken a form that puts at the disposal of not merely the geometer, but every enthusiast and above all the teacher, a concise introduction, a rigorous schooling, and at the same time abundant material for teaching preparation. The author would certainly have worked through and perhaps enlarged the inevitable new edition. Now, after his death in 1962, it seems appropriate that this work should be made available with the same carefully executed design and layout as before, for which the publishers are to be thanked.

Dr. Ernst Schuberth has taken the trouble to correct a few small errors and misprints. Some figures have, at his suggestion, been redrawn by Herr Arnold Bernhard. The bibliographical references have been expanded and, particularly with regard to the works of Locher and Adams, updated. Thanks are expressed to all those involved in this undertaking, with the wish that this important textbook may find a wide circulation amongst teachers.

— Dr. Georg Unger
on behalf of the
Mathematisch-Astronomische
Sektion am Goetheanum

Editor's Foreword

The Association Waldorf Schools of North America is grateful to Paul Courtney for his excellent translation of this significant work by Louis Locher-Ernst.

Teachers of projective geometry in Waldorf eleventh grades will benefit most directly from this book now being available in English. However, all striving individuals who recognize the value of geometric study for the development of their own thinking capacities will also be appreciative.

— David Mitchell
Boulder, CO
2003

Preface

This book is about the beauty and the reality of the polar formations of space. It was written with three aims in view: first, to offer to a wider public a spiritually fitting introduction to modern geometry; second, to provide new material for teaching in the highest classes of genuinely progressive schools and colleges; third, to awaken an awareness of the polar formation of everything spatial. Being closely connected, the aims can be dealt with together.

What is presented in these chapters differs so essentially from distinguished textbooks of greater and lesser scope that their publication seemed justified. A number of facts that hardly find mention in other books are described in detail here. A particular effort is made to bring the line field to consciousness just as much as the point field, likewise plane space as much as point space. As a result, the life of mental imagery—today one-sidedly attuned to point space—is fundamentally enriched. This extended consciousness leads us to ask about the effectiveness of forces for which plane space plays a role similar to that of point space for central forces. I hold the view, that with suitable training, modern geometry offers an instrument for pictorializing and applying realms of forces which have not hitherto been grasped mathematically. A merely formal training is not enough, though; rather, the time is ripe for a vigorous expansion of our capacity for visualizing.

The book is divided into four parts: Fundamentals, Schooling, Theory, and References. At a first reading, I recommend that you avoid getting stuck in the details. Try first to get a general overview of Part One, which is devoted to fundamentals. To get a clear picture of the facts described is all-important.

In teaching, too, whether in schools or at higher levels, the appropriate mental images should be awakened before the logical connections are explored in detail. A preliminary understanding of the basics, by working on them in a lively way free of pedantry, can be reached in ten to twenty hours. Then you can turn to Part Two, the Schooling. Here the wording is concise, intended as a stimulus; this part can be regarded as an extended collection of exercises. Having selectively worked through as drawing exercises some of what is outlined, you will be able to catch up on much of what was left unclear from the fundamentals. You can read Part Three, assigned to theory of the first order, without difficulty once you have studied the fundamentals and schooling. Part Four provides a number of references which many readers will welcome.

Everyone should be able to enjoy successfully accomplishing the few elementary exercises. These are put at the end of each chapter and directly apply the preceding material. I have deliberately refrained from setting more openended questions, which most readers would find burdensome.

I am aware that objections will come from two sides. Some, the more generally interested readers, will think the treatment too rigorous. Others, those attuned to formal mathematics, will find the pictorial nature of the presentation and the style of a number of sections to be unfamiliar in today's technical language. In answer, it may be said that these views have constantly been borne in mind. In many cases, particularly in Part Two, a long weighing of words preceded the present version.

Those who might be satisfied with an easy-going, less exact presentation should bear in mind that it is an exigency of the times to penetrate the empirical world, even the geometrical experiences mediated by "movement organisms" with ideas.

On the other hand, there are scientists for whom mathematics means merely the theory of formal structures, that can be used as a means for handling natural forces. I am well aware of what this attitude may say to the present account. Agreement is possible if the following is borne in mind. Two spheres that in reality differ can have the same formal structure throughout; nevertheless that should not mislead one into disregarding the essential difference. For example, considered formally, Euclidean geometry and the geometry polar to it can be fitted into the same structural schema, yet refer to essentially different spheres of reality.

Ever since the faculty of thought awoke in the human being, it has been preoccupied with questions about the mysterious nature of space and time. As natural clairvoyance died away and the ancient mysteries fell into decay, human beings found themselves relying purely on the power of thought to find a solutions. How many answers have been given to these riddles in the last more than two thousand years—to most, a limited justification will hardly be denied—and how much sagacity has been brought to bear on them by numerous investigators! And yet we all feel that we know exactly what is meant by the two words "space" and "time" and are convinced that basically everyone else means the same thing by them. This suggests that space, and time, is something that is in some way common to every human being, and in which we all have a common interest.

The modern era led man to a perception of space that can perhaps be characterized as follows. "There is a container, which we call 'space,' in which material things are contained. The universe is largely empty; the accumulations of physical matter are no more than minute islands in the emptiness. The emptiness is without God."

As a result of this view a gulf arose between moral and natural world orders that exercises a profound effect on modern man, even if this is not brought to consciousness in clear concepts. This is true for the farmer as much as for the teacher and the factory worker.

This "empty space," as the ideal form that moulds our empirical knowledge is called, works today like an irresistible spell. Recent developments, however, make it possible to break the spell by means of cognition. They show that space is far from being a mere container, made available by a *deus ex machina,* in which events can run their course. Space is a concept with whose help thinking can penetrate the world we perceive. Once you have realized the completely conceptual nature of space, this represents a first step towards overcoming the illusion of a universe devoid of spirit. The latter picture is unreal; it is a thought-up creation derived from one-sided experience. A vivid grasp of the way space and counterspace work together already enables us to form a more comprehensive picture of reality. Pursuing the path thus opened leads to a raising of space and counterspace to a higher unity, and with it a reintegration of the spatial into spiritual (in the widest sense) life.

To the imagination, the deep-rooted idea of space as a lifeless container represents one of the chains with which Ahriman holds man fettered. To the extent that this fallacy is seen through, the chain loses its spell-binding power.

Dr. Mario Howald-Haller helped with reading the proofs and also made several good suggestions. I take this opportunity to thank him warmly for his painstaking assistance.

— L. Locher-Ernst

PART ONE: FUNDAMENTALS

Chapter 1

The Archetypal Phenomena of Incidence

Space has three different elements: *points, straight lines,* and *planes.* We all associate particular mental pictures with these names. When we say "this edge runs in a straight line" and "this is a plane surface" we are connecting certain sense-perceptions with the concepts straight line and plane. Even if they are only vaguely outlined in our consciousness, we must, in our thinking, have grasped these concepts independently of any perception.

The deeper we penetrate these three concepts, the more manifold in nature they turn out to be. The relationships they have to each other reveal a harmony that we would never have suspected.

The concepts point, straight line, and plane are the elements we want to grasp. We shall not try to find immediate definitions for them, however. Rather, we shall state certain relationships between these concepts that are suffcient to characterize them. In the course of our investigations we shall have the opportunity to make these initial, basic characterizations more complete.

A point can lie in a line,[1] a line can lie in a plane; a plane can go through a line, a line can go through a point. Since it is, as we shall see, appropriate to do so, we shall often combine the two statements:

The point P lies in the line ℓ

The line ℓ goes through the point P

in the single statement:

The point P and the line ℓ belong to each other.

Similarly, the statement:

The point P and the plane S belong to each other means both that the plane goes through the point and that the point lies in the plane.

Once we start using the concept of mutual belonging,[2] different aspects

[1] Here and throughout this book a line means a straight line.

[2] Mutual belonging is sometimes called *incidence.*

of each of the basic elements point, line and plane become apparent. We see the line in such a way that each line has both points and planes belonging to it, namely the points lying on it and the planes going through it, and the point in such a way that each point has both lines and planes belonging to it, namely the lines and planes going through it, the plane in such a way that each plane has both points and lines belonging to it, namely the points and lines lying in it.

These reciprocal relationships are such that the three basic elements unite in seven different basic forms, and in the following pages we will show the diversity of the interplay between these basic forms. We establish the following as archetypal phenomena [*Urphänomene*] of mutual belonging:

1. To each point there belong infinitely many planes; seen thus, the point appears as a *bundle of planes.*

2. To each plane there belong infinitely many points; seen thus, the plane appears as a *field of points.*

3. To each point there belong infinitely many lines; seen thus, the point appears as a *bundle of lines.*

4. To each plane there belong infinitely many lines; seen thus, the plane appears as a *field of lines.*

5. To each line there belong infinitely many planes; seen thus, the line appears as a *sheaf of planes.*

6. To each line there belong infinitely many points; seen thus, the line appears as a *range of points.*

7. 8. If a point and a plane belong to each other, then there are infinitely many lines belonging to both, that is, which go through the point and lie in the plane; these form a *pencil of lines.*

Two elements of the same kind, that is, two points, two planes, or two lines, cannot belong to each other unless they are identical.

A point is first of all an element in its own right. If we take note of the elements that belong to it, namely the lines and planes going through it, then it appears as a bundle of lines and as a bundle of planes. When we consider both aspects of the point, that is, the point as line bundle and as plane bundle, we speak simply of a bundle. Thus a bundle is a point with all the elements belonging to it; each point is the carrier of a bundle. To say that the whole, that is, the point together with all the elements belonging to it, is greater than its parts loses its usual meaning here.

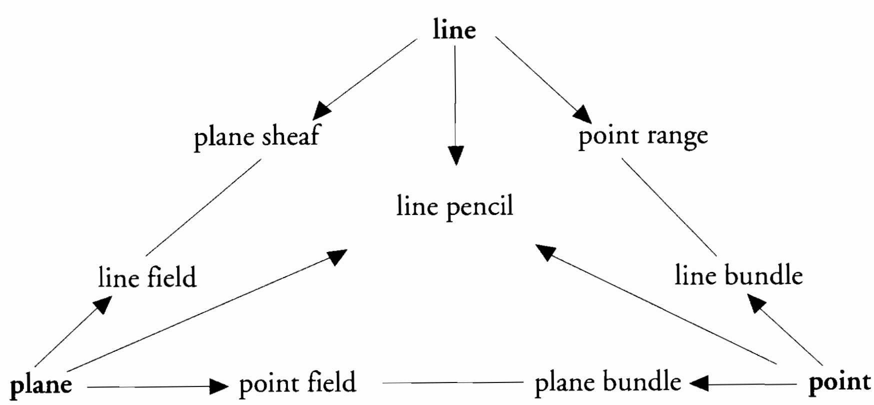
A plane, too, is in the first place an element in its own right. If in addition we consider the elements that belong to it, namely the lines and points lying in it, then it appears as a field of lines and as a field of points. If we have occasion to consider both aspects together, we speak simply of a *field*. A field is thus a plane with all the elements belonging to it; each plane is the carrier of a field.

Finally, a line does not appear merely as an element in its own right; it is also the carrier of a range of points and a sheaf of planes. The elements belonging to it are the points lying in it and the planes passing through it.

The pencil of lines is the basic form in which point and plane are evenly balanced. The essence of a point and a plane that belong to each other is to be the carrier of a line pencil. Its members are all the lines which belong to this carrier.

The relationships between the seven basic forms are summarized in the diagram on the next page. Already in these most basic relationships between the elements point, line, and plane, we can begin to experience a ruling harmony, though it does not come to light immediately. The three basic forms, point-range, line-pencil and plane-sheaf are called first-degree basic forms, the others second-degree basic forms.

Infinitely many first-degree basic forms are contained in each second-degree basic form. Each field contains infinitely many point ranges and line pencils, and each bundle infinitely many plane sheaves and line pencils. Space itself has a triple aspect: it can be looked at as point space, as plane space, and as line space. When we talk about point space, from space we are singling out its points as elements; with plane space we have in mind in particular its planes as elements; with line space, its lines. It makes sense to call point space and plane space third-degree forms, whereas line space turns out to be a fourth-degree form.



It is evident that our mental-picturing faculty is able to represent only finite pieces of lines and planes. It cannot form the picture of a whole plane or a whole line. Thinking grasps the pure concepts line and plane, and the concept of a point as the carrier of a bundle, whereas mental picturing only works piecemeal.

EXERCISES

Form, as clearly as possible, mental pictures of the basic forms in (for example) the following colors:

the line and plane bundles in yellow,
the point and line fields in blue,
the point range and plane sheaf in green,
and the line pencil in red.

Chapter 2

The Common Elements of Two Basic Forms

In each of the following pairs, the two elements can be brought into relationship with each other: two points, two lines, two planes,

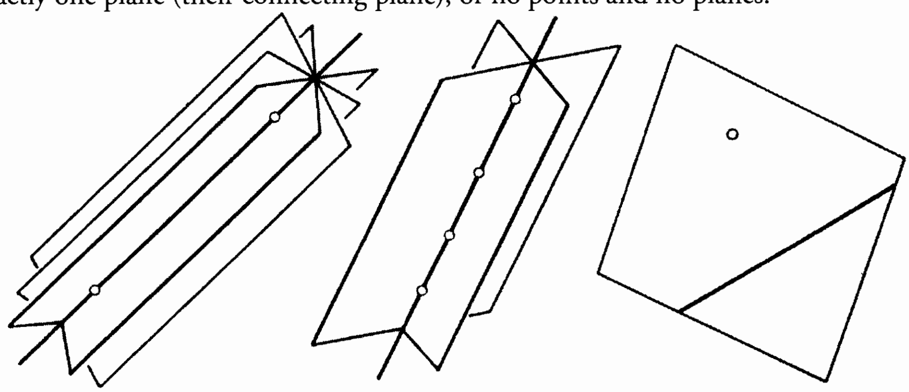
a point and a line that does not belong to it,
a plane and a line that does not belong to it,
a point and a line that belongs to it,
a plane and a line that belongs to it,
a point and a plane that does not belong to it,
a point and a plane that belongs to it.

For example, two points have exactly one line in common, as well as all the planes that go through this line. To put this somewhat differently: two bundles have in common firstly a unique line—the line joining the carriers of the bundles—and secondly all the planes belonging to this line. If we examine the possible combinations listed at the beginning, we discover the following archetypal phenomena:

9. Two points have in common a unique line (their connecting line) and all the planes belonging to this line (Figure 1).

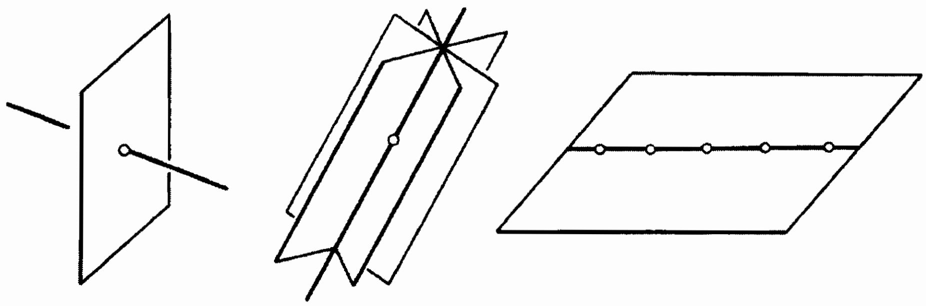
10. Two planes have in common a unique line (their line of intersection) and all the points belonging to this line (Figure 2).

11. 12. Two lines have in common either exactly one point (their point of intersection) and exactly one plane (their connecting plane), or no points and no planes.



Figures 1, 2, and 3

13. A point and a line that does not belong to it have a unique plane (their connecting plane) in common (Figure 3).
14. A plane and a line that does not belong to it have a unique point (their point of intersection) in common (Figure 4).
15. A point and a line that belongs to it have all the planes of this line in common (Figure 5).



Figures 4, 5 and 6

16. A plane and a line that belongs to it have all the points of this line in common (Figure 6).

Propositions 13 and 15 are about the common elements of a bundle and a plane sheaf, Propositions 14 and 16 are about the common elements of a field and a point range. The appropriateness of numbering some of the propositions twice will shortly become apparent.

17. 18. A plane and a point have in common either no lines at all or a line pencil.

The second part of 17. 18 has already been stated in Proposition 7. 8.

The formation of common elements comes about through the operations of connecting and intersecting, as they are called:

the connection of two points is the line common to their bundles;

the intersection of two planes is the line common to their fields;

the connection of a point with a line is the plane common to the bundle of the point and the sheafof the line;

the intersection of a plane with a line is the point common to the field of the plane and the point range of the line.

Looking again at the facts expressed in these propositions, we see that there are two cases in which a pair of basic elements shows no common elements: according to 11. 12 and 17. 18 we can have

first, two lines with no point in common and no plane in common; two such lines are called skew;
second, a plane and a point which do not belong to each other.

We mention here a well-established fact that shows that there is an essential difference between point and plane space on the one hand and line space on the other:

A point and a plane that do not belong to each other confront each other with nothing in common as opposite poles, the tension between them at first unresolved. Any additional element, however, be it point, line, or plane, has something in common with at least one member of the pair; a second point can connect with the first point, a second plane can intersect with the first plane, and a line can connect with the point and intersect with the plane.

Quite otherwise is a pair of skew lines: we can find a third line which is skew to both of them, a fourth line which is skew to each of the first three, and so on. In short, there are any number of pairwise skew lines, no pair of which thus has a common element.

Within the bounds of our normal, everyday perception there are exceptional cases where some of the propositions are not true. For example, within these bounds two planes can have no line in common: in this case we say they are parallel. And, in contradiction to Proposition 11. 12, there are lines which have a connecting plane but no point of intersection: such lines are called parallel. These anomalies are dealt with fully—and removed—in the next chapter.

First we state some facts which follow from the relationships already described.

19. If a line and a plane have two distinct points in common, then all the points of the line belong to the plane; the line is an element of the field carried by the plane.
20. If a line and a point have two distinct planes in common, then all the planes of the line belong to the point; the line is an element of the bundle carried by the point.

Proposition 19 is clear because, according to 14, a plane and a line not belonging to it have only one point in common; so if there are two common points the line must belong to the plane and, by 16, have all its points in common with the plane. Proposition 20 follows similarly.

21. Three points (three bundles) which do not belong to the same point range have a unique plane, their connecting plane, in common.

22. Three planes (three fields) which do not belong to the same plane sheaf have a unique point, their point of intersection, in common.

In fact, any two of the three points have just one line in common, which does not contain the third point. This point and the line determine a unique plane. Actually, by 19, this plane also contains the line common to the first and third points, as well as the line belonging to the second and third points. 22 follows similarly: two of the three planes have exactly one line in common, which does not belong to the third plane. But this plane and the line determine a unique point.

23. Given three or more lines, if every pair of them has a point in common but no three belong to the same bundle, then they are elements of a line field.

24. Given three or more lines, if every pair of them has a plane in common but no three belong to the same field, then they are elements of a line bundle.

Proposition 23 comes about as follows. We single out two of the lines in question; these two have a point in common and hence a plane in common as well (11. 12). By assumption, any third line does not contain the former point but has a point in common with each of the two lines. Thus, by 19, it must belong to the plane.

Proposition 24 follows similarly: any two lines we choose have a plane and hence also a point in common (11. 12). By assumption, any third line does not belong to the plane but has a plane in common with each of the two lines. Thus, by 20, it must belong to the point.

The 24 propositions describe the reciprocal behavior of the basic elements and basic forms. Natural and straightforward as the facts appear, they never theless form, as will be shown, the basis for the most remarkable formations of space. First of all, though, we must establish the fact that the 24 propositions are true without exception.

We can speak of ten elementary forms of space, namely line space, plane space and point space, the four second-degree basic forms and the three first-degree basic forms.

Point space is the totality of points, determined by all the non-skew pairs of lines of line space. Each point contains a line bundle of line space.

If we connect each pair of lines in a line bundle then we obtain a plane bundle. Each line contains a plane sheaf of the plane bundle.

Plane space is the totality of planes, which are determined by all the non-skew pairs of lines of line space. Each plane contains a line field of line space.

If we intersect each pair of lines in a line field then we obtain a point field. Each line contains a point range of the point field.

All the lines of a line bundle lying one particular plane form a line pencil.

If we intersect the planes of a plane sheaf with a plane not belonging to the line carrying the sheaf then we obtain a line pencil.

All the lines of a line field going in through one particular point form a line pencil.

If we connect the points of a point range with a point not belonging to the line carrying the range then we obtain a line pencil.

EXERCISES

Visualize Propositions 9 to 18 with the help of the diagram below. Adjacent positions on the circle denote elements (P = point, g = line, E = plane) that belong to each other; non-adjacent positions, elements that do not belong to each other.

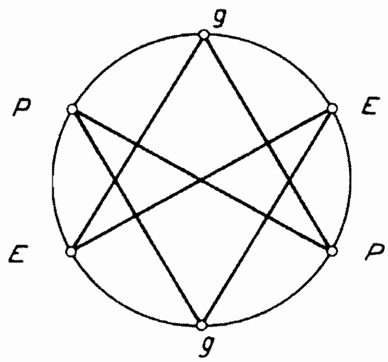


Figure 7

NOTATION

In this book, single capital letters are used for both points and planes, whereas small-case letters always represent lines.

If P, Q, R represent points, then PQ means the line common to points P and Q, and PQR is the plane common to all three points.

If P, Q, R are planes, then PQ represents the line common to planes P and Q, PQR is the point common to all three planes.

Where misunderstandings could occur we have used a superscript plus or minus sign according to whether a point or a plane is meant.

Thus $P^+Q^+R^+$ is a plane, $P^-Q^-R^-$ a point, whereas P^+Q^+ and P^-Q^- mean the line connecting P^+ and Q^+ and the line of intersection of P^- and Q^- respectively.

If the lines a, b are not skew, then ab means both the point of intersection and the connecting plane of the two lines. When it is desirable, we distinguish point from plane by writing $(ab)^+$ and $(ab)^-$.

In certain contexts it will be advantageous to use + and – signs also as subscripts to distinguish between two different lines, e.g. a_+ and a_-.

ℓ A means the point of intersection of line ℓ with plane A or the connecting plane of line ℓ with point A. When necessary we write more accurately ℓA^- and ℓA^+.

Chapter 3

Limit Elements in Space

To space as it is presented to ordinary mental picturing, we are going to add some special forms. In this extended space, the 24 propositions given so far will have unrestricted validity. The special forms are the so-called infinitely distant points and lines of space. We shall use the expression *limit elements.*

Two parallel lines do have a plane but do not have a point in common. Yet in our mental pictures, two parallel lines clearly exhibit something else common to them apart from the plane: the word "direction" suggests what it is. But this expression is not yet quite correct. One speaks after all of being able to move along a line in one of two opposite directions. In this situation we shall say "one of two opposite senses," and from now on the word direction will denote what these two opposite senses of movement have in common.

From our visualization of space it is evident to begin with that:

A point and a direction determine a unique line, which contains the point and has the given direction.

We recognize, furthermore, on the basis of the mental pictures we associate with parallel lines, the truth of the following important proposition:

> If a first line and a number of other lines of space exhibit the property that all the latter lines are parallel to the first, then any two of them are parallel.

Given any direction, through each point of space there is precisely one line with this direction, and any two such lines are parallel. We now define a new concept. Whatever it is that these parallel lines have in common that is suggested by the expression "same direction," we regard as a new element common to them and call it their common limit point. In consideration of our experience of tactile space, it is also called the point at infinity common to the lines.

Each line possesses one and only one such limit point. Instead of saying "a point and a direction determine precisely one line," we now say

A point and a limit point have precisely one line in common.

Not only is this conceptual construct useful, it is also entirely natural, as we shall see.

Two parallel planes have no line in common. Yet in our mental pictures these too exhibit something common to both of them; the word "attitude" suggests what this is. All horizontal planes, for example, have the same attitude in space. Our visualization of space shows us immediately that

> A point and an attitude determine precisely one plane, that contains the point and has the given attitude.

From this it follows that:

> If a first plane is parallel to a second and the second is parallel to a third, then the first and third planes are also parallel.

For if the first and third planes *did* have a common line, and hence common points, then such a point would contain two planes with the same attitude, namely that of the second plane.

Whatever it is that a collection of parallel planes have in common that is suggested by the expression "same attitude," we grasp in the concept of the common limit line of these planes. Each plane possesses one and only one limit line, namely the limit line that it has in common with all planes that are parallel to it. Like the limit point, it has an alternative expression, being also called the line at infinity of the plane.

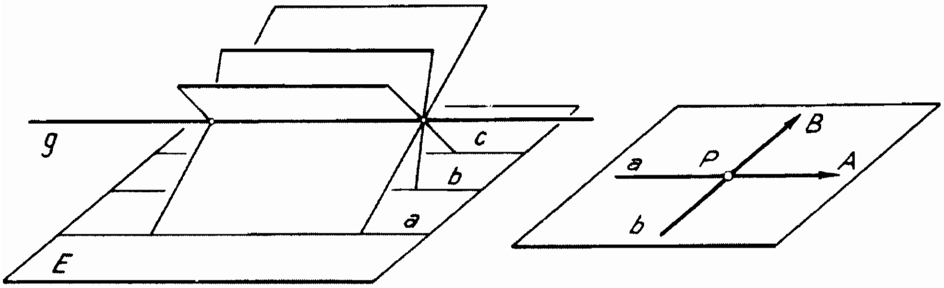
Instead of the proposition "a point and an attitude determine precisely one plane," we now say

> A point and a limit line have precisely one plane in common.

If a line belongs to a plane, the same is true of all the points of the line. For that reason it is appropriate also to consider the limit point of the line as an element it has in common with the limit line of the plane. The limit points of the lines of a field constitute the limit line of the field.

Finally, as far as visualizing space is concerned, it can happen (in contradiction to Proposition 14) that a line and plane have no point in common; they are then said to be parallel.

Suppose the line g is parallel to the plane E. Then there are infinitely many lines $a, b, c, \ldots$ of this plane that all run parallel to g (Figure 8). This can be seen by considering the sheaf of planes belonging to g. Generally speaking these planes have one (ordinary) line each in common with the given plane E. If a is one such line, then a and g could have no ordinary point in common, since such a point would also be common to g and E. The parallel lines $g, a, b, c, \ldots$ do however have a limit point in common, which belongs to the limit line of E. If a line and a plane are parallel, they thus have exactly one element in common, namely the limit point of the line; this lies on the limit line of the plane.



Figures 8 and 9

We now prove the following facts:

Two limit points determine exactly one limit line, which contains both limit points.

Two limit lines determine exactly one limit point, which belongs to both limit lines.

To help us to see the first proposition, we take any ordinary point *P* of space (Figure 9). Let *A* and *B* be the limit points in question. Now *P* and *A* have exactly one line *a* in common, and *P* and *B* exactly one line *b*. Since *a* and *b* have a point in common they also possess a plane in common. The plane's limit line contains *A* and *B*. It is not yet intuitively obvious that this is the only limit line containing *A* and *B*. If there were a second one, then the latter together with *P* would determine exactly one plane. This plane would also contain, together with *A*, *B* and *P*, the lines *a* and *b*. But the plane that *a* and *b* have in common is unique. This second plane would thus be the plane *ab* already considered. The supposed second limit line thus coincides with the one found in the first place, namely with the limit line of plane *ab*.

To see that two limit lines *u* and *v* have exactly one limit point in common, we again make use of an arbitrary ordinary point *P* of space (Figure 10). A unique plane is determined through *P* and *u*, likewise through *P* and *v*. These two planes have a line *p* in common. *p* is an ordinary line since it contains *P*. The limit point of *p* belongs to both of the respective limit lines of these planes, that is, to the limit lines *u* and *v*. It is the only limit point lying on both *u* and *v*. For if there were a second, then the latter and the former would have two different limit lines in common, which, as we just saw, is impossible.

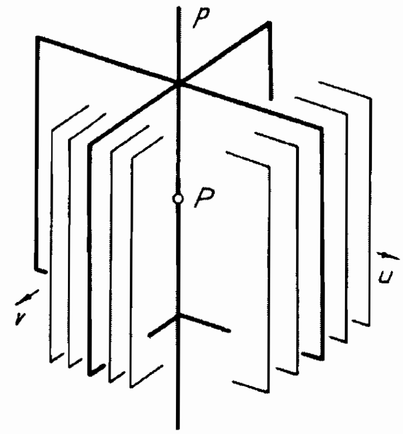


Figure 10

An interesting question arises at this point, namely: what is the nature of the form which carries all the limit lines and limit points of space? In other words, what is the limit form of space?

We have seen that any two limit lines always have exactly one limit point in common. On the other hand, by no means do all of the limit lines go through one and the same limit point. Now for ordinary lines, Proposition 23 says that given three or more such that if every pair of them have a point in common but no three go through the same point, then all of these lines belong to the same field. For this reason it is appropriate, and will prove true, to speak of the limit form of space as a field. This is the limit plane of space, also called the plane at infinity.

The limit plane of space is the particular field to which all limit lines and limit points belong.

At the beginning of this chapter we said that we were going to extend the space of ordinary mental picturing. This consists in adding to the ordinary planes the limit plane just defined, and adding to the ordinary points and lines the elements of the limit form, namely the limit lines and limit points contained in it.[3]

In space completed in this way—projective space as it is called—the 24 propositions given so far have unrestricted validity. Indeed we have demonstrated this in various examples, though not for every proposition in detail; all the same, what we have said shows that it appears to make sense to take the 24 propositions as a starting point for all further developments. As mentioned before, we shall see reason later to add some more facts which cannot be deduced from the propositions given so far.

From now on, whenever points, lines and planes are spoken of, limit elements are always included unless explicitly stated otherwise. In many instances it makes no difference whether or not limit elements are included among the elements being considered.

[3]The limit elements are also called ideal points, lines, and planes. (This refers to the non-Euclidean geometries and are not covered in this book. — Ed.)

NOTE 1. Let P be a point and ℓ be a line not containing it. We started from the assumption that in the plane $P\ell$ there is one single line through P parallel to ℓ. But we could be in doubt about whether there is actually only one such. This certainly cannot be deduced from any mental picture tied to sense perceptions. As a matter of fact if the possibility of more than one parallel is followed up, we arrive at a limit structure of space that can no longer be called a planar field. We can in fact make sense of and use this possibility, as will be shown later.

NOTE 2. The limit elements of space are not accessible to mental picturing; indeed in a certain sense they actually contradict it. Yet oddly enough, the same mental-picturing faculty is so constituted that through it we are led to the limit elements. Picture in a given plane a point P and a line ℓ not containing P. Consider in this plane the lines of the pencil carried by P. If we move through this pencil, we can see how the points of intersection of its lines with ℓ move ever further away until they disappear from mental view. To one line, namely the line parallel to ℓ, we then assign the limit point of ℓ as the point of intersection. The limit elements are thus purely ideal forms.

NOTE 3. As already mentioned, generally we can only form mental images of pieces of lines and planes. And to close observation, even in doing this we tinge or modify the forms with some sensory quality, for example, a color. All geometric forms are of a purely ideal nature. They must be clothed in color (for example) before they can appear in the world of mental images.

NOTE 4. The 24 propositions reviewed can be deduced from a smaller system of propositions. For example we inferred Propositions 19 to 24 from the first 18 propositions. It is a very real issue as to whether a system of propositions includes neither too few facts nor too many, in the sense that they can be deduced from fewer propositions. It is less important for an introduction to geometry, however, since dealing with it would require an extensive knowledge of relevant facts. What is important is to see exactly how far our 24 propositions go by themselves, and what other facts we might still have to include.

1. Form, as far as is possible, clear mental pictures of

   a) a line bundle belonging to a limit point (a collection of parallel lines);
   b) a line pencil belonging to a limit point (a collection of parallel lines in a plane);
   c) a plane sheaf belonging to a limit line (a collection of parallel planes);
   d) a limit line together with its points (a mental picture can only indirectly suggest this; imagine any plane through the limit line and a line pencil in this plane);
   e) A line pencil in the limit plane of space (a mental picture can only indirectly suggest this; imagine any ordinary line whose limit point carries the pencil in question, and consider all the planes of this line; these cut a line pencil out of the limit plane.)

2. Go through Propositions 1 to 24 selectively and interpret them clearly when some of the elements in question are limit elements.

3. Prove the proposition: If lines *a* and *b* are skew then there is exactly one plane through *a* and exactly one plane through *b* such that the two planes are parallel.

4. Make clear to yourself that the concept of direction cannot be applied to a limit line and that it is meaningless to speak of the attitude of the limit plane.

5. Determine the line that meets three given mutually skew lines *a*, *b*, and *c*, that is, the line that has a point in common with each of *a*, *b*, and *c*. (Choose any point *A* on *a* and form the plane *Ab* through *A* and *b*. This has just one point *C* in common with *c*. Since it lies in a plane with *b*, the line $p = AC$ determined by *A* and *C* also meets the line *b* in a point *B*.)

   Picture the same basic construction for the following special layout. *a* is the vertical line through the observer, *b* the limit line of the horizontal plane on which the observer is standing, and *c* is a line running at a slant in front of the observer.

   Let *A* move along *a*.

Chapter 4

The Polar Structure of Space

Going through Propositions 1 to 24, it is not long before we discover a peculiar fact. If, throughout, the concept "point" is replaced by the concept "plane," and the concept "plane" is replaced by the concept "point," but the concept "line" is unchanged, then Propositions 1, 3, 5, 7, . . . give rise to Propositions 2, 4, 6, 8, . . . and vice versa. At the same time, the names *point range* and *plane sheaf, plane bundle* and *point field, line bundle* and *line field* must be exchanged appropriately. And if the words "go through" and "lie in" occur (in place of the general expression "belong to each other"), these must be exchanged, likewise the operations connecting and intersecting. For example (Propositions 9 and 10):

Two points have in common a unique line (their connecting line) and all the planes belonging to this line.	Two planes have in common a unique line (their line of intersection) and all the points belonging to this line.

The propositions with two numbers, namely 7 and 8, 11 and 12, 17 and 18, are transformed into themselves by this interchanging of words; that is why they were counted twice.

All 24 facts hold good without exception in "projective" space—ordinary space augmented by adding the limit elements. This space thus shows a selfpolar (usually called dual) structure. Therefore, for each property of a spatial form traceable back to Propositions 1 to 24, there is a polar property belonging to a polar form. If the further propositions that are not reducible to the 24 also exhibit a polar structure, then we shall have a polarity in space that is universal. As we shall see, in elementary geometry as usually studied, on the whole only one side of the totality of space is considered.

The form polar to point space is plane space, whereas line space is self-polar. Of the seven basic forms, the line pencil is the only one that is self-polar.

Through the operations of connecting and intersecting we can get from point space to plane space and from plane space to point space: any three points that do not belong to the same line determine a plane (21); any three planes that do not belong to the same line determine a point (22). In this way we obtain all the planes of plane space from point space and all the points of point space from plane space.

EXERCISES

1. Find the proposition that is polar to the following: Three points not in line determine a unique plane, and any two of them a connecting line; the three connecting lines lie in the plane (triangle $\longrightarrow$ trihedron).
2. Do the same for the proposition: Three lines that go through a point but do not all lie in the same plane form a "three-edge"; any two edges determine a connecting plane; the three connecting planes go through the point (three-edge $\longrightarrow$ trilateral).
3. Give the construction polar to the construction in Exercise 5 (page 28). (Hint: *a, b, c.* Any plane *A* through *a*, point of intersection *Ab.* This has a plane *C* in common with *c.* The line *p* determined by *A* and *C.*)

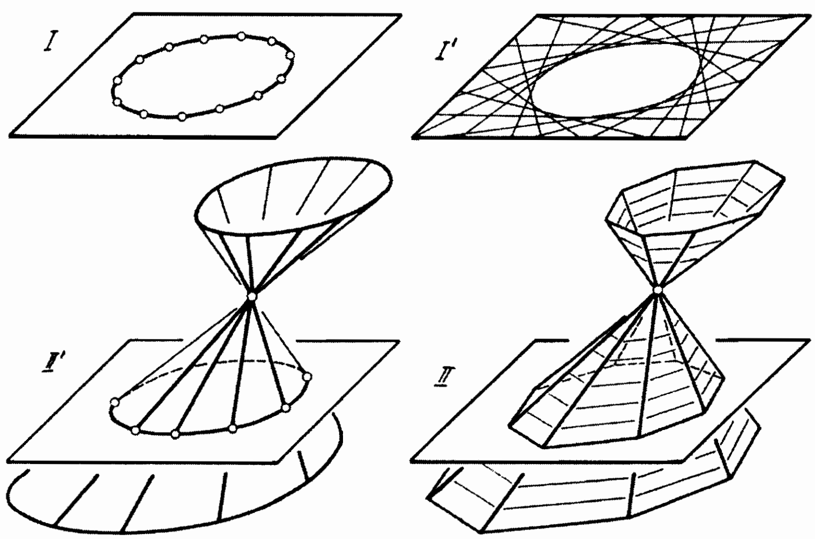
A plane can be chosen and "plane geometry" studied in its field. The elements of this geometry are the points and lines of the plane, its basic forms are point ranges and line pencils.

The phenomenon of polarity gives us, polar to the geometry of a field, the geometry of a bundle—a "point geometry." Its carrier is a point, its elements are the planes and lines of the point, and its basic forms are line pencils and plane sheaves.

For any given sequence of points of the plane there is a corresponding sequence of planes of the point [I (Figure 11), II (Figure 14)].

For any given sequence of lines of the plane there is a corresponding sequence of lines of the point [I′ (Figure 12), II′ (Figure 13)].

A point moving in the chosen plane gives rise to a *planar curve* (I). Correspondingly, a plane moving in the bundle in question gives rise to the *envelope of a cone* (II); that is, the planes envelop a conical form.



Figures 11 and 12 (above), 13 and 14 (below)

A line moving in the plane gives rise to a *planar envelope of a curve* (I′); that is, the lines envelop a curve. Correspondingly, a line moving in the bundle in question gives rise to a *conical surface* (II′), which is swept out by the moving line.

I and II are mutually polar, as are I′ and II′. If form I is connected with any point outside the plane then form II′ is obtained. Conversely, if a conical surface II′ is intersected with any plane which does not go through the carrier of the conical surface, then a plane curve I is obtained.

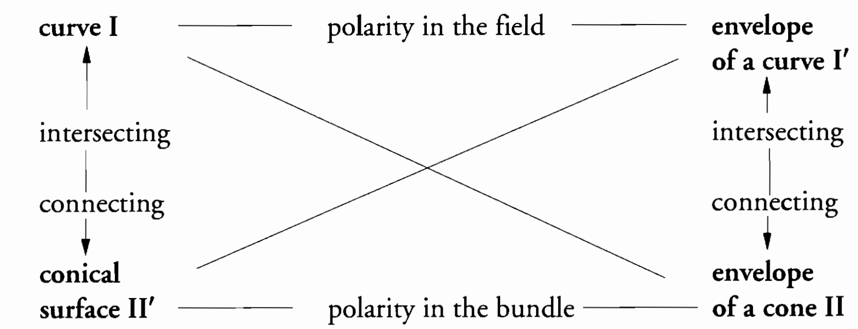
On the other hand if the envelope of a curve I′ is connected with any point outside the plane then the envelope of a cone II is obtained. And intersecting the envelope of a cone II with any plane not containing the carrier of this envelope gives rise to the envelope of a curve I′.

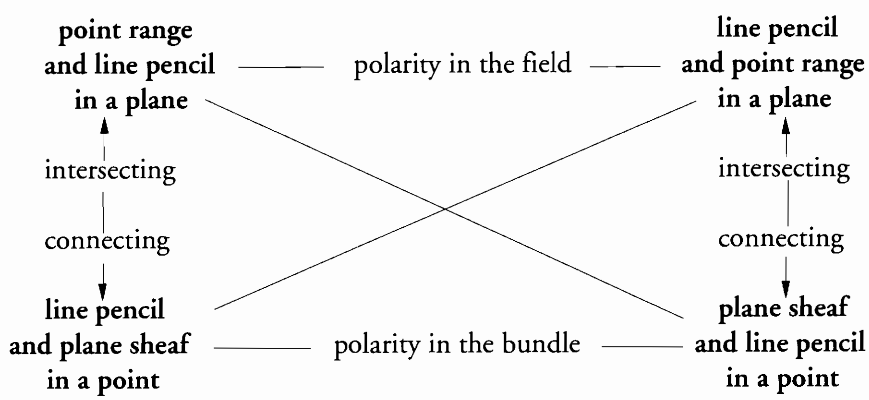
Thus the polarity of space, together with the operations of connecting and intersecting, provide geometry in the plane and geometry in the point each with a polar structure:

$$\mathrm{I} \longleftrightarrow \mathrm{I}' \text{ and } \mathrm{II} \longleftrightarrow \mathrm{II}'.$$

In the plane, the elements point and line are mutually polar and the basic forms are point range and line pencil. In the point, the elements plane and line are mutually polar and the basic forms are plane sheaf and line pencil.

To begin with this gives us little more than an empty scheme. The riches contained in it will be revealed in the following pages.





Note One. As a result of present-day educational methods, we are used to hearing about geometry in the plane, whereas geometry in a bundle seems strange. This is connected with deeper questions. Geometry in a field has a pictorial character, whereas bundle geometry does not. Since we should like to make sense of the latter, we take a plane section of the bundle, thereby changing over to the pictorial. The geometry of a bundle has an active character, which is precisely why it is harder to capture in a mental picture. In looking at a colored picture, the eye creates a bundle of light in complementary colors, as can be seen by the well-known method of looking away from a colored picture onto a moderately bright white surface. As a result, a cross-section of the bundle that is being actively created by the eye is seen on the surface.

Note Two. As far as our mental picturing is concerned, point space and plane space show an essential dissimilarity. In plane space we have a unique plane that is distinguishable from all the others: the limit plane of space. In point space we cannot immediately distinguish one point, though the limit points of the limit plane are indeed special. But if we realize that mathematical models of reality do not constitute but only regulate the reality, and that they thus provide a means of going deeply into it, then the question to ask is: Is there, within the realm we want to penetrate with our understanding, a unique point or plane that must be regarded as a limit element?

Note Three. The strangely belated discovery of the law of duality—what we call polarity—was made only during the years 1810 to 1820, mainly by the French mathematicians Gergonne and Poncelet. In the second half of the nineteenth century people began to study n-dimensional space (n greater than 3), if only as a formal generalization of the three-dimensional space given to us. The general, formal aspect of the law of duality was then discovered, according to which the basic forms of dimension one and $n-2$, two and $n-3$, and so on, correspond to each other. In linear algebra these properties are almost trivial. Perhaps this very fact is one of the reasons why the law that comprehends everything spatial has been treated hitherto as an all but insignificant phenomenon.

Certainly, as far as the given three-dimensional space of our mental pictures is concerned, it is still true that all practical applications to date tend to take into consideration, quite one-sidedly, only the one aspect of space. Yet there have been isolated thinkers who have had a profound sense of the mystery connected with polarity. The mathematician Chasles, for example, remarks:

> Other laws of duality based on other principles will be found in all the different branches of the mathematical sciences; and, as we have already mentioned in our note on the definition of geometry, one will be led to admit, so we believe, that a universal dualism is the grand law of nature, and reigns over all the branches of knowledge of the human spirit.

> Can one even foresee where the consequences of such a principle of duality would end? Having paired up all the phenomena of nature and the mathematical laws that govern them, would this principle really not go back to the very causes of these phenomena? And could we then say that to the law of gravitation there would not correspond another law that would play the same role as Newton's and serve like it to explain celestial phenomena? And if, on the contrary, this law of gravitation was its own correlative in the two doctrines, just as a proposition of geometry can be its own correlative in the duality of formed space, then this would be a clear proof that it is indeed the supreme and unique law of the universe. (M. Chasles: *Aperçu historique sur l'origine et le développement des méthodes en géométrie*, third edition, 1889, p. 408 f.)

Two figures that correspond with one another according to the law of duality are called in French "correlative." Chasles makes a beautiful remark about this:

> The word *correlative* being used in a general way in a thousand contexts, it would be highly desirable to have another adjective, one derived from the word duality. For this reason we had thought to substitute for *duality* the word *diphany* which would have expressed the double genre of properties shown by all forms in space; we should have spoken of the *principle of diphany*, and called *diphanic* those figures that would have had the mutual relationships prescribed by this principle. But we had no wish to allow ourself to substitute a new designation for the generally accepted one.

Today's physics is increasingly referred to by introducing so-called tensor quantities involving operations between mutually polar spaces. Admittedly this is still done in a purely formal, superficial way. For example, the following sentence can be found in a recent book:

> It is often convenient in physics to make correspond to real space a 'reciprocal space,' in which lengths are equal to the inverses of the real lengths. (E. Bauer: *Champs devecteurs et de tenseurs*, Masson et Cie., Paris, 1955, p. 36.)

NOTE FOUR. What is the relation between mathematics and reality? Is modern man a fisherman who casts his net made out of concepts and catches . . . nothing, the thing-in-itself remaining closed to him? If this were so, as agnosticism would have us believe, then the aim of the exact sciences would be only to apply "conceptual nets" to make the forces of nature serve man's claim to power.

On the one hand, mathematics certainly appears to be man's creation. (We shall disregard the primitive theory of the abstraction of mathematical con cepts from the phenomenal world.) But what is created by thinking proves to be something we can use; it is usable precisely to the extent that the corresponding structures are applicable in the phenomenal world. Anything going beyond thatn is nonexistent for the applicable net. From that the true state of affairs is evident. That which is grasped by thinking is also effective in the world, and whatever lives in the world manifests in thinking. Thinking as such lies beyond the realm of separation into object and subject. This separation is created by thinking itself. Mathematics and the applicability of mathematics testify to this fact. This of course does not apply to so-called formal thinking, which has little to do with true think ing and can be left to a machine, but to real thinking in which an ideal content is experienced.

Chapter 5

The Fundamental Structure

Having introduced the phenomenon of polarity, we now look at an important example; we shall also be interested in its development later.

Given three points in space, each pair of them determines a connecting line; the three connecting lines lie in a plane, which is common to the three points. Given nothing else, the figure offers no opportunity for further construction.

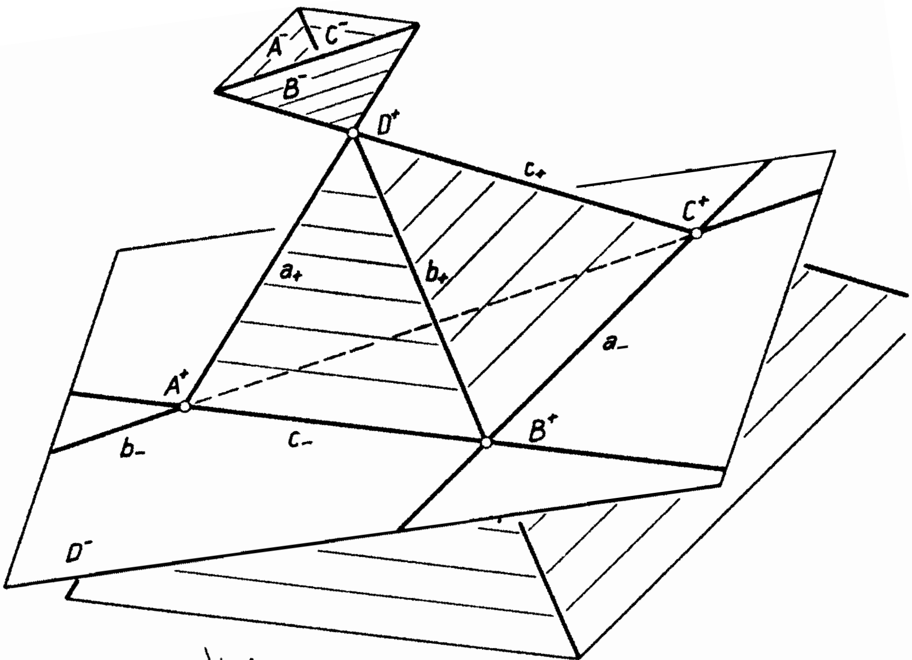
Suppose we have four points A^+, B^+, C^+, D^+ in space that are generally positioned, that is, four points of which no three belong to the same line, and which do not all belong to the same plane. Then we are at the same time given four planes, namely the planes

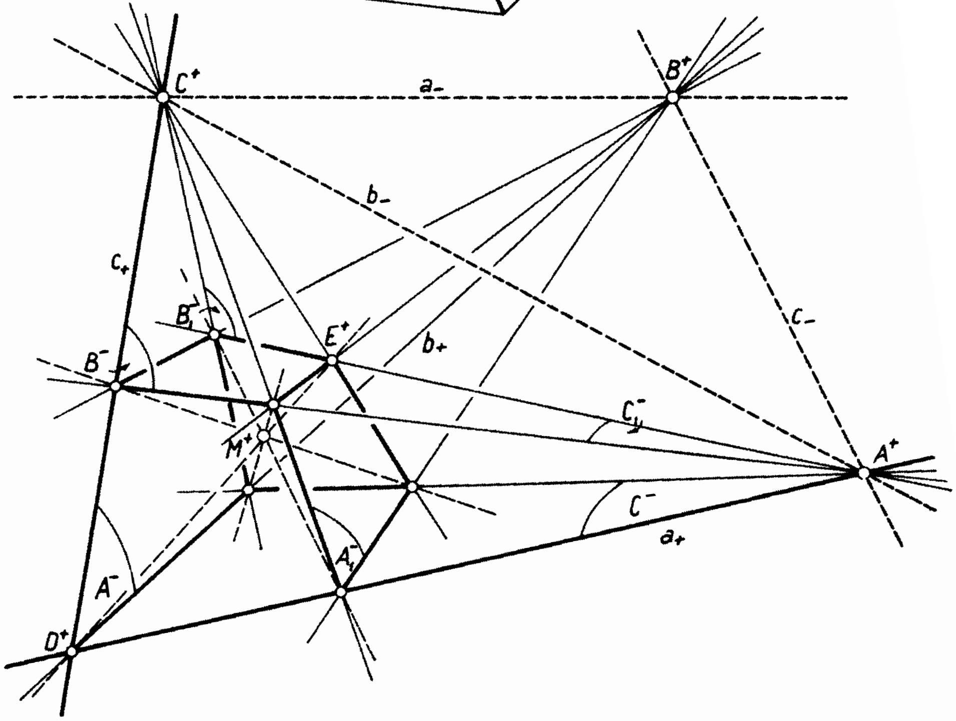
$$A^- = B^+C^+D^+, \quad B^- = C^+D^+A^+, \quad C^- = D^+A^+B^+, \quad D^- = A^+B^+C^+$$

each common to three points. Furthermore, each pair of points has a line in common; in all there are three pairs of skew lines (Figure 15), which we name as follows:

$$\begin{aligned} a_+ &= A^+D^+ = B^-C^-, & a_- &= A^-D^- = B^+C^+, \\ b_+ &= B^+D^+ = C^-A^-, & b_- &= B^-D^- = C^+A^+, \\ c_+ &= C^+D^+ = A^-B^-, & c_- &= C^-D^- = A^+B^+. \end{aligned}$$

This form is called a *tetrahedron.* (A regular tetrahedron is produced when the points have particular positions in relation to each other.) The tetrahedron with its four vertices, six edges and four faces we think of not as a bounded solid, but as a form whose edges are whole lines and whose faces are whole planes. It is self-polar. The face A^- corresponds to the opposite vertex A^+, the edge a_- corresponds to the edge a_+ which is skew to it; a_+ is the line connecting vertices A^+ and D^+, but also the line of intersection of planes B^- and C^-, whereas a_- represents both the line of intersection of planes A^- and D^- and the line connecting vertices B^+ and C^+.





Figures 15 and 16

Had we started out with four generally positioned planes A^-, B^-, C^-, D^-, that is, four planes with no three belonging to the same line and with no one point in common, then the same form would have been produced. Any pair of the four planes determines a line, any three a point.

The tetrahedron as a whole is a harmonious form, complete and perfect. It offers no opportunity for further construction.

If we add a fifth point, E^+, that does not lie in any of the four planes, then the harmony is disturbed. Various constructions immediately suggest themselves. To make this easier to picture, we imagine the fifth point E^+ in the "interior" of the tetrahedron (Figure 16; we shall clarify the concept "interior" in detail later on).

We choose one of the four faces of the tetrahedron, say $D^- = A^+B^+C^+$, and connect E^+ with the vertices A^+, B^+, C^+ of this face. We now have two tetrahedra, $A^+B^+C^+D^+$ and $A^+B^+C^+E^+$, that have in common three vertices and the face D^- that these define. The other faces of the two tetrahedra group themselves into three pairs:

$$
\begin{array}{lcl}
A^- = a\_D^+ & \text{and} & A_1^- = a\_E^+, \\
B^- = b\_D^+ & \text{and} & B_1^- = b\_E^+, \\
C^- = c\_D^+ & \text{and} & C_1^- = c\_E^+.
\end{array}
$$

These three pairs of planes, A^-, A_1^-, B^-, B_1^-, and C^-, C_1^-, each of which belongs to one of the edges $a\_$, $b\_$, $c\_$, determine a hexahedron. Because of the general positioning of the five points we started with, it looks like a deformed cube.

The six planes mentioned group themselves into fours, each set of four having a point in common, namely A^+, B^+, and C^+, respectively.

Apart from the lines $a\_$, $b\_$, $c\_$, the six planes have three times four other lines of intersection, the edges of the hexahedron, of which four go through A^+, four through B^+, and four through C^+.

Apart from the points A^+, B^+, C^+, the six planes determine eight other points of intersection, the vertices of the hexahedron.

The form consisting of the three pairs of planes, the eight vertices and twelve edges we call a *simple hexahedron.*

We also consider the four diagonal lines, which are the connecting lines of opposite vertices of the hexahedron. D^+E^+ is one such diagonal line. Each pair of these diagonal lines belongs to a diagonal plane, which is the plane connecting two opposite edges. Two such diagonal planes go through each of the three points A^+, B^+, C^+, for example the planes through A^+ formed by pairs of edges through A^+ that do not belong to the same face of the hexahedron.

Thus we see that of the four cross lines, any pair has a plane in common, but they do not all four belong to the same plane. Hence, by Proposition 24, they belong to the same bundle. Let the bundle's carrier be called M^+.

The *complete hexahedron* consists of 12 points, 12 + 1 planes and 16 + 3 lines, that is:

8 vertices,	A^+, B^+, C^+,	and	the "middle point" M^+;
6 faces,	6 diagonal planes,	and	plane $D^- = A^+B^+C^+$;
12 edges,	4 diagonal lines,	and	a_-, b_-, c_-.

When D^- is the limit plane, then the opposite faces of the hexahedron will be parallel; the hexahedron becomes a parallelepiped. If in addition A^+, B^+, C^+ take up certain positions in relation to each other, a *cuboid* is formed. In a special case a cube is produced.

We started off with a tetrahedron and disturbed its harmony by adding a fifth point. To obtain the form polar to the complete hexahedron we now carry out the polar disturbance by adding to the tetrahedron $A^-B^-C^-D^-$, not a fifth point E^+, but a fifth plane E^-. Before, we considered the two tetrahedra $A^+B^+C^+D^+$ and $A^+B^+C^+E^+$. Now, polar to this, we look at the two tetrahedra $A^-B^-C^-D^-$ and $A^-B^-C^-E^-$, that have three faces and the vertex D^+ determined by these faces in common (Figure 17). The other vertices of the two tetrahedra group themselves into three pairs:

$$A^+ = a_+D^- \quad \text{and} \quad A_1^+ = a_+E^-,$$
$$B^+ = b_+D^- \quad \text{and} \quad B_1^+ = b_+E^-,$$
$$C^+ = c_+D^- \quad \text{and} \quad C_1^+ = c_+E^-.$$

The three pairs of points, A^+, A_1^+, B^+, B_1^+, and C^+, C_1^+, each of which belongs to one of the edges a_+, b_+, c_+, determine an *octahedron.* Only for certain particular positionings of the planes is it regular.

The six points mentioned group themselves into fours, each set of four having a common plane, namely A^-, B^-, and C^-, respectively.

Apart from the lines a_+, b_+, c_+, the six points have three times four other connecting lines, the edges of the octahedron, of which four lie in A^-, four in B^- and four in C^-.

And apart from the planes A^-, B^-, C^-, the six points determine eight connecting planes, the faces of the octahedron.

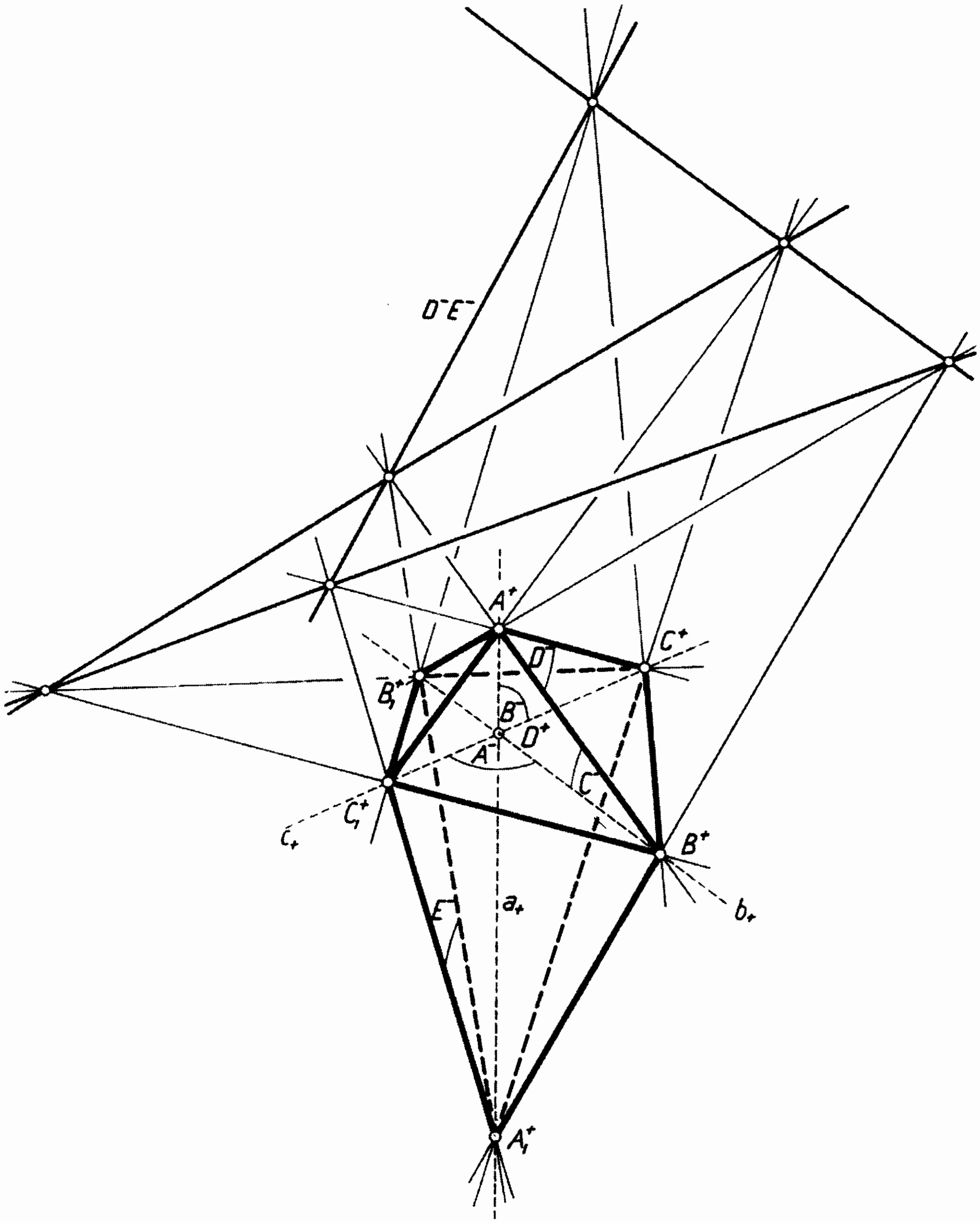


Figure 17

The form consisting of the three pairs of points, the eight faces and twelve edges we call a simple *octahedron.*

We consider also the four *diagonal lines*, the lines of intersection of opposite faces of the octahedron. D^-E^- is one such. Any two of these cross lines belong to a cross point, which is the point of intersection of two opposite edges. Two such cross points lie in each of the three planes A^-, B^-, C^-; for example, the points in A^-

determined by pairs of edges in A^- that do not belong to the same vertex of the octahedron.

Thus we know that any pair of the four cross lines have a common point without all four belonging to the same point. Hence, by Proposition 23, they belong to the same field. Let the field's carrier be called M^-. (In Figure 17, the cross lines appear as the four continuous lines drawn at the top; the plane M they determine, which is not labelled in the figure, can be pictured as a horizontal plane.)

The *complete octahedron* consists of 12 planes, 12 + 1 points and 16 + 3 lines, that is:

8 faces,	A^-, B^-, C^-,	and	the "middle plane" M^-;
6 vertices,	6 diagonal points,	and	point $D^+ = A^-B^-C^-$;
12 edges,	4 diagonal lines,	and	a_+, b_+, c_+.

The two forms, the complete hexahedron and the complete octahedron, are fundamental for getting one's bearings in space, the first for point space and the second for plane space. They stand in polar opposition as follows:

Hexahedron	*Octahedron*
Six faces, eight vertices. Three sets of four edges through A^+, B^+, and C^+ respectively. Four cross lines as connecting lines of opposite vertices. The middle point M^+, that is, the bundle carrying the diagonal lines.	Six vertices, eight faces. Three sets of four edges in A^-, B^-, and C^- respectively. Four cross lines as lines of in tersection of opposite faces. The middle plane M^-, that is, the field carrying the diagonal lines.
The field D^- of the points A^+, B^+, C^+ and their connecting lines a_-, b_-, c_-.	The bundle D+ of the planes A^-, B^-, C^- and their lines of intersection a_+, b_+, c_+.

Calling M^+ the middle point of the hexahedron seems natural enough; moreover, we think of that portion of space enclosed by the six faces and containing M^+ as the hexahedron's interior.

We shall now see why, in the case of the octahedron, it is correct to speak of the plane M^- as its middle plane. To the interior of the hexahedron with respect to M^+ there corresponds a set of planes of space determined by the six vertices of the octahedron, that is, the planes that do not go through the octahedron's pointwise interior (with respect to D^+). In plane space this pointwise interior thus appears as an empty form. In plane space, all the planes that flow round this empty form therefore constitute the interior of the octahedron with respect to M^-.

In the complete hexahedron, if we disregard the plane D^- and the lines a_-, c_- but retain the points A^+, B^+, C^+, then we are left with a form consisting of twelve points, twelve planes and sixteen lines, namely

the eight vertices, the points A^+, B^+, C^+ and the middle point M^+;
the six faces and six diagonal planes;
the twelve edges and four diagonal lines.

In the complete octahedron, if we disregard the point D^+ and the lines a_+, b_+, c_+ but retain the planes A^-, B^-, C^-, then again we are left with a form with twelve planes, twelve points, and sixteen lines, namely

the eight faces, the planes A^-, B^-, C^-, and the middle plane M^-;
the six vertices and six diagonal points;
the twelve edges and four diagonal lines.

Both these forms turn out to be arranged exactly alike. In fact, through each of the twelve points go four lines and six planes of the form, in each of the twelve planes lie four lines and six points of the form, each of the sixteen lines contains three points and three planes of the form.

The form is therefore self-polar; it is known as the *Reye configuration* but might more appropriately be called the fundamental structure of space.

The fundamental structure of space is the self-polar form common to the complete hexahedron and the complete octahedron.

In Figure 16, for example, it is not hard to recognize the octahedron with M^+ and C^+ as opposite vertices and thence to see the whole form constructed as in Figure 17. It is clear, too, how the hexahedron with middle point A^+ is formed from Figure 17.

If we disregard everything but the incidence (that is, the mutual belonging) of the elements, then every point in the fundamental structure has the same rights as every other point, every plane the same rights as every other plane; likewise all the lines have equal rights. For example, you can regard each of the twelve points as the middle point of a simple hexahedron, each plane as the middle plane of a simple octahedron:

The fundamental structure contains twelve simple hexahedra and twelve simple octahedra.

That a cube, apparently so commonplace, has eleven companions is seldom acknowledged. The fact that the twelve octahedra cover the whole of point space exactly once and that the twelve hexahedra structure the whole of plane space into twelve regions is especially remarkable.

EXERCISES

1. Picture in your mind a cube and form an idea of the corresponding complete cube as well as the corresponding fundamental structure.

2. Picture in your mind a regular octahedron and form an idea of the corresponding complete octahedron as well as the corresponding fundamental structure.

3. Form, with the help of Figures 18 and 19, a mental picture of the twelve simple hexahedra given with the fundamental structure. Figure 18 shows the hexahedron with middle point C^+; its cross lines are the edges going through C^+ of the simple hexahedron emphasized in Figure 16. The interior with middle point C^+ extends over the limit plane of space, whilst M^+ denotes, for this simple hexahedron, an exterior point. Find the corresponding hexahedra with A^+ and B^+ as middle points. To see the eight hexahedra whose middle points are vertices of the hexahedron emphasized in Figure 16, look at Figure 19, which shows the hexahedron with middle point E^+.

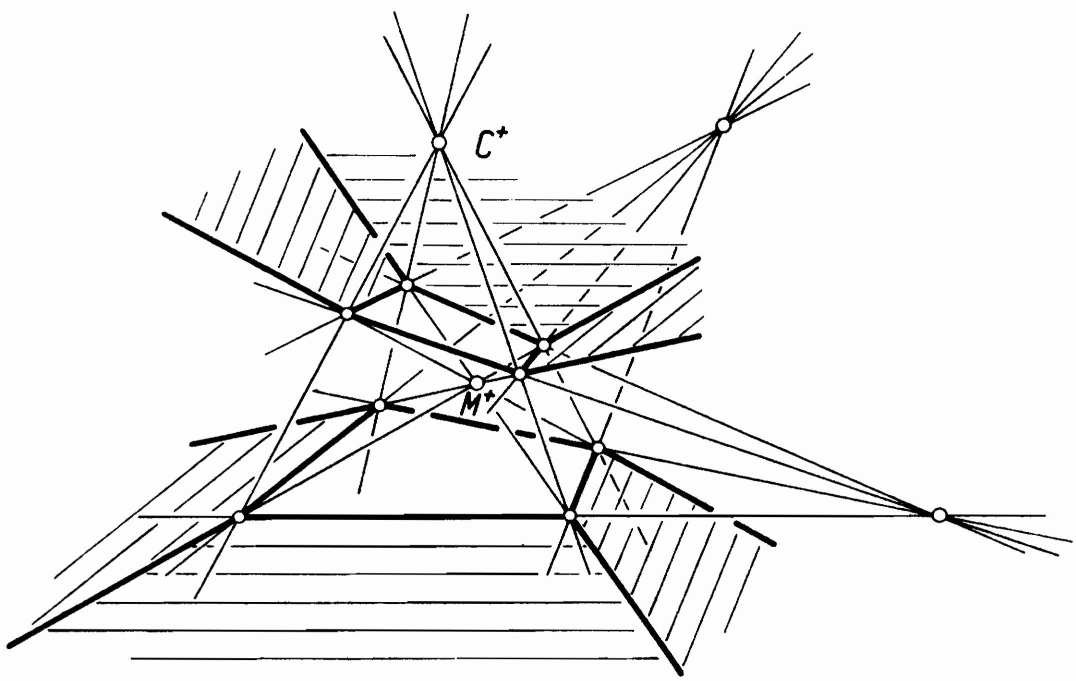


Figure 18

4. With the help of Figures 16 and 17, form a mental picture of the twelve simple octahedra given with the fundamental structure. Choose one of the twelve planes and mark the six points lying in it. If F is one of the six points, consider the two lines going through F that do not belong to the chosen plane. Each of these two lines contains two other points of the configuration apart from F. Now emphasize —say, in red—the segment between these points that does not contain F. Repeat for the other five points in the chosen plane. The octahedron with the chosen plane as middle plane will now become visible. The picture is easy to form if a face of the emphasized hexahedron in Figure 16 is chosen as middle plane. For example, the octahedron with middle plane BCD and with A^+ and M^+ as opposite vertices is immediately visible. With other middle planes, picturing the octahedron is more difficult. Figures 20 and 21 are useful for making the twelve octahedra intelligible.

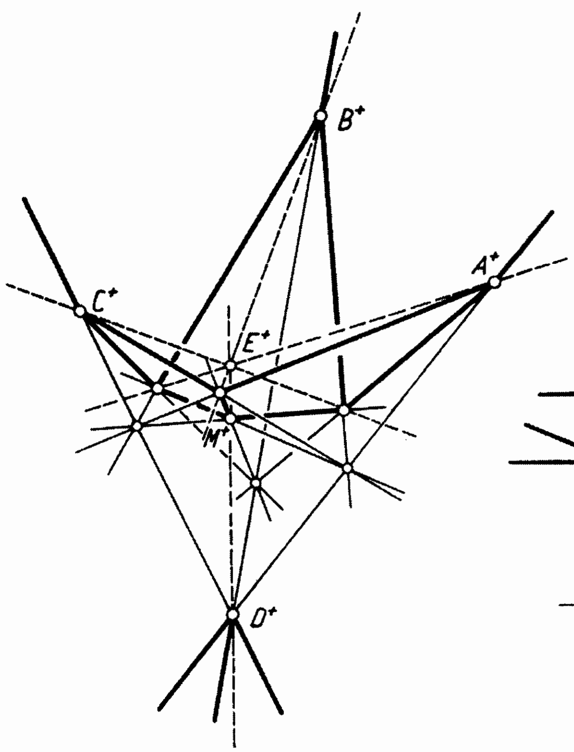
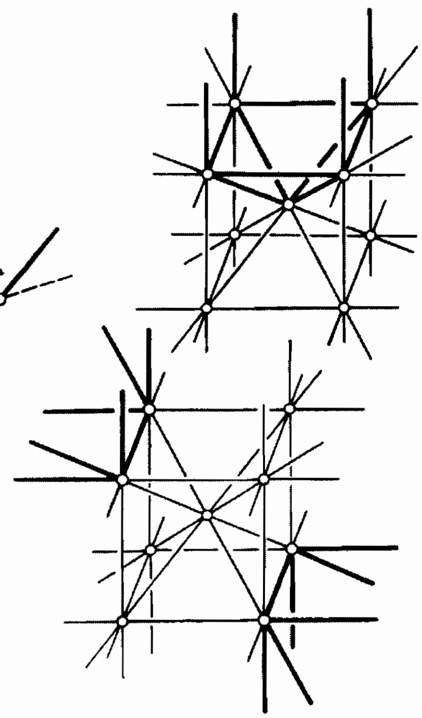


Figure 19



Figures 20 (above) and 21 (below)

Chapter 6

The Archetypal Phenomena of Ordering

The mutual relationships shown by the points of a range or the planes of a sheaf or the lines of a pencil mean we can speak of being able to "run through" these elements in their carrier in a particular sense. We also use the concepts "natural order," "lying between," "separate each other" to describe these relationships. The properties connected with the ordering of the elements in a first-degree basic form are given so directly with our inner pictures of these forms that imbuing them with concepts does not, at first, seem very interesting. For a real understanding, however, the complete "infiltration" of the immediate visualization with the right concepts is necessary.

If f, q, v, x are four lines of a line pencil then the natural order that they have in the pencil can be expressed by the corresponding sequence of the symbols that denote them. In Figure 22, for example, it is the sequence *fqvx* for running through the lines in one sense, and *xvqf* for running through them in the opposite sense. Continuing in one or the other sense we arrive back again at the same element from which we started. Thus, by running through the elements, the cycles

$$\ldots fqvx\, fqvx\, fqvx \ldots$$
$$\text{and} \ldots xvqf\, xvqf\, xvqf \ldots$$

are produced, sequences that can be continued indefinitely both ways and are opposite one another or, as we also say, of which one is the inverse of the other. Declaring the natural order of the four lines of the pencil is just a matter of giving their positions within a period of the cycle. Besides *fqvx* we could just as well choose *qvxf*, *vxfq*, or *xfqv* as the period; for the opposite sense, as well as *xvqf* we could also have *vqfx*, *qfxv*, or *fxvq*.

We use the expressions $(fqvx)$ and $(xvqf)$ respectively to represent the cycles as wholes.

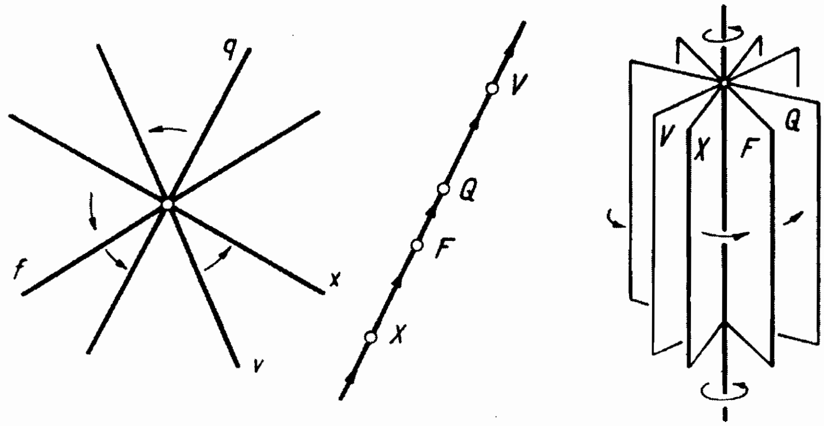
Thus $(fqvx)$, $(qvxf)$, etc. denote the same cycle.

If F, Q, V, X are four points of a range then they appear to be embedded in exactly two mutually opposite cycles. For Figure 23 it is the sequences

$$\ldots FQVX\, FQVX\, FQVX \ldots = (FQVX)$$
$$\text{and} \ldots XVQF\, XVQF\, XVQF \ldots = (XVQF).$$

In Figure 23, starting from F and going via Q to V and beyond, if we continue in the same sense across the limit point of the line we arrive back at X again. Four planes of a sheaf are also contained in exactly two mutually opposite cycles (Figure 24).

With regard to the elements they carry, the first-degree basic forms, namely the point range, the line pencil, and the plane sheaf, are complete forms. In the case of the point range, this is true only through the addition of its limit point.



Figures 22, 23 and 24

Given only three elements of such a basic form, for example F, Q, V, one can write them down in any order and expand the sequence periodically: the natural order in one sense or the other always results. Indeed in the cycles

$$(FQV) = \ldots FQV\,FQV\,FQV \ldots \text{ and } (VQF) = \ldots VQF\;\; VQF\;\; VQF \ldots$$

are contained all six possible permutations

$$FQV,\ QVF,\ VFQ \text{ and } VQF,\ QFV,\ FVQ.$$

With four elements this is not the case. For example $(FQVX)$ and $(QFVX)$ are not opposite, yet they are distinct:

$$(FQVX) = \ldots FQVX\,FQVX\,FQVX \ldots = (QVXF) = (VXFQ) = (XFQV),$$

whereas

$$(QFVX) = \ldots QFVX\,QFVX\,QFVX \ldots = (FVXQ) = (VXQF) = (XQFV).$$

With two elements F, Q, the sense of running through the elements cannot be understood from the cycle $\ldots FQFQFQ \ldots$ of the two symbols. Because, for example, one can get from F to Q in one sense (in Figure 23 without meeting V

and X) or the other (in Figure 23 via X, the limit point and V). The statements FQ and QF give no information about which of the two senses of running through the elements is to be understood.

A cycle of three elements A, B, C fixes the running-through sense in the form in question: (ABC) indicates one running-through sense, (BAC) the opposite running-through sense.

A cycle of more than three elements indicates a running-through sense only if the sequence represents the natural ordering of the elements.

Having made these preparatory comments we summarize the facts about ordering in some propositions. The concept of cycle is used as a means of doing this. (To avoid repetition, we allow upper case letters to denote lines as well as points and planes here.)

The Archetypal Phenomena of Ordering

Suppose $A, B, C, \ldots, K$ are at least three elements of a point range or a plane sheaf or a line pencil, and that all the elements are distinct. It then follows that

a) If the cycle

$$\ldots ABC \ldots KABC \ldots KABC \ldots K \ldots = (ABC \ldots K)$$

indicates the natural ordering of the elements $A, B, C, \ldots, K$, then the only other cycle representing the natural ordering of these elements is the opposite cycle $(K \ldots CBA)$.

b) Suppose X is an element of the basic form in question distinct from $A, B, C, \ldots, K$. If the cycle $(ABC \ldots K)$ indicates the natural ordering of the elements $A, B, C, \ldots, K$, then exactly one of the cycles $(AXBC \ldots K)$, $(ABXC \ldots K)$, …, $(ABC \ldots XK)$, $(ABC \ldots KX)$, and its inverse, represents the natural ordering of the elements $A, B, C, \ldots, K, X$.

c) Suppose the elements $A, B, C, \ldots, K$ are connected or intersected with some element, and the connecting elements or elements of intersection are $A', B', C', \ldots, K'$, respectively. If the cycle $(ABC \ldots K)$ represents the natural ordering of the original elements, then the cycle $(A'B'C' \ldots K')$ indicates the natural ordering of the elements $A', B', C', \ldots, K'$.

d) If A, B, C are three distinct elements of a first-degree basic form, then this form also contains elements X for which $(AXBC)$ represents the elements' natural ordering.

Note One a) captures a way of characterizing natural ordering. b) asserts that, with respect to $A, B, C, \ldots, K$ in the natural ordering, an element X occupies a unique fixed place in the period in question. Statement c) expresses an intuitively obvious fact. Suppose, for example, that the points $A, B, C, \ldots, K$ of a range are connected with a point S outside it and $a, b, c, \ldots, k$ are the connecting lines. Then, if $(ABC \ldots K)$ gives the points' natural ordering, $(abc \ldots k)$ indicates the natural ordering of the lines of the pencil S. Statement c) can be expressed in the following concise form:

Natural ordering is carried over from one form to another by the operations of connecting and intersecting.[4]

d) means that the elements are dense, something that is explained in detail in Chapter 10. Incidentally, d) turns out on closer inspection to be a consequence of a) b) c) and earlier propositions.

NOTE TWO. Propositions a) b) c) d) are self-polar. The whole of geometry evolves out of the 24 propositions given earlier and a) b) c) d), together with just one more proposition, Proposition e), which we explain later. All other concepts connected with ordering now reduce to the concept of cycle. This happens in the following way.

Let A,B,C be any three elements of a first-degree basic form. The cycle (ABC) represents one sense of running through the form in question, (BAC) the other. Let $P, Q, R, S, T, \ldots$ be arbitrary elements of the same form. We now imagine the two opposite cycles indicating the natural ordering of the elements $A, B, C, P, Q, R, S, T, \ldots$ Of these cycles, if we take just those that contain A, B, C in the order

$$\ldots A \ldots B \ldots C \ldots$$

then, by b), the elements P, Q, R occur in all of these cycles either in the order $\ldots P \ldots Q \ldots R \ldots$ or in the order $\ldots Q \ldots P \ldots R \ldots$ In the first case, (PQR) represents the same running-through sense as (ABC); in the second case, (PQR) and (ABC) indicate opposite running-through senses. In this way, the concept of "running-through sense" is based on the concept of cycle. Suppose that $(ABCD)$ gives the natural ordering of the elements A, B, C, D. Visualizing these elements then leads us to say that the two elements A and C are separated by the two elements B and D. We denote this by AC/BD.

If $(ABCD)$ indicates the natural ordering, then it follows from the concept of cycle and from Proposition a) that the only cycles that give it are

$$(ABCD) = (BCDA) = (CDAB) = (DABC)$$
$$\text{and } (DCBA) = (CBAD) = (BADC) = (ADCB).$$

[4] Or: Natural ordering is preserved by the operations of connecting and intersecting.

Thus if A and C are separated by B and D, that is, if AC/BD is true, then

$$BD/CA,\ CA/DB,\ DB/AC,\ DB/CA,\ CA/BD,\ BD/AC,\ AC/DB$$

are also true.

Thus, four distinct elements A, B, C, D of a first degree basic form *organize themselves into exactly two pairs of elements that separate each other.* In this way the concept "pair of mutually separating elements" is reduced to the concept of cycle.

Furthermore, if A, C and B, D are mutually separating pairs of elements, then (ABC) and (ADC) represent the two opposite senses of running through the elements of the form. And, conversely, if (ABC) and (ADC) indicate opposite running-through senses, then AC/BD holds. From this we can see how the concepts "mutually separating" and "running-through sense" are connected with each other.

This leaves the concept "between." If $(ABCD)$ gives the natural ordering, then visualizing the elements A, B, C, D leads us to say:

B lies between A and C with respect to D.

This statement has the same meaning as the expression AC/BD. In projective space the statement "B lies between A and C" is ambiguous. For example if A, B, C are points of a range, then we can get from A to C either in such a way that we do not meet the limit point, or by going via the limit point. Thus the statement "B lies between A and C" as used in school geometry is understood as: B lies between A and C with respect to the limit point of the line.

In essence all these facts are more or less self-evident. And yet astonishing insights into the formation of space result from them. In the above, our intention has been to indicate how the concepts of separating, between, running through, and natural ordering are connected and how they can be introduced with the help of the concept of cycle.

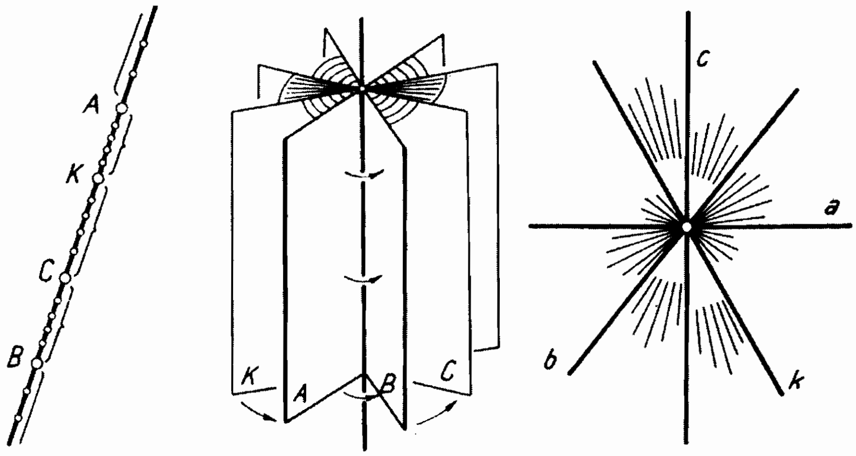
REMARK. Alternatively, the phenomena of ordering could all be reduced to the concept of the mutual separation of two pairs of elements. We chose the cycle as the basic concept because it appeals more strongly to direct experience.

PARTITION THEOREM. The following facts are now immediately evident. Let $A, B, C, D, \ldots, K$ be n distinct elements (n greater than 1) of a first-degree basic form, and suppose $(ABC \ldots K)$ indicates their natural ordering. Then, according to whether they are points, planes, or lines (Figures 25, 26, 27), these n elements divide

the line as point range into exactly *n segments*;
the line as plane sheaf into exactly *n angle spaces*;
the point as line pencil into exactly *n angle fields*.

This is a consequence of Proposition b). Each such segment or angle space or angle field contains all elements X of the form in question whose natural ordering is given by just one of the cycles (*AXBC . . . K*), (*ABXC . . . K*), (*ABCX . . . K*), . . . or (*ABC . . . KX*), and also two boundary elements.

Remark. Notice that an angle field contains entire lines, not, as with the usual angle of school geometry, just half-lines. The same is true for an angle space; it contains entire planes, not just half-planes.

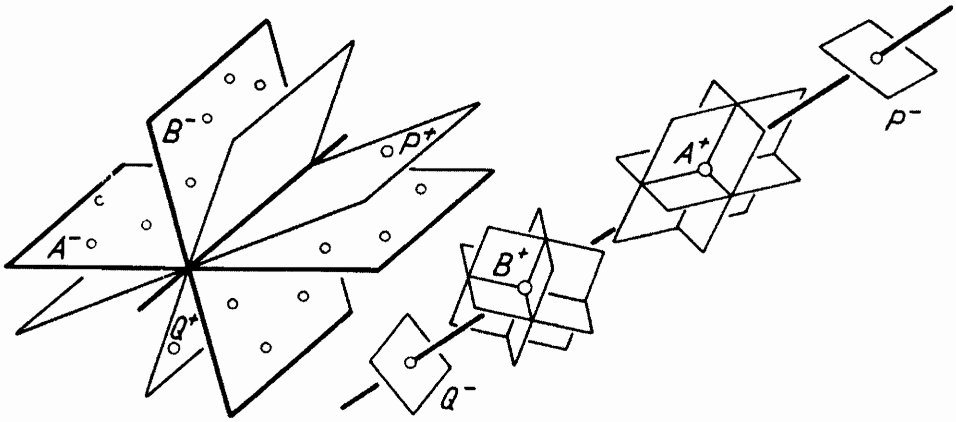


Figures 25, 26 and 27

Now it is not only the first-degree basic form itself that is divided by (for example) two of its elements into two parts; the surrounding space also undergoes a partitioning. We describe what happens in the six possible cases. It is important to build up an exact picture of these remaining absolute fundamentals.

1. Two planes *A, B* in point space (Figure 28). The two planes produce the following partitioning of point space. First we single out all points in *A*, all points in *B*, and in particular all points common to both planes, that is, the point field (*A*), the point field (*B*) and the point range *AB*. All remaining points of space are divided into two domains of points: firstly, the domain of all point fields whose carrying planes belong to the interior of one angle space determined by *A* and *B*; secondly, the domain of all point fields whose carrying planes belong to the interior of the other angle space determined by *A* and *B*. Points in the same domain lie in interior planes of the same angle space. The two fields (*A*) and (*B*) form the boundary of these domains of points; the point range *AB* in particular is part of the boundary. A point of one domain cannot move to a point of the other domain without coming to lie in *A* or *B* at least once, that is, without it coinciding at least once with a boundary element.

Two points P^+, Q^+ belong to the same domain if and only if they are not separated by the planes A^-, B^-, that is, if and only if the planes connecting P^+ and Q^+ with the line of intersection *AB* are not separated by A^-, B^-.



Figures 28 and 29

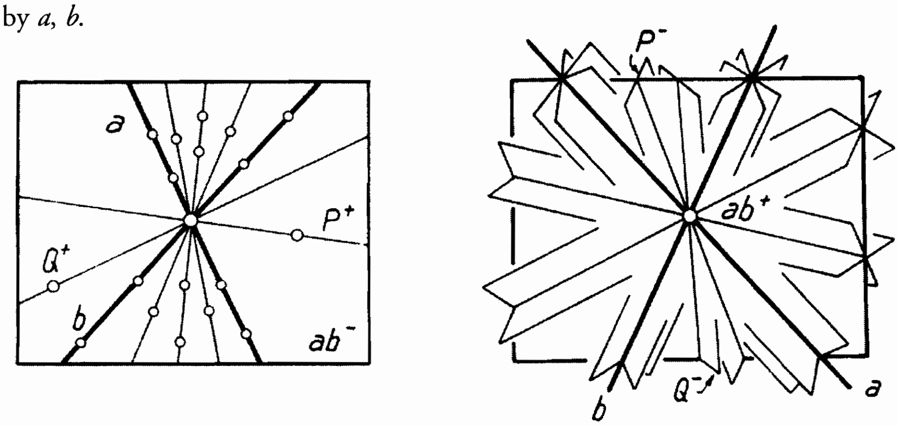
2. Two points A, B in plane space (Figure 29). The two points produce the following structuring of plane space. First we "highlight" all the planes through A, all the planes through B, and in particular all the planes common to both points, in other words plane bundle (A), plane bundle (B) and plane sheaf AB. All the other planes of space are structured into two regions of planes: firstly the region of all plane bundles carried by points belonging to the interior of one of the segments determined by A and B; secondly, the region of all plane bundles carried by points belonging to the interior of the other segment determined by A and B. Planes in the same region meet interior points of the same segment. The two bundles (A) and (B) form the boundary of these regions of planes; the sheaf AB in particular is part of the boundary. A plane of one region cannot be moved into a plane of the other region without meeting A or B at least once, that is, without it coinciding at least once with a boundary plane.

Two planes P^-, Q^- belong to the same region if and only if they are not separated by the points A^+, B^+, that is, if and only if the points of intersection of P^- and Q^- with the connecting line AB are not separated by A^+, B^+.

We now explain the division of the second-degree basic forms by two of their elements. There are four cases, I, II, I′, II′, all of which arise from one of them in accordance with the law of polarity.

I. Two lines a, b structure a plane in which they both lie into two planar domains of points (Figure 30). Such a domain consists of the points of all the ranges in this plane whose carrying lines belong to the interior of one of the two angle fields determined by a and b. Alternatively, one such domain consists of all those points whose connecting lines with ab^+ belong to the interior of the same angle field. The boundary is formed by point ranges (a) and (b); ab^+ in particular is a boundary point. Within the field it is not possible for a point of one domain to be moved to a point of the other domain without it coinciding at least once with a boundary point.

Two points P, Q in the plane in question (that is, ab^-) belong to the same domain if and only if they are not separated by the lines a, b, that is, if and only if the lines connecting P and Q with the point of intersection ab^+ are not separated by a, b.



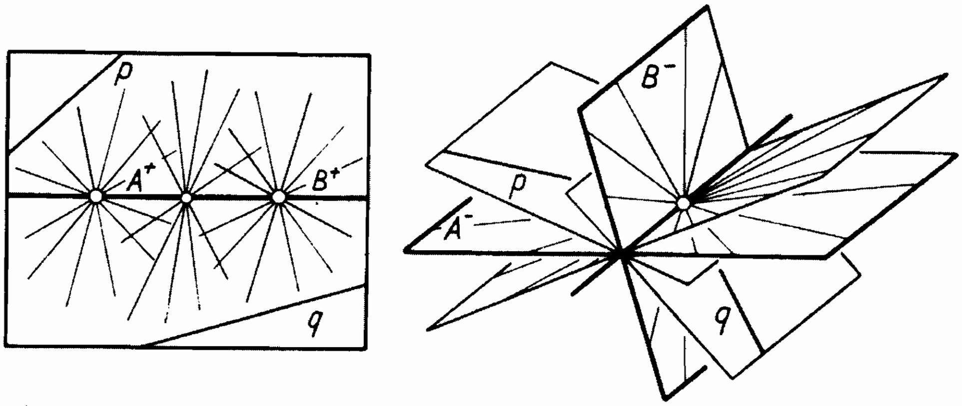
Figures 30 and 31

II. Two lines a, b structure a point through which they both go into two punctual regions of planes (Figure 31). Such a region consists of the planes of all the sheaves in this point whose carrying lines belong to the interior of one of the two angle fields determined by a and b. Or, one such region consists of all those planes whose lines of intersection with ab^- belong to the interior of the same angle field. The boundary is formed by plane sheaves (a) and (b); ab^- in particular is a boundary plane. Within the bundle, it is not possible for a plane of one region to be moved into a plane of the other region without it coinciding at least once with a boundary plane.

Two planes P, Q in the point in question (that is, ab+) belong to the same region if and only if they are not separated by the lines a, b, that is, if and only if the lines of intersection of P and Q with the connecting plane ab^- are not separated by a, b.

I′. Two points A,B structure a plane in which they both lie into two planar regions of lines (Figure 32). Such a region consists of the lines of all the pencils in this plane whose carrying points belong to the interior of one of the two segments determined by A and B. Thus we could also say: one such region consists of those lines of the plane in question that meet the interior of the same segment. The boundary is formed by two line pencils (A), (B); the line AB in particular is a boundary line. Within the field, it is not possible for a line of one region to be moved into a line of the other region without it coinciding at least once with a boundary line.

Two lines p, q in the plane in question belong to the same region if and only if they are not separated by the points A, B, that is, if and only if the points of intersection of p and q with the connecting line AB are not separated by A, B.



Figures 32 and 33

II′. Two planes A, B structure a point through which they both go into two punctual regions of lines (Figure 33). Such a region consists of the lines of all the pencils in this point whose carrying planes belong to the interior of one of the two angle spaces determined by A and B. Or, expressing it another way: one such region consists of all those lines of the point in question that lie in interior planes of the same angle space. The boundary is formed by two line pencils (A) and (B); the line AB in particular is a boundary line. Within the bundle it is not possible for a line of one region to be moved into a line of the other region without it coinciding at least once with a boundary line.

Two lines p, q in the bundle in question belong to the same region if and only if they are not separated by the planes A, B, that is, if and only if the planes connecting p and q with the line of intersection AB are not separated by A, B. The following diagram shows how the four cases I, II, I′, II′ are connected:

I. Two lines in a point field	—— polarity in the field ——	**I′. Two points in a line field**
↑ intersecting | connecting ↓		↑ intersecting | connecting ↓
II′. Two planes in a line bundle	—— polarity in the bundle ——	**II. Two lines in a plane bundle**

The structurings given in Propositions 1, 2, I, II, I′, II′ follow directly from the Partition Theorem. It is sufficient to show this for 1, for example. Let A^- and B^- be the two planes, P^+ and Q^+ any two points which belong to neither A^- nor B^-. We connect P^+ and Q^+ with the line of intersection s = *AB* and consider the connecting planes *Ps* and *Qs* obtained. By the Partition Theorem, the planes *A, B* form exactly two angle spaces in the plane sheaf *AB*. Therefore, either the connecting planes *Ps, Qs* both belong to the same angle space, or one plane belongs to one angle space and the other plane to the other. There is no other alternative. In the first case A^-, B^- and P^+, Q^+ do not separate each other—in the sense defined above—and in the second case a separation of these pairs occurs.

EXERCISES

1. Form a vivid picture of how a point runs repeatedly through a point range, plane runs repeatedly through a plane sheaf, and a line runs repeatedly througha line pencil. Note that the first two processes are mutually polar. Later we shall resolve the following peculiar problem. Suppose

   a point X^+, starting from A^+, runs through a range until it coincides with A^+ again for the first time;
   a plane X^-, starting from A^-, runs through a sheaf until it coincides with A^- again.

   In the second event it is immediately clear that, after a "half turn," X^- *as an element* becomes identical with A^-, but at the same time an exchange of two "half fields" of X^- occurs. In comparison with this, what can we say in the case of X^+ ?
   (Half bundle)

2. Make structuring 1 of point space clear to yourself for the case where one of the two elements *A, B* is the limit plane. Make structuring 2 of plane space clear to yourself for the case where one of the two elements *A, B* is a limit point.

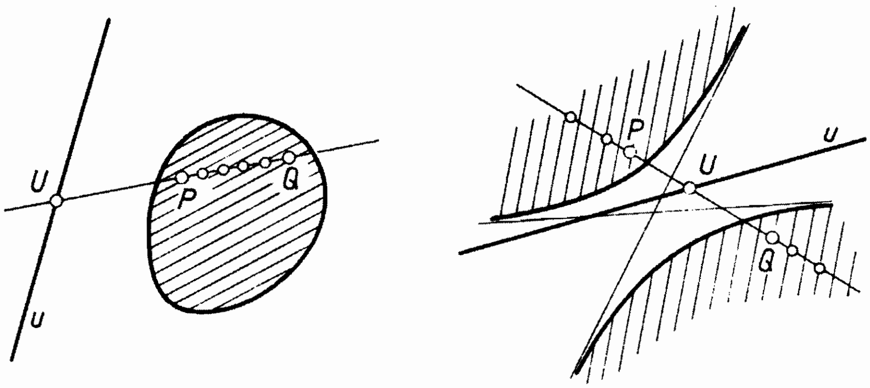
3. The partitioning of second-degree basic forms by two elements arises from structurings 1 and 2 of the third-degree basic forms. In fact

I arises from 1 by intersection with a plane that does not go through *AB*;
II arises from 2 by connection with a point that does not lie in *AB*;
I′ arises from 2 by intersection with a plane that does go through *AB*;
II′ arises from 1 by connection with a point that does lie in *AB*.

Chapter 7

Surrounds and Cores in a Plane

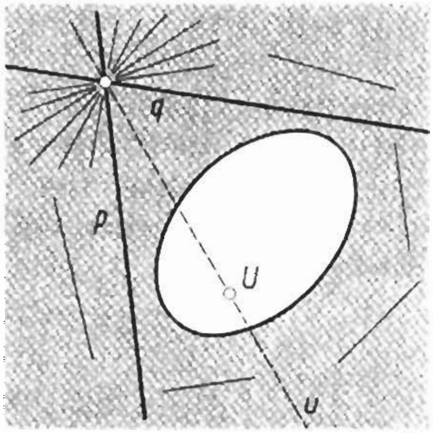
We now describe some facts connected with ordering, in the first place for geometry in a plane. Subsequently it will not be difficult to look at the corresponding situation for geometry in a point. We shall be concerned here with certain fundamental sets of points and sets of lines in the plane in question; we speak of domains of points and regions of lines.

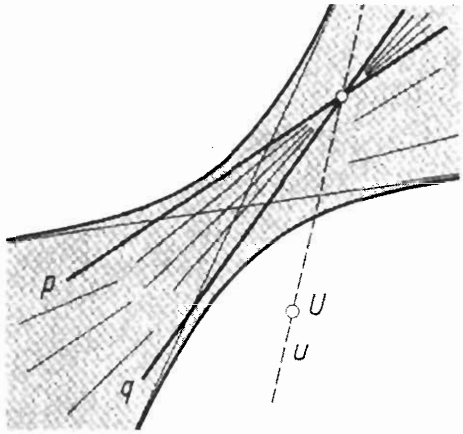


Figures 34 and 35

Convex domains are especially simple domains of points; we call them core domains or cores for short (Figures 34, 35). A set of points is a core domain if there exists a fixed reference line u, of the plane with the following property. For any two points P and Q of the set, if U is the point of intersection of the line PQ with the reference line u then all the points lying between P and Q with respect to U also belong to the set. Thus, along with P and Q, all the points that are separated from U by P and Q belong to the domain. A core domain can extend over the limit line of the plane.

By the law of polarity in the plane, a concave region of lines, which we call *a surround of lines* or *surround region,* is defined as follows (Figures 36, 37). A set of lines is a surround of lines if a fixed reference point U with the following property can be found. For any two lines p and q of the region, if u is the connecting line of the point PQ with the reference point U then all those lines which lie between p and q with respect to u belong to the set. Thus, along with p and q, all the lines that are separated from u by p and q belong to the region.





Figures 36 and 37

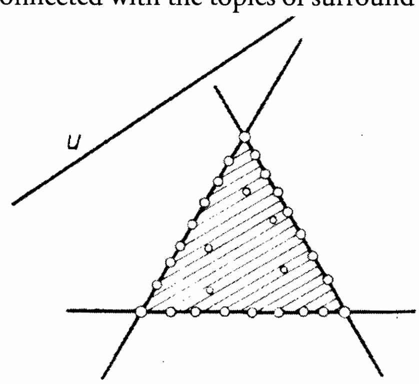
The simplest core domains are those formed by three lines (Figure 38). The simplest surround regions are determined by three points (Figure 39). The boundary of a three-sided core is given by three line segments; the boundary of a three-cornered surround by three angle fields.

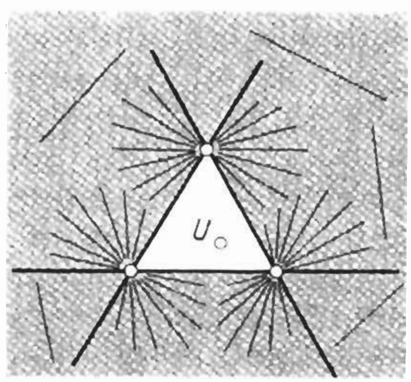
The conceptual constructs "core" and "surround" appeal directly to our visualizing ability. Nevertheless the important thing is that such constructs arise with inner necessity from the archetypal phenomena of space, without anything arbitrary being introduced.

Suppose we have a core domain. Then all the lines that go round it, that is, that contain none of its points, form a surround region. Conversely, any surround region creates a core domain, that is, the domain consisting of all those points that contain no line of the region.

We shall now show, at least for the simplest cases, how these concepts are connected with the fundamental propositions given in the preceding chapters.

The familiar triangle that we learn about in elementary school has the following property: a line that does not go through any vertex has either no points in common with its perimeter, or two and only two. This proposition is closely connected with the topics of surround and core just touched on, as we now show.

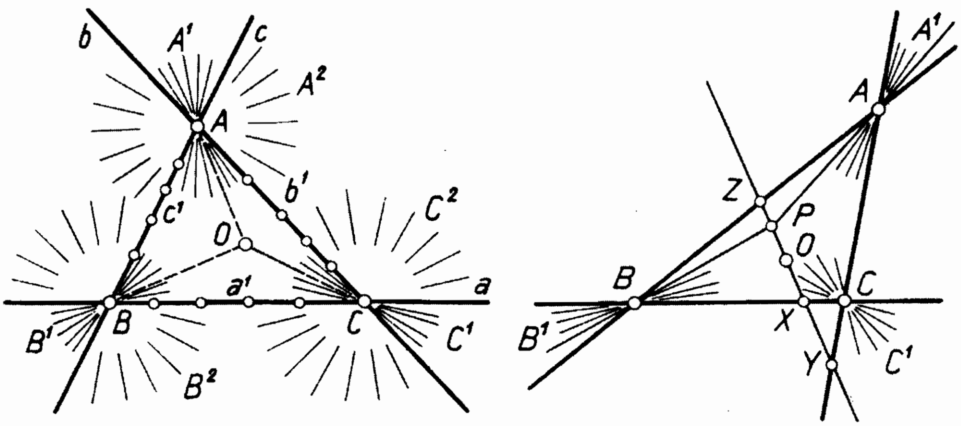




Figures 38 and 39

Let a, b, c be three lines of a field that do not all go through one point. Let their points of intersection be $A = bc$, $B = ca$, $C = ab$. The line pencil (A) is divided into two angle fields by the lines b and c, likewise the line pencil (B) by c and a and the line pencil (C) by a and b. The points B and C form two segments on the range (a), as do the points C and A on b, and the points A and B on c.

We now introduce appropriate notations for these angle fields and line segments (Figure 40). Let A^1, A^2 be the two angle fields in pencil (A), and let B^1, B^2 and C^1, C^2 have corresponding meanings. Let the line segments determined by B and C be called a^1 and a^2, the corresponding segments on (b) and (c) being b^1 and b^2, c^1 and c^2 respectively. The question of which angle fields and which segments are to be labelled number 1 is still left open. A suitable way forward is as follows. Take as reference point any point O of the plane that does not belong to any of the three lines. Let the angle field to which the line OA belongs be given the number 1; similarly the angle fields to which OB and OC belong are B^1 and C^1 respectively. Furthermore let the respective segments which are met by OA, OB and OC be given the number 1. Using point O ensures consistency in the labelling of the angle fields and segments.



Figures 40 and 41

Now for any point P of the plane not belonging to any of the lines a, b, c, we can establish to which of the angle fields the connecting lines AP, BP, CP belong. For the point O these angle fields are, as we agreed, A^1, B^1, C^1. If AP belongs to A^1, BP to B^2, and CP to C^2, then we say P has signature 122.

Similarly, each line p of the plane not running through any of the points A, B, C gets a signature. Thus if p intersects for example the line segments a^2, b^1, c^1, we say p has signature 211.

The signature shows a remarkable regularity:

A point P not belonging to any of the lines a, b, c can have only one of the four signatures:

111, 122, 212, 221;

and a line p not containing any of the points A,B,C can bear only one of the following signatures:

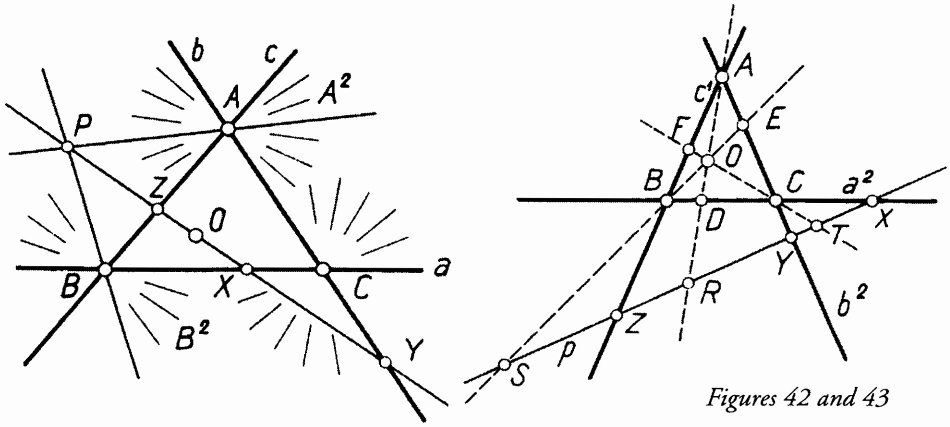
222, 211, 121, 112.

This should be checked graphically with several points and several lines. The proof, that is, the reduction of the rule to earlier propositions, needs a little care.

First we take a line in A^1 and a line in B^1. Their point of intersection P thus has a signature beginning with 11 (Figure 41). We find the points of intersection X, Y, Z of the line OP with the lines a, b, c, assume for the moment that OP does not go through any vertex, and hence that X, Y, Z are distinct. Since O and P lie on lines in A^1, it follows that O, P do not separate Y, Z. Since O and P lie on lines in B^1, they do not separate Z, X either. Now O and P determine exactly two line segments in OP. Since neither Y, Z nor Z, X are separated by O, P it follows that X and Y and Z all belong to the same segment. This implies that the pairs X, Y and O, P do not separate each other either, which means P lies on a line in C^1. Thus P has signature 111.

There are still three special cases to consider. If OP goes through C there is nothing more to prove: the third number is 1 from the outset. If OP goes through A, then Y and Z coincide with A, and since A, X and O, P do not separate each other, it follows at once that CP lies in C^1. If OP goes through B then Z and X coincide with B; again it follows immediately that CP lies in C^1, since B, Y and O, P do not separate each other.

We now assume that the number 2 appears twice in P's signature. Let P be for example the point of intersection of a line in A^2 with a line in B^2 (Figure 42). We have to establish that the third number of the signature is necessarily 1. Again we make use of the points of intersection X, Y, Z of the line OP with a, b, c. Since AP belongs to A^2 and BP to B^2, OP cannot go through A or B. If OP went through C there would be nothing more to prove. Thus we may assume that X, Y, Z are distinct. By assumption, O, P separate both Y, Z and Z, X. Therefore X and Y both belong to the same one of the two segments determined by O and P. Thus O and P are not separated by the lines a and b, and hence CP belongs like CO to C^1. So the signature of P is 221. This proves the first part of the above proposition.

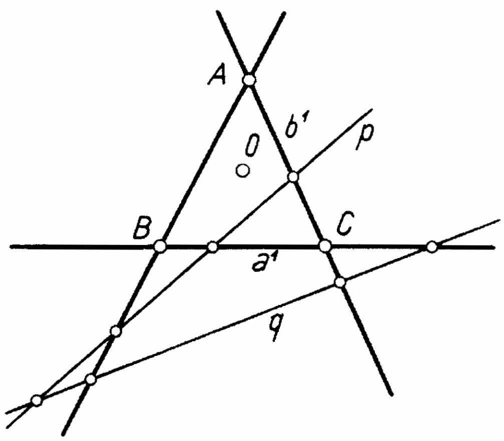


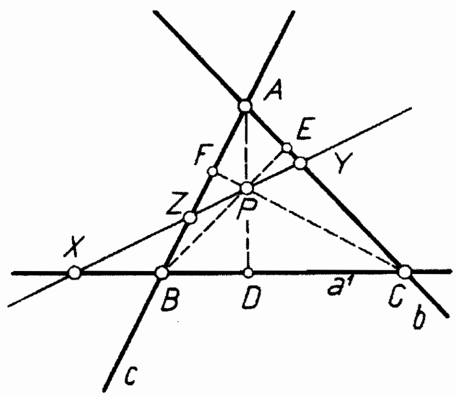
Figures 42 and 43

To see the second part, we consider to begin with a line p that meets the segments a^2 and b^2 in X and Y, say (Figure 43). We have to show that p necessarily meets c^2, that is, p has signature 222.

Let D be the intersection of AO with a, E the intersection of BO with b, F the intersection of CO with c. By assumption B, C and D, X separate each other as do C, A and E, Y. Intersecting the lines from O to B, C and D, X with p, we see that the corresponding pairs of points of intersection S, T and R, X separate each other. Hence $(RSXT)$ gives the natural ordering. If we intersect the lines from O to C, A and E, Y with p, then the corresponding pairs of points of intersection T, R and S, Y separate each other. Hence $(RSTY)$ indicates the natural ordering. Together, $(RSXT)$ and $(RSTY)$ give $(RSXTY)$ as the natural ordering.

Since, by assumption, the pairs of lines BA, BC and BE, BY separate each other, the same is true of the corresponding points of intersection of these lines with p, that is, the pairs Z, X and S, Y separate each other and hence $(SXYZ)$ indicates the natural ordering. Taken together $(RSXTY)$ and $(SXYZ)$ imply that $(XTYZ)$ represents the natural ordering of the points X, T, Y, Z. Thus the pairs Y, X and Z, T separate each other. If we connect C with these four points and intersect the connecting lines with c it follows that A, B and Z, F separate each other. Since F lies on c^1, Z must lie on c^2. Thus the signature of p is 222.





Figures 44 and 45

Finally we assume that the number 1 appears twice in p's signature. We have to prove that the third number is necessarily 2. We connect for example a point on a^1 with a point on b^1 (Figure 44) and show that the connecting line p meets the line c in c^2. To that end we consider a line q with signature 222; such lines exist as we have just seen. If the point of intersection pq happens to lie on c there is nothing more to prove. So we can assume that p and q meet the line c in distinct points. Now by assumption p, q and A, C separate each other as do p, q and B, C. The lines p, q form exactly two angle fields. Because of the separation just mentioned, A and C lie on lines belonging to different angle fields, just as B and C belong to lines of different angle fields. Hence A and B necessarily belong to lines of the same angle field, that is, p, q and A,B do not separate each other. Thus p intersects c^2 and hence has signature 112, as was to be shown. This proves the proposition in all cases.

REMARK. It might be objected that the meaning of this proposition is so transparently clear that a proof is superfluous. Undeniably, this objection has some justification. The proposition could, like some facts mentioned earlier, be added to the archetypal phenomena. But then something would have remained undiscovered, namely the insight that this curious proposition has already been given, in essence, by the phenomena of order referred to in the last chapter. Besides, the proof provides an example of thinking that is modern and axiomatic in the best sense, applied to a fundamental fact.

We can now show that

All points with the same signature form a core domain, and all lines with the same signature a surround region.

This result is connected with the fact that a line that goes through a point with signature 111 (122, 212, 221) cannot have the complementary signature 222 (211, 121, 112 respectively). *A point and a line with complementary signatures cannot belong to each other.*

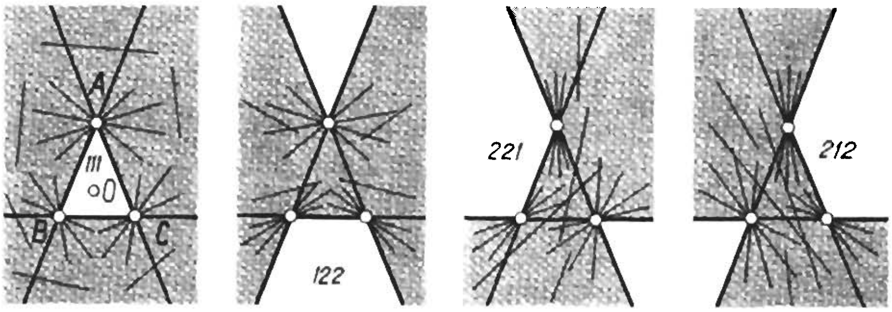
If we connect for example a point P with signature 111 with a point X on a^2 then the line PX necessarily has signature 211, whereas 222 is impossible. This is shown by the following considerations (Figure 45). Let PX intersect b in Y and c in Z; let D be the point of intersection of AP with a, E the point of intersection of BP with b, and F the point of intersection of CP with c. By assumption, B, C and D, X separate each other. Hence the same is true of the lines AB, AC and AD, AX and also for the pairs of points Z, Y and P, X in which PX intersects these lines. Connecting C with Z, Y and P, X and intersecting the four lines with c we find that Z, A and F, B separate each other as well. A, B and F, Z, therefore, do not separate each other, since four elements determine exactly two pairs that separate each other. Hence Z and F belong to the same segment on c, which means that PX meets c^1. So the signature of PX ends with 1 and is, therefore, necessarily 211. In exactly the same way it turns out that, for a line through a point with signature 111, apart from 211, only the signatures 121 and 112 are possible. Therefore, 111 points and 222 lines can never belong to each other.

Two points P, Q with the same signature are, by definition, separated neither by a, b nor by b, c nor by c, a. Thus if we look at the two segments formed by P and Q, we see that all three of the points of intersection of a, b, c with PQ lie in the same segment of these two. Therefore all points of the other segment have the same signature as P and Q.

For the points P, Q with signature 111, we can choose as reference line u a line with signature 222. Then all points between P and Q with respect to the point of intersection U of PQ with u have signature 111 as well. Thus the points with signature 111 form a core domain. The same argument can be used for the other signatures. In conclusion we state the following result:

Three lines in a plane split the plane as point field up into four three-sided core domains; their points of intersection structure the plane as line field into four three-cornered surround regions. Each of the four core domains has exactly one of the four surround regions flowing round it.

This is indicated in Figures 46 to 49.



Figures 46, 47, 48 and 49

The cores 111, 122, 212, 221 appear as blank spaces left by the surrounds 222, 211, 121, 112 respectively. The core domains are bounded by the segments

$$a^1b^1c^1,\ a^1b^2c^2,\ a^2b^1c^2,\ a^2b^2c^1.$$

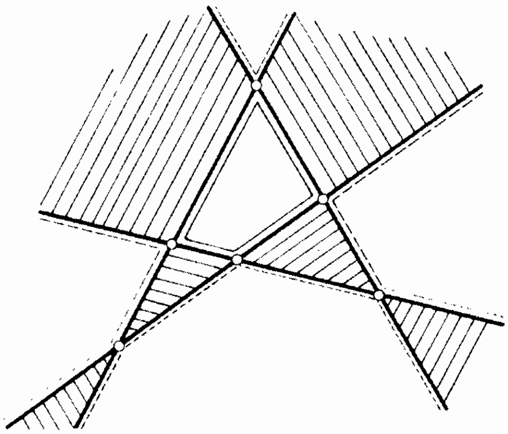
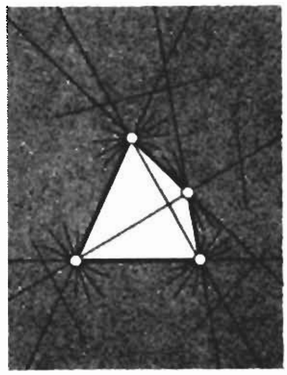
The boundaries of the surround regions consist of the angle fields

$$A^2B^2C^2,\ A^2B^1C^1,\ A^1B^2C^1,\ A^1B^1C^2.$$

As far as applying mathematics meaningfully is concerned, it is highly significant that the forming of core domains and of surround regions are two of the most fundamental processes of spatial formation.

The usual concept of the "interior" of trilateral *abc* means, were it to be expressed more accurately, the core domain whose corresponding surround region contains the limit line of the plane. But if, for example, we choose a line with signature 211 (Figure 47) as a distinguished line, single it out for special honors so to speak, then the interior of trilateral *abc* with respect to this distinguished line is core domain 122. Each of the four cores can represent the interior of the trilateral *abc*; it just depends on which line is chosen as reference line.

Directing our attention to the line field rather than the point field, we shall have to say: The line-interior of triangle ABC with respect to a fixed point is that surround region whose signature complements that of the point. For example, if in some situation the point *O* in Figure 46 were taken as reference point then the line-interior of triangle *ABC* would be the region containing the limit line of the plane.

Figures 50 and 51

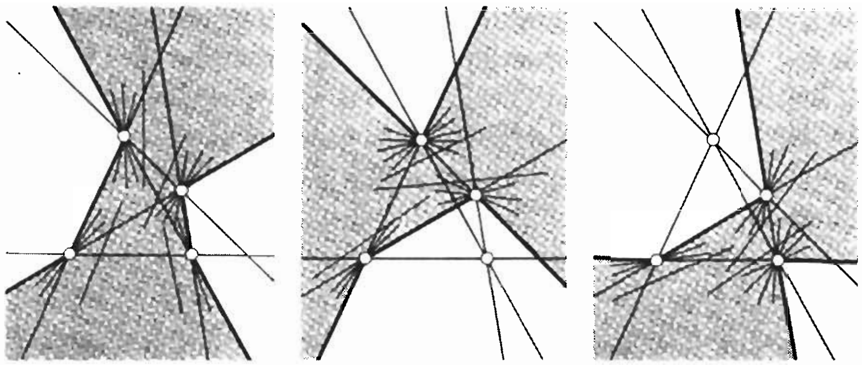
Having dealt with the partitioning of the field by three lines and by three points in detail, it is not difficult to assess the corresponding situation with four elements.

We picture the splitting up of the point field by three lines *a*, *b*, *c* and add a fourth line *d* which does not go through any of the points of intersection of the first three. It turns out that *d* has no points in common with just one of the four three-sided core domains. The three other core domains are each divided up by *d* into one four-sided and one three-sided domain. Thus:

Four lines of a plane, no three of which go through the same point, split the plane as point field up into seven core domains: four three-sided and three four-sided (Figure 50).

Correspondingly:

Four points of a plane, no three of which lie in the same line, structure the plane as line field into seven surround regions: four three-cornered and three four-cornered (Figures 51 to 54).



Figures 52, 53 and 54

In general, the number of domains and regions which come into being with *n* lines or *n* points is as follows:

Number of dividing lines (points)	1	2	3	4	5	6	7	...
Number of points of intersection (connecting lines)	0	1	3	6	10	15	21	...
Number of domains (regions)	1	2	4	7	11	16	22	...

Thereby two points *P*, *Q* belong to the same domain if and only if among the dividing lines there are no two that separate *P* and *Q* from each other. And two lines *p*, *q* belong to the same region if and only if among the dividing points there are no two that separate *p* and *q* from each other.

We can only speak of cores and surrounds if the number of dividing elements is greater than two.

The situation for five dividing elements will be explained later. With five, as with three or four elements, there is only one way of dividing up the field. With more than five elements, though the number of domains or regions is indisputably given by the above, there are, as it turns out, different ways in which the same number of dividing elements can structure the field. An exact statement of the different ways, even for twelve dividing elements, has so far proved impossible to find. This is a difficult open problem.

REMARK. The thorough treatment we have given these simple partitionings of the plane was for a good reason. It becomes ever more apparent that our consciousness is biased towards the point field and point space. While we can immediately grasp the dividing up of a point field, a certain amount of effort is required to think our way into how line regions behave. We tend immediately to think of a line as consisting of points. In short, we are pre-eminently pointminded.

We should develop a taste for cultivating line-mindedness one-sidedly in contemplating a point, the point to be seen immediately as the carrier of a line pencil. A major task lies in cultivating, along with point-consciousness, the corresponding line-consciousness and plane-consciousness. How these rather abstract modes of expression are to be understood will become ever clearer in context.

EXERCISES

1. In Figure 40 we chose as reference point a point O belonging to the interior of abc with respect to the limit line of the plane. Assume O is in another domain, give appropriate signatures to the angle fields and segments and make yourself clear that as a result, the ideas followed through above are not changed at all.

2. In the plane containing lines a, b, c choose any fourth line o as reference line and label those segments that are met by o: a^1, b^1, c^1 and the complementary segments: a^2, b^2, c^2. The angle fields should be signed as follows: the lines through A that meet a^1 form $A1$, the lines through B meeting b^1 form B^1, etc. Then the signatures for the four surround regions are 111, 122, 212, 221, and those for the cores that they flow round are the complementary signatures 222, 211, 121, 112.

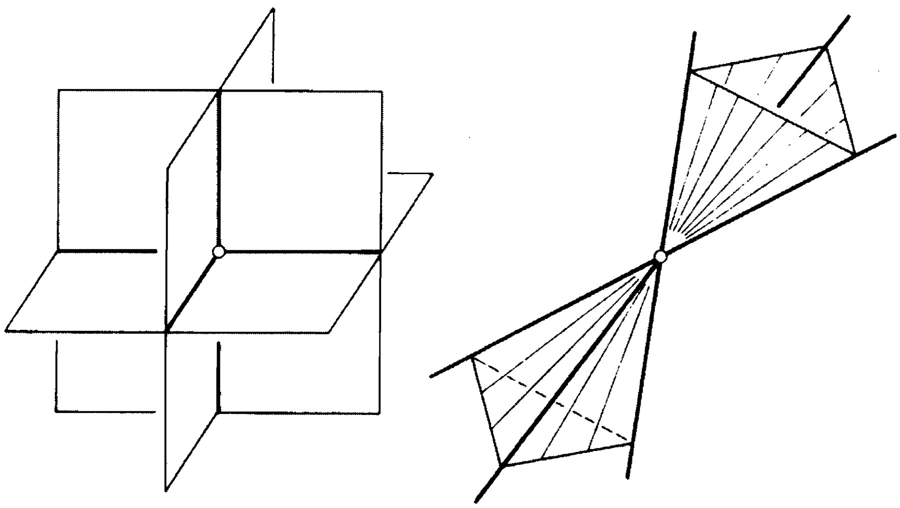
3. Make the seven core domains obtained with four dividing lines a, b, c, d clear to yourself for the case where they form a square, and also for the case where a, b, c form an equilateral triangle and d is the limit line.

4. Consider the divisions of a planar field by a, b, c and A, B, C, connect the figure with a point outside the field and thereby make the meaning of the following propositions clear to yourself: Three planes of a bundle (L) that do not belong to the same sheaf split the line bundle up into four regions of lines in a point. Three lines of a bundle (L) that do not belong to the same pencil structure the plane bundle into four three-edged regions of planes. Each of the four regions of lines is surrounded by one of the four regions of planes.

Chapter 8

Surrounds and Cores in Space

In this chapter we look at the most fundamental structures of space, regarded both as point space and as plane space. The methods of reducing the facts described to the axioms of ordering are the same as those which were used in such full detail in the last chapter.

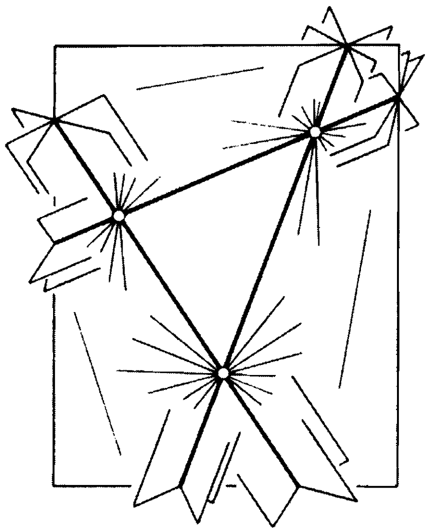


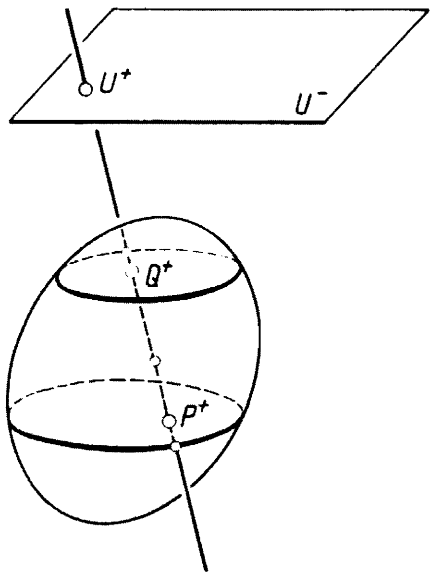
Figures 55 and 56

Three planes split point space up into four three-faced sets of points (Figures 55, 56). The boundary of one such set consists of the points of the three lines of intersection of the three planes and of the points of the lines of three angle fields. If all the points of one such three-faced point set are connected with the point common to the three planes, a three-faced region of lines in a point is obtained (Exercise 4 of Chapter 7).

Three points split plane space up into four three-cornered sets of planes. The boundary of one such set consists of the planes of the three connecting lines of the three points and of the planes of the lines of three angle fields. (Figure 57 shows such a set.) If all the planes of such a three-cornered set are intersected by the plane common to the three points, a three-cornered surround of lines is obtained. The four three-cornered sets of planes mentioned are not difficult to see with the help of the four three-cornered surrounds of lines determined by the three points.

These point and plane sets still do not represent spatial cores and surrounds. In the plane, cores and surrounds arise only with at least three dividing elements; in space, this happens only when there are at least four dividing elements.





Figures 57 and 58

In general, we speak of a spatially convex set of points, or core, in space, if a fixed reference plane U^- with the following property is given (Figure 58). For any two points P^+, Q^+ of the set, if U^+ is the point of intersection of PQ with the plane U^-, then all the points between P and Q with respect to $U+$ also belong to the set. That is, all points separated from U^+ by P, Q belong, with P and Q, to the set.

We speak of a spatially concave set of planes or a *surround of planes* if we can produce a fixed reference point $U+$ with the following property. For any two planes P^-, Q^- of the set, if U^- is the connecting plane of P^-Q^- with the reference point U^+, then all the planes between P^- and Q^- with respect to U^- also belong to the set. That is, all planes separated from U^- by P^-, Q^- belong, with P^- and Q^-, to the set.

If we imagine an egg shape and a plane U^- with no points in common with the egg, we can see that together with two points P^+, Q^+ in the egg, all those points separated from U^+ by P^+, Q^+ also belong to the egg, where U^+ is the point of intersection of P^+, Q^+ with U^-. If we imagine all the planes of space having no point in common with the egg, then these form a surround of planes; we can take any point of the egg as reference point.

Every core has a surround of planes flowing round it; every surround of planes creates a core.

The simplest cores and surrounds are produced by the tetrahedron. The tetrahedron is the self-polar form (Figure 59) consisting of four planes A^-, B^-, C^-,

D^- that do not belong to the same point, and no three of which go through the same line. It also consists of the planes' points of intersection

$$A^+ = B^-C^-D^-, \quad B^+ = C^-D^-A^-, \quad C^+ = D^-A^-B^-, \quad D^+ = A^-B^-C^-$$

and the six connecting lines

$$a_+ = A^+D^+ = B^-C^-, \quad b_+ = B^+D^+ = C^-A^-, \quad c_+ = C^+D^+ = A^-B^-,$$
$$a_- = A^-D^- = B^+C^+, \quad b_- = B^-D^- = C^+A^+, \quad c_- = C^-D^- = A^+B^+.$$

A remarkable structuring both of point space and plane space is brought about by the tetrahedron, a structuring we shall talk about in detail, firstly because it is the most primal structuring of space, and secondly because it offers a deeper insight into the Fundamental Structure introduced in Chapter 5.

A tetrahedron structures point space into eight tetrahedral cores and eight tetrahedral surrounds of planes, each of the eight surrounds creating exactly one of the eight cores.

What we have here is the partition of space, on the one hand by four planes, the faces of the tetrahedron, and on the other by four points, the vertices of the tetrahedron.

The fact that space necessarily comes to be divided into eight parts is seen at once if we first picture the partition of space by three planes (Figures 55, 56) and then note that each of the four three-faced sets of points is split up into two pieces by the addition of a fourth plane.

To get a better insight into this structuring, we direct our attention to the tetrahedron's edges taken in the order $a_+b_+c_+a_-b_-c_-$; we shall always consider them in this sequence. Each edge is intersected by exactly two of the four planes, namely the two that do not go through the edge. Each edge is thus divided into two segments by the four planes. Altogether we therefore have twelve segments

$$a_+^1,\ a_+^2,\ b_+^1,\ b_+^2,\ c_+^1,\ c_+^2,\ a_-^1,\ a_-^2,\ b_-^1,\ b_-^2,\ c_-^1,\ c_-^2.$$

Which segments are to be assigned number 1 must still be declared. To do this we choose an arbitrary reference point O not belonging to any of the four planes, connect it with each edge, and intersect the connecting planes obtained with the respective opposite edges. The intersected segments are given the number 1. With the reference point O chosen in Figure 59, the segments numbered 1 are all those that do not meet the limit plane.

Those points which are not separated from O by any two of the planes form a tetrahedral core whose boundary contains the segments a_+^1, b_+^1, c_+^1, a_-^1, b_-^1, c_-^1. For this reason we use the sequence of numbers 111 111 as signature for this core. The tetrahedral core that shares the boundary segments a_-^1, b_-^1, c_-^1 with the core with signature 111 111 is given the signature 222 111, since the segments a_+^2, b_+^2, c_+^2 belong to its boundary. By checking which segments determine each core we find that there are eight cores. Their signatures are:

(1) 111 111	(5) 222 111
(2) 122 122	(6) 211 122
(3) 212 212	(7) 121 212
(4) 221 221	(8) 112 221

The boundary of each core consists of the four vertices of the tetrahedron, six segments, and four three-sided planar core domains.

Core (2) has only the two boundary segments a_+^1 and a_-^1 in common with core (1). While for example (1) and (7) have in common the three boundary segments a_+^1, c_+^1 and b_-^1, hence also the three-sided planar core domain that they enclose.

If we put cores (1), (2), (3), (4) in a first group and the others in a second group, then the following is true: Two cores of the same group have two boundary segments in opposite edges in common, while a core of one group and any core of the other group have a three-sided planar core domain in common (two such cores together making up a three-faced set of points).

To see the two groups, all we need to do is start with any tetrahedral core (for example 111 111) and take those four cores that adjoin its four boundary-face domains. These form one group. The first core together with the remaining cores, which each have two segments in opposite edges in common with it, makeup the second group.

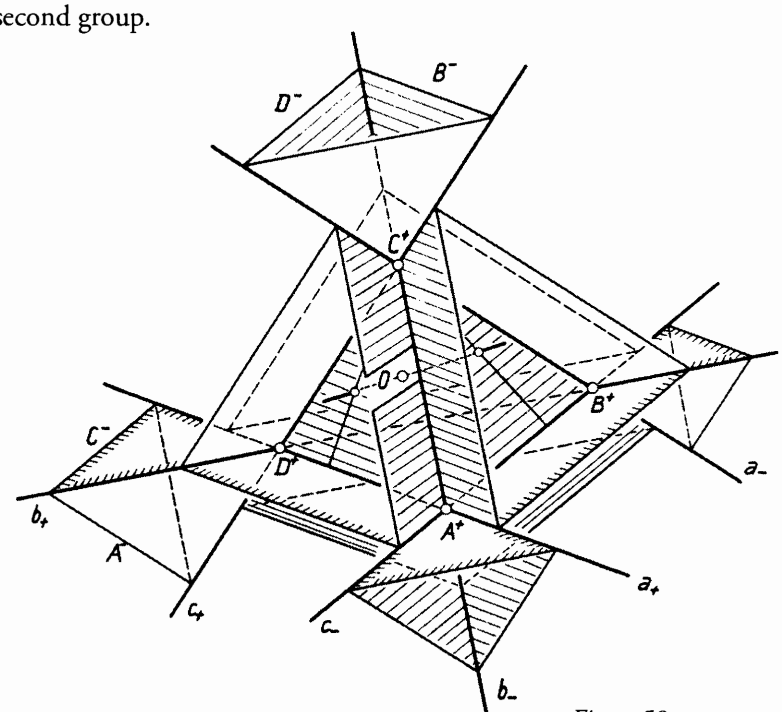


Figure 59

The eight surrounds of planes we identify as follows. For any plane P^-, we name the segments on the six edges it meets. The numbers of the segments met, quoted in the order $a_+b_+c_+a_-b_-c_-$, gives the signature of the plane P^-. The only signatures possible are those which complement the cores' signatures, where, for example, 211 211 and 122 122 are complementary signatures.

The eight surrounds are given by the signatures:

(1) 222 222	(5) 111 222
(2) 211 211	(6) 122 211
(3) 121 121	(7) 212 121
(4) 112 112	(8) 221 112

The boundary of each surround consists of the four planes of the tetrahedron, six angle spaces, and four three-edged regions of planes in a point (Exercise 4 of Chapter 7).

The eight tetrahedral surrounds of planes also form two groups of four. Two surrounds of the same group have in common two boundary angle spaces in opposite edges. A surround of one group has a three-edged region of planes in a point as common boundary with each surround of the other group; two such surrounds together make up a three-cornered set of planes.

We can also view the eight surrounds as follows. Let P^-, Q^- be two planes that do not contain any vertex of the tetrahedron. These produce two angle spaces. We connect the vertices of the tetrahedron with the line of intersection P^-Q^-. Three cases are now possible. Either no two vertices are separated by P^-, Q^-, or one vertex is separated from the other three, or two vertices are separated from the other two. In the first case, P^-, Q^- are planes of the same surround, in the second they are planes from surrounds belonging to different groups, in the third they are planes from surrounds of the same group. Suppose we choose a plane of surround (1) with signature 222 222 as P^-. Then the particular separation of the tetrahedron's vertices brought about by P^-, Q^- determines the surround (1), (2), . . . to which Q^- belongs, as follows:

(1) *ABCD* (no separation)	(5) *D.ABC*
(2) *AD.BC*	(6) *A.BCD*
(3) *BD.CA*	(7) *B.CDA*
(4) *CD.AB*	(8) *C.DAB*

REMARK. Given a tetrahedron on its own, we cannot speak without more ado of the tetrahedron's interior, because each of the eight core domains could be looked upon as its "inner chamber." If, however, a plane U^- is given as well, then one of the eight chambers is singled out as a result: that core domain that contains no points of the plane. This core is the point-interior of the tetrahedron with respect to U^-. Of the surrounds of planes determined by the tetrahedron, we take the one to which U^- belongs, we see that the point interior with respect to U^- is the core domain round which this surround flows. We can only speak of the interior of a tetrahedron with regard to point space once a plane has been singled out for special honors, so to speak. For point space as we normally picture it, the limit plane is such a distinguished plane. Hence, in this space, we are never in any doubt about what is meant by the interior of a tetrahedron.

For ordinary mental picturing, of the eight surrounds of planes, the one containing the limit plane is a distinguished surround. If we remove purely in thought the special honors accorded to the limit plane, then all eight surrounds appear on an equal footing. But if some point U^+ is given in addition to the tetrahedron, then a surround of planes is singled out, namely the one that flows round the core to which U^+ belongs. For "plane-consciousness," that surround must be regarded as the (plane-) interior of the tetrahedron with respect to U^+.

We should note, therefore, when using the concepts "inside" and "outside," that they have different meanings for point space and for plane space, and that the existence of a distinguished plane or point must first be assumed.

We now give, without proof, the number of domains of points and regions of planes created by *n* planes and *n* points in point space and plane space, respectively. We are always assuming that no three of the *n* planes go through the same line and no four of them go through the same point, and polar to this that no three of the *n* points lie in the same line and no four of them lie in the same plane.

n planes (points)	1	2	3	4	5	6	7 . . .
Number of their points of intersection (their connecting planes)	0	0	1	4	10	20	35 . . .
Number of domains of points (regions of planes)	1	2	4	8	15	26	42 . . .

The sequence of differences 1, 2, 4, 7, 11, 16, . . . between successive numbers of domains is the same as the sequence of numbers of domains for the plane.

We can speak of cores and surrounds only if *n* is greater than three.

Two points P^+, Q^+ belong to the same domain if and only if among the dividing planes there is no pair which separate P^+, Q^+. And two planes P^-, Q^- belong to the same region if and only if among the dividing points there is no pair that separates P, Q^-.

As a result of our investigations, we note that even the simplest, most elementary forms show that the formation of surrounds and cores represents a basic principle of the creation of spatial forms.

In conclusion, we look at the structuring of space by the hexahedron and by the octahedron.

A hexahedron is formed in the following way (Figure 16). Imagine three lines $a_{\_}$, $b_{\_}$, $c_{\_}$ that do not belong to the same point in a plane D^- and put two planes, neither of which coincides with D^-, through each line. Then the three pairs ofplanes form a hexahedron (6-plane). This is not the general 6-plane, however; with the latter the lines of intersection of pairs[5] of the six planes are skew. The word hexahedron will always be used for the special 6-plane formed as we have just explained. As is shown in Chapter 5, it possesses a middle point M^+. Its six faces split point space up not, like the general 6-plane, into 26 cores but into 23 cores, which are not difficult to see with the help of Figure 16:

1. A six-faced core with purely four-sided boundary faces. This is the hexahedral body or hexahedral solid in the usual sense. If X^+ is any point of this core then no two of the six faces are separated from each other by M^+ and X^+.

2. Six five-faced cores each with one four-sided and four three-sided boundary faces. These are the pyramids on the faces of the hexahedral body. If X^+ is a point of one such core then M^+ and X^+ separate exactly one of the faces from the other five.

3. Twelve tetrahedral cores. Each edge segment of the hexahedral body is an edge segment of one such core. (For example, the tetrahedron with the following vertices: the front top right-hand vertex of the hexahedron and A^+, C^+, E^+.) If X^+ is a point of one such core then M^+ and X^+ separate two faces with a common edge from the other four.

4. Four six-faced cores with purely three-sided boundary faces. (One of these can be seen as follows. Take the three points A^+, B^+, C^+ and picture the triangle ABC that is completely visible within Figure 16 as the base of the three-sided pyramid with apex E^+ as well as the one with apex D^+, the latter extending over the limit plane; the two pyramids together form one of the domains in question.) Each such core has A^+, B^+, C^+ and two opposite vertices of the body of the hexahedron as vertices. Such a core can be characterized by saying that it consists of all the points X^+ which together with M^+ separate the three planes of one of the vertices from the three planes of the opposite vertex.

In a hexahedron, both a point, the middle point M^+, and a plane, namely the plane D^- of the three lines carrying the pairs of planes of the hexahedron, are thus singled out. Thus the hexahedron itself determines both an interior for point space and an interior for plane space. The point-interior is the core that holds $M+$. The plane-interior of the hexahedron is the set of planes that flow round this core; it contains in particular the distinguished plane D^-.

The octahedron is formed as follows (Figure 17). Imagine three lines a^+, b^+, c^+ with a common point D^+ but not all lying in one plane, and choose two points distinct from D^+ in each line. The three pairs of points form an octahedron. It possesses a middle plane M^-, in which the lines of intersection of opposite faces

[5] Mutually exclusive pairs.

lie. This is not the general spatial 6-point, however, because with the latter the connecting lines of pairs[6] of the six points are skew. That is why the six vertices of the octahedron structure plane space into not 26 regions, but 23.
These are the following surrounds of planes:

1. A six-cornered surround with purely four-edged boundary vertices; the middle plane M^- belongs to it. This surround consists of all planes that do not meet the body of the octahedron (or octahedral solid) in the usual sense. It contains all planes X^- with the property that no two vertices are separated by M^- and X^-.

2. Six five-cornered surrounds of planes, each with one four-edged and four three-edged boundary vertices. Think of a plane X^- that together with M^-, separates one particular vertex from the other five vertices of the octahedron. All such planes X^- form one of the surrounds.

3. Twelve four-cornered surrounds. Here you should consider a plane X^- that together with M^-, separates two vertices belonging to the same edge from the other four. All such X^- form a four-cornered surround.

4. Four six-cornered surrounds of planes with purely three-edged boundary vertices. One such surround contains all planes X^- with the property that M^- and X^- separate the three vertices of a face from the three vertices of the opposite face.

In an octahedron, both a plane, the middle plane M^-, and a point, namely the point D^+ of the three lines that carry pairs of vertices, are singled out. The plane-interior of the octahedron is the surround containing the plane M^-; the core it flows round is the point-interior, to which D^+ belongs.

Remark. Here we feel justified in referring for the first time to a concept that pointed the way for what is presented in this book. With any spatial form we can start from point space as a system in which to embed the form, referring all individual elements to this space. In accordance with the phenomenon of polarity, plane space, too, can be regarded as a primal system in which everything else is embedded. Thus it is the most primitive of spatial phenomena that lead us to speak of a *space* and its *counterspace.* To be sure, point space and plane space only become true space and counterspace if further determining elements are added. Rudolph Steiner outlined the idea of counterspace in a direct and vivid way, both for the arts and for natural scientific application. What is presented here is a mathematical working out in detail of what Steiner outlined. As will be shown, the hexahedron can be used as a basis for orientation in space, the octahedron as a basis for orientation in counterspace.

EXERCISES

1. Make the eight tetrahedral core domains clear to yourself for the case when the three planes of the tetrahedron belonging to the vertex *D* intersect each other at right angles and the fourth plane is the limit plane.

2. Imagine a tetrahedron and two points *P*, *Q* neither of which belong to a plane of the tetrahedron. The connecting line of the two points is intersected by the planes of the tetrahedron in four points. These can be distributed in different ways in the two segments determined by *P* and *Q*. Either all four points of intersection belong to the same segment, or three lie on one and the fourth on the other, or two lie on one and two on the other. In the first case, *P*, *Q* lie in the same core, in the second case in cores belonging to different groups, in the third case in two cores of the same group. Thus, the cores can be represented by the eight possible ways of separating the four planes.

3. Consider four lines *a*, *b*, *c*, *d* generally positioned in a plane. Let *P* be a point in one of the four-sided domains determined by the lines. For an arbitrary, further point *Q* of the plane not lying in any of the four lines, one can check whether and how the four lines are separated by *P* and *Q*. Of the eight possible ways of separating the lines, namely *abcd* (no separation), *ad.bc*, *bd.ca*, *cd.ab*, *d.abc*, *a.bcd*, *b.cda*, *c.dab*, it turns out (in contrast to what happens in space—see previous exercise) that the second, third, or fourth case is impossible. Why is this?

4. Make clear to yourself the structuring of space by a cube and by a regular octahedron into 23 cores and 23 surrounds, respectively.

5. A plane that does not go through any vertex of a tetrahedral core meets the core in either three or four or none of its boundary segments. This shows what signatures a plane can have. In the first case the plane splits the core into a tetrahedral domain and a five-faced core domain bounded by three four-sided and two three-sided planar domains. In the second case the core is divided into two five-faced cores. With the help of this observation, the division of point space by five planes into 15 cores is not hard to see. (To make this easier to picture, take the limit plane as one of the five planes). There will actually be five tetrahedral and ten five-faced core domains.

Chapter 9

The Complete Spatial 5-point and 5-plane

The simplest spatial figures are the complete n-points and the complete *n*-planes polar to them. A complete spatial *n*-point is the form determined by *n* points generally positioned in relation to each other; they are called the vertices of the *n*-point. Any two of these points determine a line; these connecting lines are the sides of the complete *n*-point. Any three of the vertices determine a plane; these connecting planes are the faces of the complete *n*-point.

For example, a complete spatial 5-point is given by five points of which no three belong to the same line and no four belong to the same plane. If we connect every pair of these points, which we call 1, 2, 3, 4, 5 for short, we get the ten sides 12, 13, 14, 15, 23, 24, 25, 34, 35, 45 of the 5-point. Any three points have a plane in common. The ten planes determined in this way, namely 345, 245, 235, 234, 145, 135, 134, 125, 124, 123, are its faces.

Polar to this, a complete spatial 5-plane is given by five planes of which no three belong to the same line and no four belong to the same point. If we intersect every pair of these planes we get the ten edges of the 5-plane. Any three of the planes have a common point. The ten points determined in this way are the vertices of the 5-plane.

The complete spatial 4-point has six sides and four faces. The four faces form a complete spatial 4-plane whose vertices and edges are the vertices and sides of the 4-point. This is the self-polar tetrahedron. As already mentioned, the tetrahedron is a form complete in itself. It presents no opportunity for further construction out of itself, so to speak. Starting from it, we constructed in Chapter 5 the hexahedron and the octahedron by adding a fifth point and a fifth plane respectively. We shall now look at this figure from another point of view.

If to a tetrahedron with vertices *A, B, C, L* we add a plane not going through any vertex, the result is a complete spatial 5-plane. Of its ten vertices, four are the vertices *A, B, C, L* of the tetrahedron. The other six we shall name as follows (Figure 60):

Let *A′, B′, C′* be the points of intersection of the fifth plane with the edges *AL, BL, CL* respectively;

Let *U, V, W* be the points of intersection of the fifth plane with the edges BC, CA, AB respectively. Furthermore let ℓ be the line of intersection of the fifth plane with the plane ABC.

We now direct our attention to the triangles A, B, C and A', B', C'. Let their sides be called

$$a = BC, \quad b = CA, \quad c = AB, \quad a = B'C', \quad b' = C'A', \quad c' = A'B'.$$

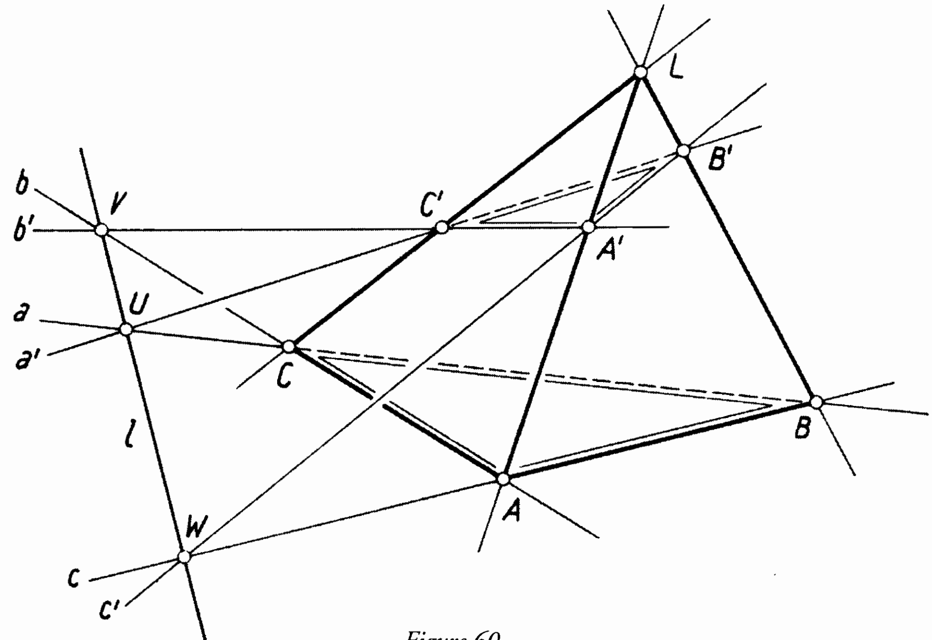


Figure 60

The two triangles are in perspective both with respect to a point and with respect to a line. By this we mean that

1. The connecting lines AA', BB' and CC' of correspondingly named vertices belong to one point (the center L of the perspectivity).

2. The points of intersection aa', bb', cc' of correspondingly named sides belong to one line (the axis ℓ of the perspectivity).

These two properties necessitate each other. That is, if two triangles have the first property, they also possess the second property; and the first property follows necessarily from the second. In short we can say:

(D 1.) *If two triangles are in perspective with respect to a point, then they are also in perspective with respect to a line; and, conversely, two triangles in perspective with respect to a line are also in perspective with respect to a point.*
(This theorem is named after Gérard Desargues, 1591 – 1661, a pioneer of modern geometry.)

The truth of this can be seen almost immediately. Suppose first that two triangles ABC, $A'B'C'$ in different planes have Property 1): the connecting lines AA', BB', CC' intersect in pairs, that is, they go through the same point L. Thus we have a tetrahedron $ABCL$ and the plane $A'B'C'$, that is, a 5-plane. Since the lines a

and a' lie in the same plane *BCL* of the tetrahedron, they intersect; *b* and b' intersect likewise, as do *c* and c'. Hence the three points of intersection aa', bb', cc' lie in plane *ABC* as well as in plane $A'B'C'$, thus they lie in the line of intersection of the two planes. This proves Property 2).

Conversely, suppose two triangles *abc*, $a'b'c'$ in different planes have Property 2): the sides *a* and a', *b* and b', *c* and c' intersect, the three points of intersection thus belonging to one line, namely the line of intersection of the planes of the triangles. Again we have a 5-plane, namely the planes of the two triangles, the plane aa' of the lines *a* and a', and likewise the planes bb' and cc'. The lines AA', BB' intersect since they belong to plane cc'; BB' and CC' intersect likewise, as do CC' and AA'. But since these three lines do not belong to the same plane they must go through the same point *L* common to the three planes aa', bb', cc'. The first property thus follows from the second.

Figure 60 shows a complete spatial 5-plane with its ten edges AA', BB', CC', *a*, *b*, *c*, a', b', c' and ℓ and its ten vertices *A*, *B*, *C*, A', B', C', *U*, *V*, *W* and *L*. We have distinguished the vertex *L* as center of perspectivity of two triangles. But any vertex of the 5-point has the same rights, so to speak, as any other. In fact

The complete 5-plane contains ten different illustrations of Desargues' Theorem.

Some examples: If vertex *V* is singled out as center of the perspectivity, then $AA'W$, $CC'U$ are the perspective triangles; the corresponding sides AA', CC' intersect in *L*, as do $A'W$ and $C'U$ in B', and *WA* and *UC* in *B*; and *L*, B' and *B* all belong to one line. But if, for example, *A* is chosen as center of the perspectivity then *BCL* and WVA' are the perspective triangles, whose corresponding sides intersect in the three points *U*, C' and B', respectively, points belonging to one line.

Whichever of the ten vertices of the 5-plane is chosen as center, the axis of the perspectivity is determined uniquely as the line of intersection of the two planes not going through that vertex.

The figure of Desargues' Theorem, the Desargues Configuration, contains ten lines and ten points, each line carrying three points and each point carrying three lines.

The proof is essentially dependent on the fact that the two triangles in question lie in different planes. How is it then for two triangles *ABC* (*abc*), $A'B'C'$ ($a'b'c'$) in the same plane? For any two triangles drawn in a plane, suppose the vertices of one are called *A*, *B*, *C* and the vertices of the other $A'B'C'$ (and the sides are named appropriately: $a = BC$ etc.). Then the connecting lines AA', BB', CC' will not in general go through the same point but form a trilateral. And the points of intersection aa', bb', cc' will not in general belong to the same line, but form a triangle.

As we shall show in a moment, Desargues' Theorem actually holds in this case as well. For two triangles in the same plane, the situation can also be expressed as follows:

(D 2.) *If one of the two forms: trilateral AA', BB', CC' and triangle aa', bb', cc' degenerates, then so does the other.*

The trilateral degenerates when its sides belong to a pencil; the triangle degenerates when its vertices belong to a range.

Just as Desargues' Theorem in space represents a property of the spatial 5-plane, so Desargues' Theorem in the plane turns out to be a property of the spatial 5-point.

To see this we consider a spatial 5-point with vertices P, Q, R, S, T (Figure 61; to make this easy to see, vertex T is assumed to be in the limit plane) and intersect it, that is, its ten sides and ten faces, with any plane X^- that does not contain a vertex.

First we divide the 5-point into the two tetrahedra $PQRS$ and $PQRT$, which have the face PQR in common. Now let A, B, C be the points of intersection of the three edges PS, QS, RS of the first tetrahedron with the plane X^- and let A', B', C' be the points of intersection of the three edges PT, QT, RT of the second tetrahedron with X^-. The lines AA' and ST intersect because they both lie in the face PST of the 5-point; similarly BB' and ST intersect as lines in the face QST, as do CC' and ST as lines in the face RST. But AA', BB', CC' all lie in plane X^- and thus all three intersect ST in the same point L, that is, the point of intersection of the side ST of the 5-point, with X^-.

The lines $a = BC$ and $p = QR$ also intersect since they both lie in face QRS; furthermore, $a' = B'C'$ and p intersect since these two lines belong to face QRT. Thus a, a', p meet each other in the point of intersection U of the side p with X^-. In exactly the same way $b, b', q = RP$ meet each other in a point V and $c, c', r = PQ$ in a point W of X^-. The points U, V, W belong both to plane X^- and to plane PQR and therefore to the line of intersection ℓ of these planes.
Thus we see that

The plane section of a complete spatial 5-point gives a Desargues Configuration.

Its ten lines are the lines of intersection of the intersecting plane X^- with the ten faces of the spatial 5-point; its ten points are the points of intersection of the ten sides of the spatial 5-point with X^-.

The organizing of the Desargues Configuration into two perspective triangles arose from letting side ST of the spatial 5-point play a special role. If we transfer this role to some other side then its point of intersection with X^- turns out to be the center of the perspectivity of two triangles. Like the complete 5-plane,

The Desargues Configuration in a plane contains ten illustrations of Desargues' Theorem.

Desargues' Theorem is just as easy to prove for the plane once we see that the plane section of a complete spatial 5-point produces a Desargues Configuration.

We assume first that two triangles ABC, $A'B'C'$ in the same plane have the property of being in perspective with respect to a point L, that is, AA', BB', CC' belong to the point L. We have to show that in this case the points of intersection ad', bb', cc' lie in a line. To that end we take any line through L not belonging to the plane of the triangles and choose two points S and T on it (Figure 61). We then consider the lines SA, SB, SC and TA', TB', TC'. SA and TA' intersect since they belong to the same plane TLA; let P be their point of intersection. Likewise SB and TB' intersect in a point Q, SC and TC' in a point R. Thus we have recovered the spatial 5-point $PQRST$. As shown above, ad', bb', cc' lie in a line, namely the line of intersection of the plane PQR with the plane of the triangles.

We now assume that two triangles ABC, $A'B'C'$ in the same plane have the property of being in perspective with respect to a line ℓ, that is, ad', bb', cc' belong to the line ℓ. We have to show that in this case the connecting lines AA', BB', CC' go through a point. To do this we take a plane through ℓ other than the plane of the triangles but otherwise arbitrary, and choose in this plane a line p through ad', a line q through bb' and a line r through cc'. These three lines p, q, r form a triangle with vertices $P = qr$, $Q = rp$, $R = pq$. Since the triangles ABC, PQR in different planes are, by construction, in perspective with respect to ℓ it follows that the connecting lines AP, BQ, CR go through the same point S; and since $A'B'C'$, PQR are in perspective with respect to ℓ, it follows that $A'P$, $B'Q$, $C'R$ go through the same point T. (We are thus applying Desargues' Theorem in space.) Again a spatial 5-point $PQRST$ is produced. Hence AA', BB', CC go through the point of intersection L of ST with the plane of the given triangles.

Remark. Notice that only the archetypal phenomena that led to the 24 propositions of the first four chapters were used to prove Desargues' Theorem: the ordering laws are not necessary. This is thus a theorem grounded purely on the phenomena of mutual belonging of basic elements, and the connecting and intersecting of these elements. It can be shown, however, that it is not possible, by these means alone, to prove Desargues' Theorem in the plane without using the space outside the plane.

Desargues' Theorem is about two triangles. The polar theorem in space is the corresponding statement about two 3-planes. A 3-plane is the form consisting of three planes not belonging to the same line; it has three edges. Just as a triangle also represents a trilateral, so a 3-plane also represents a 3-edge (three lines of a bundle not in the same plane).

Given two 3-planes ABC, $A'B'C'$ (upper case letters here denote planes), they can have the special property of being in perspective with respect to a plane L; that is, the lines of intersection AA', BB', CC lie in the plane L (Figure 62).

But two 3-planes can also be in perspective with respect to a line ℓ. That is, if their edges are $a = BC$, $b = CA$, $c = AB$ and $a' = B'C'$, $b' = C'A'$, $c' = A'B'$ respectively, then the connecting planes of corresponding edges, namely the planes aa', bb', cc', can belong to a line ℓ.

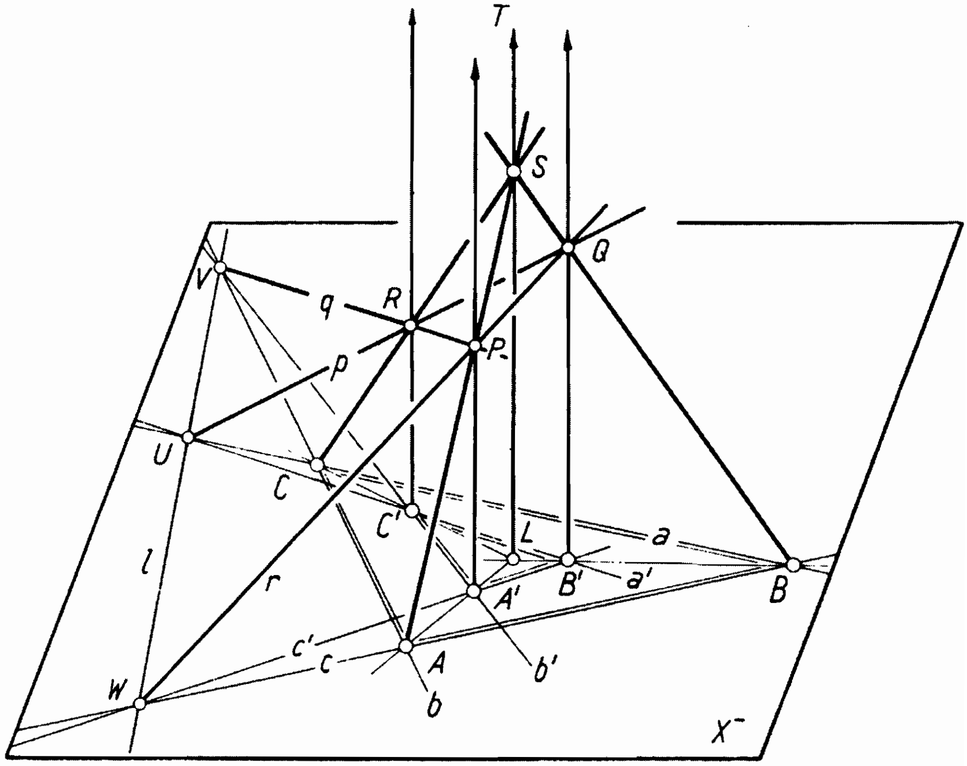


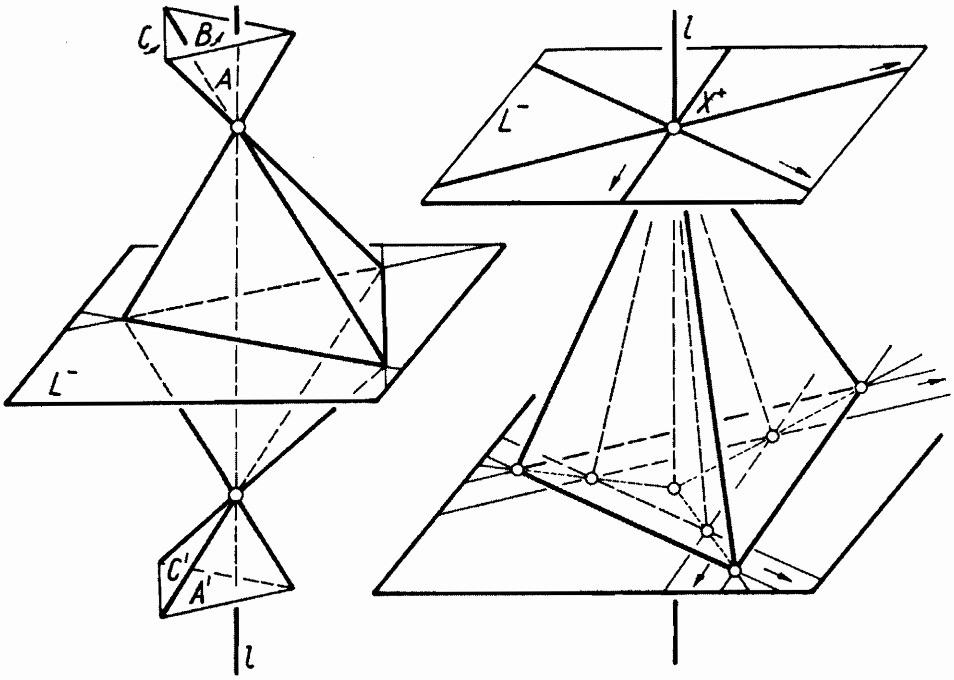
Figure 61

The one property implies the other. That is, the following theorem, polar to the theorem about perspective triangles, holds true:

(D 3.) *If two 3-planes are in perspective with respect to a plane then they are in perspective with respect to a line; and, conversely, if two 3-planes which are in perspective with respect to a line they are also in perspective with respect to a plane.*

This is a property of the complete spatial 5-point (Figure 62). Since our archetypal phenomena are self-polar, and since we have proved the polar theorem, the truth of the above statement is guaranteed from the outset. The spatial 5-point provides ten illustrations of it.

The case in which the two 3-planes belong to the same bundle (Figure 63, in which for clarity the bundle is intersected with a plane) corresponds polarly to the special case of Desargues' Theorem in which the two triangles belong to the same field.



Figures 62 and 63

If we wanted to show directly that D3 is true in this case too, we could polarize the proof given above. That is, we would now have to use what is polar to the section of a complete 5-point, namely the projection of a complete 5-plane. By this is meant the following. Let *P, Q, R, S, T* be the five planes of a spatial 5-plane. Let X^+ be any point not belonging to any of the faces of the 5-plane. Connect X^+ with the ten vertices and ten edges of the 5-plane. A form consisting of ten lines and ten planes in the bundle with carrier X^+ is obtained as a result. This form is the projection of the 5-plane in X^+; it is the configuration in a point polar to the Desargues Configuration in a plane.

We still need carefully to consider the (incidentally particularly powerful) theorem applying to the bundle. Suppose the bundle carried by the point X^+ has two 3-planes *ABC, A′B′C′* in it. Then the lines of intersection *AA′, BB′, CC′* of corresponding faces will in general form a 3-edge in X^+; and the connecting planes *aa′, bb′, cc′* of corresponding edges will in general constitute a 3-plane in the bundle. However:

(D 4.) *If one of the two forms: 3-edge AA′, BB′, CC′ and 3-plane aa′, bb′, cc′ degenerates, then so does the other.*

The 3-edge degenerates when its edges *AA′, BB′, CC′* belong to the same plane, the 3-plane degenerates when its planes *aa′, bb′, cc′* belong to a sheaf.

Given two perspective 3-planes in a bundle, if we intersect this configuration with a plane, then a planar Desargues Configuration is produced. And if we form, in a point, the projection of a planar Desargues Configuration, the corresponding configuration in a point is obtained.

EXERCISES

1. Make the meaning of Theorem D2 clear with a some drawings.

2. In Figure 61 let the plane intersecting the complete 5-point move about and see what changes the Desargues Configuration undergoes as a result.

3. In Figure 61 imagine that the vertices *P, Q, R* of the complete 5-point and the intersecting plane are held fixed, but give vertices *S* and *T* different positions and see what changes the Desargues Configuration undergoes as a result.

4. Verify that D1 holds true even if two corresponding vertices or two corresponding sides of the two triangles coincide.

5. For a given Desargues' Configuration, illustrate the ten ways in which it can be interpreted as a pair of perspective triangles.

6. Prove Theorem D4 by forming the projection of Theorem D2.

7. Form a vivid mental picture of Theorem D4.

8. To help acquire an insight into how the theorems in this chapter are related to one another, complete the following statements:

I. Two triangles in a field that are in perspective with respect to a point . . . .

II. Two 3-planes in a bundle that are in perspective with respect to a plane . . . .

I′. Two trilaterals in a field that are in perspective with respect to a line . . . .

II′. Two 3-edges in a bundle that are in perspective with respect to a line . . . .

The complete 5-plane gives the theorem about two triangles (trilaterals) in different fields. Forming a projection produces the theorem for perspective 3-edges (3-planes) in a bundle.

The complete 5-point gives the theorem about two 3-planes (3-edges) in different bundles. Taking a cross-section produces the theorem for perspective triangles (trilaterals) in a field.

Chapter 10

Continuity

In Chapter 6 we mentioned that the whole of geometry could be developed from the 24 propositions elucidated in the first four chapters and the Propositions a) b) c) d) about order relationships, provided some single further thing e) about ordering is added. We now explain what this phenomenon is.

This is a matter that has preoccupied philosophers since the awakening of thinking consciousness. It is also true to say that it can be grasped very simply and with completely clear concepts. Yet the extraordinarily far-reaching consequences of this property are strangely unexpected and for the most part unclear.

Two points A, B of a point range split it into two segments. Similarly two elements of a line pencil or a plane sheaf split them into two angle fields and two angle spaces, respectively. In short, two of its elements divide a first-degree basic form into two intervals. In saying this we are uniting the concepts segment, angle field and angle space under the common designation *interval.* If the two boundary elements, are counted as part of the interval, it is called *closed*; if we want to consider only the interior elements and exclude the boundary elements we speak of an open interval. What follows applies to all three first-degree basic forms. For purposes of clarification we can therefore use any one of these three forms.

Let (a, b) be one of the two closed intervals determined by a and b (Figure 64). If x is an arbitrary element of this interval, then a sense of running through the elements of the interval is fixed by the cycle (axb). We could call a the starting element of (a,b) and speak of the sense $a \longrightarrow b$ of running through the elements.

We now deal with the properties of the set of all elements of an interval. Property a) of ordering says that this set is *ordered.* That is, any two elements x, y of this set are assumed to have a relationship, namely that of "precedence." If $(axyb)$ represents the natural ordering, we can say: "The element x precedes the element y." On the basis of a), if x precedes y and y precedes z then x precedes z. In this case y lies between x and z.

Since by (a, b) one particular interval of the two produced by a, b is understood, the word "between" needs no amplification: it can only mean between with respect to an element o not belonging to the interval.

(x, y) is understood to mean the interval determined by x and y that belongs to the interval (a, b). On account of ordering property d) we call the set in question dense. This is short for the following property: given any two elements of the ordered set, there is at least one other element of the set between them.

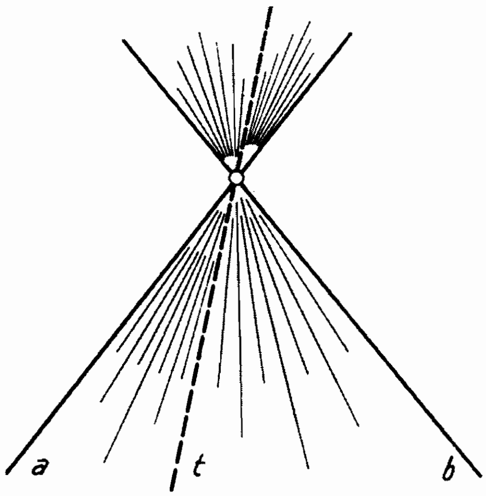


Figure 64

The elements of an interval thus represent a dense ordered set.

All the elements of a first-degree basic form together constitute a dense set, which, because of ordering properties a) b), we call cyclicly ordered.

In fact these sets are not only dense but also continuous or, to put it another way, free of gaps.

We use an auxiliary observation the better to explain this. We imagine any ordered set of elements and split it into two subsets — in fact we do it in a way that preserves order, which means every element of one subset precedes every element of the other.

If the set in question is finite, for example the set of seven elements

$$a, x_1, x_2, x_3, x_4, x_5, b$$

ordered let us say in the given sequence, then an order-preserving division into two subsets might for example be given by

$$a, x_1, x_2, x_3 \text{ and } x_4, x_5, b.$$

The first four elements make up the first subset, the remaining three the second subset. The first has a last element, namely x_3, the second a first element, namely x_4. The order-preserving division into two parts of any finite set is characterized by a jump, in the example by the jump from x_3 to x_4.

In the set in which we are interested, namely the set of elements of an interval, such jumps are impossible because of denseness. But even if it cannot have any jumps, a dense ordered set can still have gaps. To show this we construct the following dense set (Figure 65). Let t be an element of the interval (a, b). We bisect both subintervals (a, t) and (t, b). The four subintervals which result are then bisected once more. Imagine the process of bisecting continued forever. Now take the set M consisting of the elements a and b together with all the constructed bisecting elements; but t should not be included in M. M is obviously a dense ordered set if we take its ordering to be the natural ordering possessed by the elements of the interval (a, b). M can be split into two subsets M_1 and M_2 in a way which preserves order: we only need for example to include all the elements of the set M which precede t in M_1 and all the others in M_2. In this case M_1 possesses no last and M_2 no first element. Between M_1 and M_2 there is a gap. Between each element of M_1 and each element of M_2 there are certainly other elements of M; all the same the division in two of M given is perfectly well-defined. The gap, which is a fact of the division in two with the properties described, is closed by the element t. So clearly a dense ordered set is by no means in general gap-free in the sense explained.

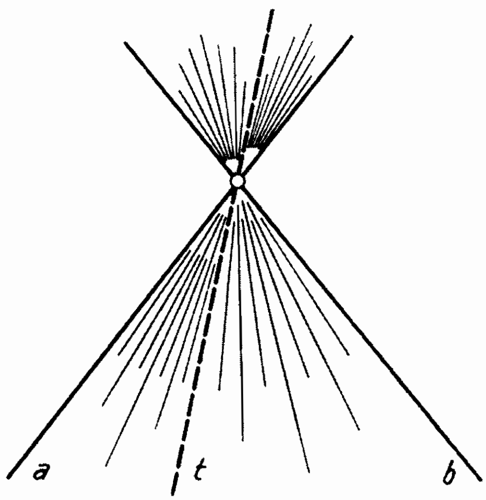


Figure 65

We now go back to the set of all elements of the interval (a, b). This set has the property that an order-preserving division into two parts with a gap, such as the one we just demonstrated for the artificially constructed set M, is impossible. We can certainly divide it into two parts in a way which is orderpreserving. Starting with an arbitrary element t, we include all elements which precede t in the first subset and all the rest in the second. Though the first subset has no last element, the second does have a first element, namely t. This element t brings about the split.

Of course we could also include t along with all the elements preceding t in the first subset. The second subset would then have no first element, but the first subset would have a last element. From this we see that, without one of its elements bringing about the division, an order-preserving division into two parts of the set of all elements of the interval (a, b) is impossible. Therein lies the difference between a continuous (gap-free) ordered set and a merely dense ordered set.

We are now in a position to state the ordering phenomenon e) announced earlier.

e) *Continuity. An order-preserving division into two parts of the set of all elements of an interval can be brought about only by a dividing element of the interval.*

Suppose it is known, in some situation, that the set of elements of an interval (*a*, *b*) has undergone an order-preserving division into two parts. Then by e) one can conclude that either the first subset possesses a last element or the second subset a first element. It is that element which brings about the division. Without such an element, a continuous set, in contrast to one which is merely dense, can never be divided into two parts in a way which preserves order.

The ordering properties a), b), d), e) can be summarized as follows:

The elements of a first-degree basic form constitute a continuous, cyclicly ordered set.

We should straight away add that this by itself is still not sufficient to characterize the way the elements of a first-degree basic form fit together, so to speak. Before explaining this in greater detail, we look at a highly significant property of a continuous set.

Imagine a line with a finite number of points on it, for example three of them. Suppose that from the given points further points of the line are constructed according to some rule (the details of which need not interest us here): first a fourth, then a fifth, sixth, etc. The procedure always produces new points and never stops. As a result we get a set of points which, though infinite, is nevertheless *countable*. We call a set with an unlimited number of elements countable if its elements can be enumerated in such a way that every single element has exactly one number, and each number is assigned to one and only one element.

Now it so happens that

An ordered set that is continuous is never countable.

Thus it is impossible from the outset to obtain all the points of a line from a finite number of them through a sequence of steps of a construction producing new points one at a time. Any such construction process picks out of the continuous set only a portion of the elements, which, as we shall show in a later REMARK, is actually vanishingly small compared with the contents of the whole set.

The proof of the above proposition is basically simple but requires a precise grasp of the concepts involved. By its very nature, the proposition can only be proved indirectly. That is, we assume that the elements of the interval (*a*, *b*) do form a countable set. In that case they can be numbered in such a way that each element *x* gets a number and each number is assigned to only one element. This enumeration could be represented by the sequence

$$x_1, x_2, x_3, x_4, x_5, x_6, \ldots \qquad (1)$$

The order of the sequence of elements in (1) need have nothing to do with the natural ordering of the elements in the interval (a, b). However, we could easily assume that x_1 precedes x_2 inside (a, b); were this not the case from the start, we could achieve this by exchanging the numbers of these two elements. Now it turns out that the assumption that the elements can be numbered leads to an order-preserving division into two parts of the set of elements of the interval (a, b) without the use of a dividing element. But by e) this is impossible. This shows that counting the elements of a continuous set is impossible as well.

In the order-preserving division into two parts just announced, call the first subset L and the second R. It arises as follows (Figure 66):

First step. For each successive element x_3, x_4, x_5, etc. of sequence (1) we ask the question: Does the element lie to the *left* of x_1 or to the *right* of x_2 or between x_1 and x_2 in the interval (a, b)? In the first case we include it in subset L, in the second case in subset R. We proceed in this way until we come across two elements for which the third case is true. These two elements do not need to be successive in (1), however.

(An example to make this clear:

x_3 left of x_1, x_4 left of x_1, x_5 between x_1 and x_2, x_6 left of x_1,
x_7 right of x_2, x_8 right of x_2, x_9 between x_1 and x_2.

In this example x_5 and x_9 would be the first two elements for which the third case is true.)

To simplify the terminology, we rename these first two elements for which the third case is true as a_1 and b_1, the leftmost one being renamed a_1. a_1 is then included in L and b_1 in R.

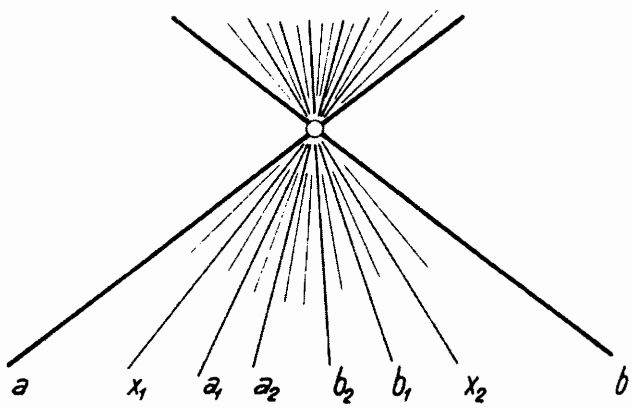


Figure 66

Second step. After having found the second element lying between x_1 and x_2 we continue going through the elements of sequence (1) and for each one ask the question: Does it lie left of a_1 or right of b_1 or between a_1 and b_1 in the interval (a, b)? In the first case we include it in L, in the second in R. We go on in this way until once again two elements have been found for which the third case is true. We rename these a_2 and b_2, the leftmost being renamed a_2. a_2 is included in L, b_2 in R.

Third step. Continuing in sequence (1) we proceed exactly as in the second step, except that now the question refers to elements a_2 and b_2.

By continuing in accordance with the procedure of the first three steps, all the elements of sequence (1), and hence by assumption all the elements of interval (a, b), are allocated to the two subsets L and R. Moreover, each element in L lies to the left of each element in R. The division into two parts is thus order-preserving. On the other hand, the division is not brought about by any element, which in a continuous set is impossible.

The importance of the insight we have just acquired is that two essentially different degrees of infinity have been demonstrated: the countably infinite and the infinite that cannot be counted. This has its consequences.

As an example let us examine the challenge: "The line segment (A,B) may be run through from A to B." Certainly no one will be in any doubt about what is to be understood by this. And yet the challenge cannot be met in objective consciousness if we demand that every single intervening position between A and B be brought to consciousness. Indeed if we have arrived at X say, we would then have consciously to grasp the point following X. But there is no such neighbor of X! Whichever point Y after X we gave, we could immediately give another point lying between X and Y. The set does not need to be continuous for this situation to occur: it occurs even with a dense set. Take, for example, the set M constructed above. With this set we can at least imagine the individual positions (elements) in a countable sequence, though of course the natural ordering is, completely destroyed in the process. Thus, making oneself conscious of each position in M is won at the cost of having to destroy the ordering. Mentally to picture every single one of the infinitely many positions is of course impossible; even so we can begin to give a sequence of acts of consciousness in which we know for certain that each element will sooner or later come to light, if only we go on long enough.

But with a continuous set even this is impossible, since a continuous set is not countable! Our basic phenomenon e) of continuity may at first appear insignificant and harmless, yet it leads to an insight that can be described more or less as follows: Even the first-degree basic forms, and thus the simplest geometrical forms, have the property that it is impossible even in an infinite sequence of acts of consciousness to imagine all their elements individually.

Obviously the above challenge to run through segment (A,B) from A to B, whether in a mental picture or actually in space, can easily be met. But it is impossible to imagine all the positions individually one after the other, even if the natural ordering is sacrificed in the process. The intention to run from A to B does indeed achieve its objective, but at a cost: not every position can be brought to consciousness.

REMARK. As a result of the continuity of our forms, geometry is plunged into a living world. All stepwise construction takes place only as it were on the surface and, by its nature, can never completely fill out the forms. In the last 30 years, consequences of continuity (in conjunction with the so-called Axiom of Choice) have become known that go beyond everything previously conceived of as possible in this area of mathematics.

We mentioned that the ordering properties of first-degree basic forms are still not adequately characterized by the statement that they are continuous, cyclicly ordered sets. To help explain what characteristic quality belongs essentially to these basic forms we make an observation.

Imagine for example two lines *a* and *b* in a line pencil. Then imagine the two lines that bisect the two intervals determined by *a* and *b*, then the four lines that bisect the four intervals created, and so on without end. Let *S* be the obviously countable set of all the lines constructed in this way. Now suppose that *x*, *y* are any two lines of the pencil: they do not need to belong to *S*. Then no matter how close the two lines are to each other, there are always lines of the set *S* lying in the interval (*x*, *y*). (This observation is intended only as a comment, not as a proof.) For this reason, we call the set *S* a skeleton of the pencil. Generally a subset *S* of the set of all elements of a point range, line pencil, or plane sheaf is called a skeleton of the basic form in question if it has the following two properties:

First, *S* should be countable;
second, every interval of the basic form should contain elements of *S*.

Later we shall discover that on the basis of our archetypal phenomena alone, without enlisting the concept of fixed length, such skeletons can actually be constructed, e) playing a crucial role. This provides a guarantee that, even if we cannot grasp every single element, we can, by stepwise construction, still infiltrate any given interval.

An ordered, continuous set possess such skeletons is called a (linear) continuum. Every interval is thus a continuum. If, since it is closed and complete, we call a first-degree basic form a cyclic continuum for short. Then, to sum up, we can say that

A first-degree basic form is a cyclic continuum, that is, a cyclicly ordered continuous set that possesses skeletons.

From any three given elements, for example, from three points of a point range, we shall be able by stepwise construction to determine a skeleton of the range. The points of the range not belonging to the skeleton are rightly called "irrational with respect to this skeleton," since they have the same relationship to the points of the skeleton as the irrational numbers to the rational numbers. The continuum thus shows itself to be the inexhaustible source of geometry.

In many applications it is advantageous to use Proposition e) in another form. To simplify the terminology, we explain things in a point range. Suppose in a line segment, which has a fixed sense of running through the points, a (nonterminating) sequence

$$P_1, P_2, P_3, P_4, \ldots$$

of points is given. If P_1 precedes P_2, P_2 precedes P_3, P_3 precedes P_4, and so on then we call the sequence *monotonic*. Now as a consequence of continuity, such a sequence determines exactly one point, the accumulation point of the sequence. This is a point H with the following property. If X is any point of the segment preceding the point H, then the interval (X, H) always contains points of the sequence. The name is self-explanatory: any neighborhood of H contains points of the sequence; these "accumulate" about H. The existence of such a point is a result of e); more than that, the following fact is equivalent to e):

Existence of accumulation points: Every monotonic sequence of points of a line segment possesses an accumulation point.

Proof: Let P_1, P_2, P_3, . . . be a monotonic sequence of points of the line segment (A, B). It produces an order-preserving division into two parts of the segment if we include in the first subset all those points that precede at least one point of the sequence, and all the rest in the second subset. Every point of the segment belongs either to the first or to the second subset. The division is obviously order-preserving. And so by e) there exists a point H that brings this about. This point H has the properties we attributed to an accumulation point.

Conversely, fact e) can be deduced from the existence of accumulation points. The proposition is important because it allows us, by stepwise construction and hence by a countable procedure, to determine points that do not arise directly from the construction sequence.

A closely related question which one might ask here is, under what conditions can we say of two infinite sets that they yield the same number of elements, or have the same power ? (This terminology originates with Georg Cantor, 1845 – 1918, the brilliant discoverer of different orders of infinity.) The following statement is almost self-evident. *Two sets are of the same power if and only if the elements of one can be mapped onto the elements of the other in a one-to-one correspondence.*

For example, every countable set has, by definition, the same power as the set of natural numbers 1, 2, 3, 4, ...

If (A, B) and (C, D) are any two intervals of a point range g then the sets of their elements have the same power! Proof (the case when (A, B) is part of

(C,D) is shown in Figure 67): Connect an arbitrary point E not on g, with the boundary points C and D and choose a point F on EC and a point G on ED so that the line FG does not meet the interval (C, D). The connecting lines from E to the points of the interval (C, D) intersect the line FG in an interval (F, G).

The points of (C, D) and of (F, G) are mapped onto each other in a one-to-one correspondence through these lines. Likewise, if H is the point of intersection of the lines AF and BG then the elements of the two intervals (A, B) and (F, G) are mapped onto each other in a one-to-one correspondence through the lines from H to the points of these intervals. Taken as a whole, the construction gives a one-to-one correspondence between the points of (A, B) and the points of (C, D), which proves that the two sets have the same power.

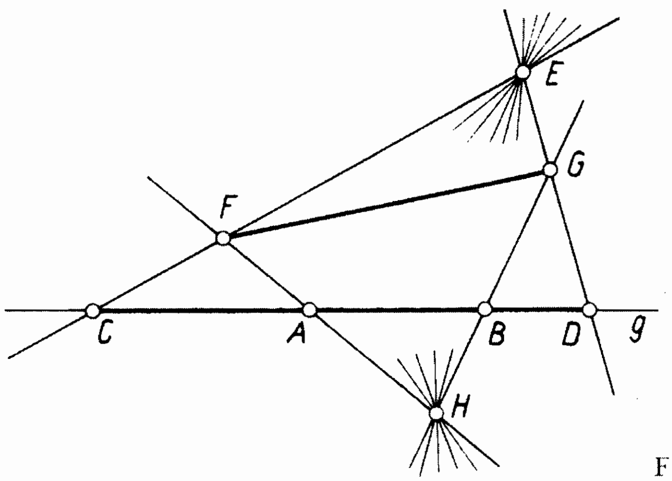


Figure 67

The characteristic difference between finite and infinite sets is that an infinite set possesses subsets having the same power as the whole set. With finite sets this is impossible. For example, the countable set of natural numbers 1, 2, 3, 4, . . . has the subset 2, 4, 6, 8, . . . ; that the latter possesses the same power as the whole set is clear to see. This is shown by the following one-to-one correspondence between the elements of the two sets:

$$\begin{array}{cccccccc} 2 & 4 & 6 & 8 & 10 & 12 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & \ldots \end{array}$$

Since the set of elements of an interval is not countable, its power is different from that of every countable set; it is the power of the continuum.

First Remark. In this remark we use, purely for purposes of illustration, the concept of the length of a line segment. Suppose (A,B) is a line segment ten meters long. We imagine a countable, dense set M of points of the segment. It could, for example, be the set of points that results from endlessly repeated bisection of the segment. Suppose the sequence

$$P_1, P_2, P_3, P_4, \ldots$$

represent an enumeration of the elements of M. We now take another segment, this time of arbitrary length s. For concreteness, suppose s is one millimeter. We cover P_1 with a segment of length $\frac{1}{2}s$ so that its mid-point falls on P_1. Then we cover P_2 with a segment of length $\frac{1}{4}s$, P_3 with a segment of length $\frac{1}{8}s$, P_4 with a segment of length $\frac{1}{16}s$ and so on. The mid-point of the covering segment should always coincide with the point covered. One has the impression, because of the denseness of the set M, that the entire segment (A, B) is going to be covered over quite thickly, perhaps even many times, by the covering segments. But if we add the covering segments together it turns out that the sum

$$\tfrac{1}{2}s + \tfrac{1}{4}s + \tfrac{1}{8}s + \tfrac{1}{16}s + \ldots .$$

has a limiting value of length s, as can readily be seen. Now s was an arbitrary length of one millimeter. This means that the dense but countable subset M is completely covered over with segments which, laid end to end, *together represent an arbitrarily small length.* This example motivates the assertion that any countable subset of the set of all elements of an interval plays only a vanishingly small part in the whole set.

Second Remark. From now on, we use **A** as an abbreviation for what was expressed in the 24 propositions of the first four chapters and **O** for the ordering phenomena a) b) c) d) e).

In the foregoing chapters we have presented, in **A** and **O**, the fundamentals from which the whole of geometry can be developed. We have come to know some important properties besides. We recall in particular that all properties **A** and **O** are self-polar. Thus they produce the geometry in which point space is the original system as well as the polar geometry in which plane space is taken as the starting system.

1. A picture to spur the imagination of the power of the continuum is given by the following.
Let M_1, M_2 and M_3 be three countable sets with elements $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ and $c_1, c_2, c_3, \ldots$ respectively. Then the set formed from the elements of all three sets is also countable. This can be seen for example from the following scheme:

$$\begin{array}{llllllll}
M_1: & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \ldots \\
M_2: & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \ldots \\
M_3: & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \ldots
\end{array}$$

That is, the elements can be counted by means of the sequence

$$a_1\ a_2\ b_1\ a_3\ b_2\ c_1\ a_4\ b_3\ c_2\ a_5\ b_4\ c_3\ a_6\ b_5\ c_4 \ldots$$

The proposition is obviously true for any finite number of countable sets. But it is also true for a countable set of countable sets. If $M_1, M_2, M_3, \ldots$ is a countable set of sets and the sets M_1, M_2, etc. are themselves all countable, then all the elements of all these sets themselves form a countable set. This follows immediately from the scheme:

$$\begin{array}{llllllll}
M_1: & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \\
M_2: & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \ldots \\
M_3: & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \ldots \\
M_4: & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

It is just a matter of counting up the elements in the sequence of diagonals indicated:

$$a_1\, a_2\ b_1\ a_3\ b_2\ c_1\ a_4\ b_3\ c_2\ d_1\ a_5\ b_4\ c_3\ d_2\ e_1\ a_6\ b_5 \ldots$$

Now imagine a line segment (A,B). Let M_1 be the set of dividing points arising from repeated bisection of the segment. A second countable set M_2 of points is formed by dividing (A,B) into three equal parts, again trisecting each third, and so on. Similarly, a countable set M_3 of points is formed by repeatedly dividing into fifths. This process can be continued by repeatedly dividing into 7, then 11, 13, 17, 19, . . . (prime numbers). In trying to picture these mutually exclusive sets $M_1, M_2, M_3, \ldots$ one after the other in the interval (A, B), one gets an impression of the denseness of the set of all elements of all these sets. Yet by the above proposition, all these dividing points together constitute only a vanishingly small part (see the First Remark above) of the continuum. The cosmos of geometry, too, is ever reborn in the chaos of the continuum.

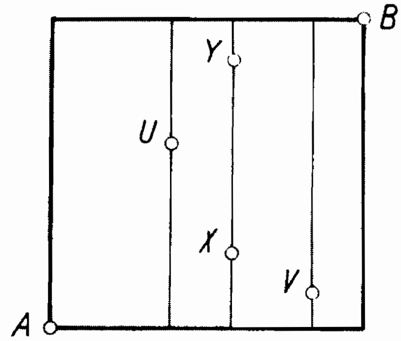


Figure 68

2. Let Q be the set of points lying inside or on the boundary of a square. It can be given an ordering as follows (the explanation is immediately clear from Figure 68). We take the lower left corner A as initial element and the upper right corner B opposite as final element. If two points of Q do not lie one below the other then the leftmost precedes the rightmost (U precedes V). If two points do lie one below the other then the lowest precedes the highest (X precedes Y). This effectively orders all the elements of Q. This ordered set is continuous; that is, it has property e). Yet it does not possesses a single skeleton (see page 106 for definition). Between any two points of the set, and hence between any two points standing one below the other, there must—according to the definition of a skeleton—be points of the skeleton. But the set of pairs of points standing one below the other is certainly not countable, since the set of vertical line segments (like the set of points of a side of the square) is not countable. This example gives an idea of the fundamental significance attached to the existence of skeletons.

   It is all the more remarkable that the set Q has the same power as the set of points of a side of the square, as is shown in the following appendix.

APPENDICES

These go somewhat beyond the scope of this book. Even so, they highlight some facts connected with continuity which are important for gaining a deeper insight into the state of modern consciousness.

FIRST APPENDIX

The set of points inside and on the boundary of a square has the same power as the set of points of any line segment.

When it was first published by G. Cantor in 1878, this theorem caused an extraordinary sensation. To prove it we use the ordinary measure of length though this is not, in fact, essential, as we shall see later.

First of all we give a method, important also for other purposes, of indicating length. Starting with a chosen fixed line segment of length s, we first take half of it; then we take half of what is left, that is, a quarter of the whole segment; then we take half of the new remainder, that is, an eighth of the whole segment; and so on. The result of this can be represented by

$$s = \tfrac{1}{2}s + \tfrac{1}{4}s + \tfrac{1}{8}s + \tfrac{1}{16}s + \ldots = (\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots)s\,.$$

We call the half, quarter, eighth, . . . parts of the segment, elemental lengths: to be precise the first, second, third, . . . elemental lengths, respectively. Any line segment ℓ not greater than s can be measured using these elemental lengths. To do this we check whether ℓ exceeds the half segment, that is, the first elemental length $\tfrac{1}{2}s$; then whether the remainder, if there is one, exceeds the quarter segment (the second elemental length $\tfrac{1}{4}s$); then whether the next remainder if any exceeds the eighth segment $\tfrac{1}{8}s$ and so on. If ℓ is smaller than $\tfrac{1}{2}s$ then the process begins with the longest elemental length which ℓ exceeds. In this way every length ℓ not greater than s can be represented by our elemental lengths, each of the latter being used at most once.

To get s itself we need the complete set of elemental lengths. To represent a shorter length some of them will not be needed. To specify a length we state which of the elemental lengths are needed; in fact all we need to do is give their numbers. For example

$$\ell = (2, 3, 5, 9, 17, \ldots)s$$

shall mean that ℓ results from joining together the second, third, fifth, ninth, etc. elemental lengths. The bracketed sequence before s represents the ratio ℓ : s. In particular

$$s = (1, 2, 3, 4, 5, 6, \ldots)s.$$

Since

$$\tfrac{1}{2}s = \tfrac{1}{4}s + \tfrac{1}{8}s + \tfrac{1}{16}s + \ldots,$$

it follows that the length $\ell = \tfrac{1}{2}s$ can be represented in two ways, either as $\ell = (1)s$ or $\ell = (2, 3, 4, 5, 6, \ldots)s$. In fact, $(1) = (2, 3, 4, 5, 6, \ldots)$, $(2) = (3, 4, 5, 6, 7, \ldots)$ and so on. Thus every segment formed out of a finite number of elemental lengths can also be pieced together from infinitely many of them, since we can replace the smallest elemental length of the finite representation by all of the still smaller elemental lengths coming after it.

In this way a one-to-one correspondence is set up in which each length ℓ not greater than s is related to a unique increasing sequence of natural numbers. If two such sequences, say $(a_1, a_2, a_3, a_4, \ldots)$ and $(b_1, b_2, b_3, b_4, \ldots)$, are given then we can relate the pair to a unique sequence $(c_1, c_2, c_3, c_4, \ldots)$, that is,

$$(a_1, a_1 + b_1, b_1 + a_2, a_2 + b_2, b_2 + a_3, a_3 + b_3, \ldots\ldots)$$

also consisting of increasing natural numbers. Conversely the two first sequences are uniquely determined by this third one by taking the difference of successive terms, as is immediately clear. Thus the sequence

$$(2, 5, 6, 8, 12, 13, 14, 18, 22, 29, \ldots)$$

gives alternate members of the sequences

$$(2, 3, 7, 8, 12, \ldots) \text{ and } (3, 5, 6, 10, 17, \ldots),$$

and generally $(c_1, c_2, c_3, c_4, \ldots)$ gives the two sequences with

$$a_1 = c_1, \quad b_1 = c_2 - a_1, \quad a_2 = c_3 - b_1, \quad b_2 = c_4 - a_2, \quad a_3 = c_5 - b_2, \ldots.$$

By this means we have arrived at a one-to-one correspondence that relates each sequence with a pair of sequences. With this we can prove our proposition, as we shall now show in detail.

Let $ABCD$ be a square with side of length s, EF a segment (Figure 69), P a point inside the square or on BC or on CD, and G the mid-point of EF.

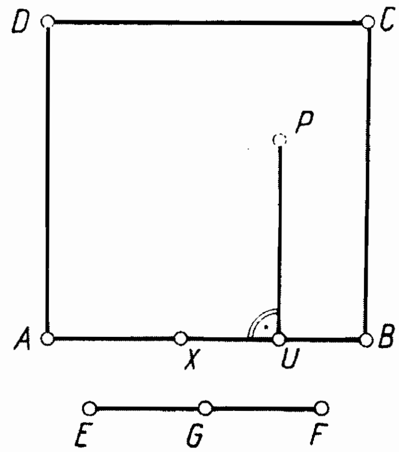


Figure 69

The two lengths

$$AU = (a_1, a_2, a_3, \ldots)s, \; UP = (b_1, b_2, b_3, \ldots)s$$

can be related to the length

$$AX = (c_1, c_2, c_3, \ldots)s$$

in a one-to-one correspondence. As a result, the points P are mapped one-to-one onto AB (excluding A). By contraction (or stretching) AB can now be mapped onto GF (excluding G), and the missing boundary DAB of the square onto the closed interval EG. As a result, a one-to-one correspondence is established between the set of all points of the square surface and the points of the arbitrary segment EF.

To obtain another picture of what was proven, imagine the square lying in a horizontal plane. Suppose we erect, from each point of the square surface, a perpendicular of a certain length. The upper endpoints of these perpendiculars then form a "terrain" above the square. If we now give to each perpendicular whatever length is related to its foot-point by the above procedure, then the resulting terrain has some very strange properties. This is because no two points of the terrain are at the same height. It is therefore impossible, starting from some point of the terrain, to move along a contour line. In fact it consists, pictorially speaking, purely of peaks (except for the curve above the portion of boundary DAB). Continuity is completely destroyed. For completeness we note, without giving a proof, that there is in fact no one-to-one mapping from the square onto a segment that is also a continuous mapping, and hence such that the terrain is everywhere "smooth."

The essential thing about the facts we have been elucidating is the insight they give into the incisive significance of continuity.

Are there more than two degrees of infinite power? To answer this question, we must first be quite clear about the essential features of the "degree," that is, what it means when one power is greater than another power. It is fairly obvious that the following criterion is the one to use. The power of set B is greater than the power of set A if, first, A has the same power as a subset of B, and secondly, B does not have the same power as any subset of A (not even A itself). With this criterion we can prove:

Starting with an arbitrary set M, the set N whose elements are the subsets of the set M has a power which is greater than the power of the set M.

(Even though the theorem is self-evident in the case of finite sets, a simple example should illustrate the concepts adequately. Let M be the set consisting of the three elements a, b, c. It has eight subsets: firstly the subsets $\{a\}$, $\{b\}$, $\{c\}$ consisting of one element of M at a time, secondly the subsets $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ consisting of pairs of elements of M, and finally the "improper" subsets $\{a, b, c\}$ and ϕ, namely the set consisting of all three elements, that is, M itself, and the "empty" set consisting of no elements.)

To prove the theorem we have to show that both the conditions mentioned are fulfilled.

1. M has the same power as a subset of the set N. For this we just need to take those elements of N—that is, subsets of M—that are formed from exactly one element of M. The set consisting of all the "singletons" $\{a\}$, $\{b\}$, $\{c\}$, . . . is a subset of N and obviously has the same power as the set M with elements a, b, c, ...
2. N cannot have the same power as any subset of M. To see this, assume N did have the same power as the subset M' of M (where M' could be identical with M). Then each element of M' could be mapped to an element of the set N—that is, a subset of M—in a one-to-one correspondence. We imagine such a mapping.

We consider one particular subset of M, namely the one consisting of every element of M' which is not contained in the set to which it is mapped. By our assumption, this special subset would, as an element of N, be mapped to a specific element of M', say the element x. We now check to see whether x belongs to this special subset. If the latter contains x, then it cannot contain x after all, because it consists precisely of those elements of M' which do not belong to the subsets to which they are mapped. But if x does not belong to the special subset, then, for the same reason, it must nevertheless belong to it! Our assumption thus leads to a contradiction. A one-to-one correspondence between the subsets of M, that is, the elements of N, and the elements of a subset M' of M, is thus impossible.

This proves that N is indeed of greater power than M. Starting from a countable set, one can thus ascend to sets of ever greater power.

PART TWO: SCHOOLING

Chapter 11

The Four Basic Metamorphoses

In this chapter we characterize certain movement-forms that arise almost directly from the simplest movement processes, that is, from running through a point range, a plane sheaf, or a line pencil.

In plane X^-, let a point Q^+ be given as a source point and a line q as a director line (Figure 70). Suppose points emerge from the source point Q which move away from it in straight lines. But they should not leave Q in a completely arbitrary way; for any two such points, A and B say, their connecting line AB is to meet the director line q in a fixed point. Thus, while A and B move away from Q in straight lines, the line AB runs through a pencil carried by a point lying on q.

Suppose at a particular moment the points emerging from Q form the perimeter of a five-sided domain $ABCDE$, say, to which domain Q belongs. Then, in the course of time, the domain's size will change; it remains five-sided, however. A second state of the process is determined by referring to a single point A' of the corresponding figure. For example, B' arises from the condition imposed on the movement as follows: AB is intersected with q and the point obtained is connected with A'; the connecting line meets the line QB in the required point B'. B',C',D', and so on are all obtained in this way. That this construction fixes the second state uniquely is a consequence of Desargues' Theorem, as will be shown later.

As the movement continues, the five-sided domain expands, extending over the limit line (provided q is not itself the limit line) and covering all but a strip along q. The strip becomes ever narrower, and its boundary comes ever closer to q, reaching a limiting position as it finally coincides with q.

We call the process planar stretching, Q its source point, q its director line. Stretching brings about a metamorphosis between the point Q as "source" and the point range q as "sink."

If we let the process flow in the opposite direction then the point range q could be thought of as the source of the points and the point Q as the sink. The transforming effect of the process on a figure is one of planar compressing.

The expressions *stretching* and *compressing* draw attention to the changes a figure undergoes. If we want to emphasize the picture of the points emerging from Q, with q as sink, we could speak of raying; for the opposite process, with the point range q as source and Q as sink, of *sucking* into Q.

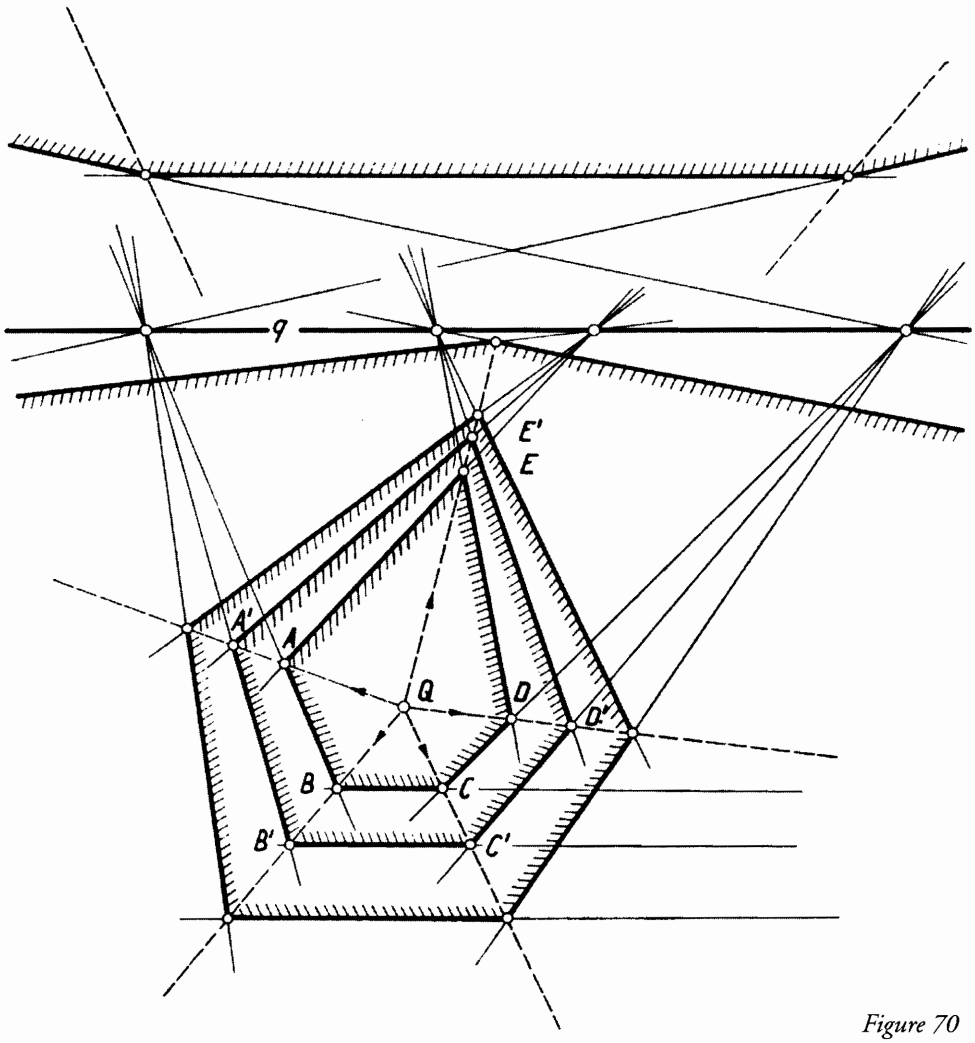


Figure 70

The process pictured in Figure 70 can be looked at from another point of view. Imagine again a plane X^{-}, and in it a line q and a point Q^{+}. This time we picture the line q as a source, not of points but of lines, the lines detaching themselves from q. Suppose also that such a line a released from q is to run through a pencil of lines, a's point of intersection with q remaining fixed, however. This corresponds, in the case of stretching, to the fact that the path QA of a point A emerging from Q remains fixed. Individual lines released from q should not move independently of each other. If a and b are any two such (Figure 71), their point of intersection ab is allowed to move on a fixed line of the pencil Q. Thus, while a and b move in their respective pencils with carrying points on q, the point of intersection ab may run along a fixed point range through Q.

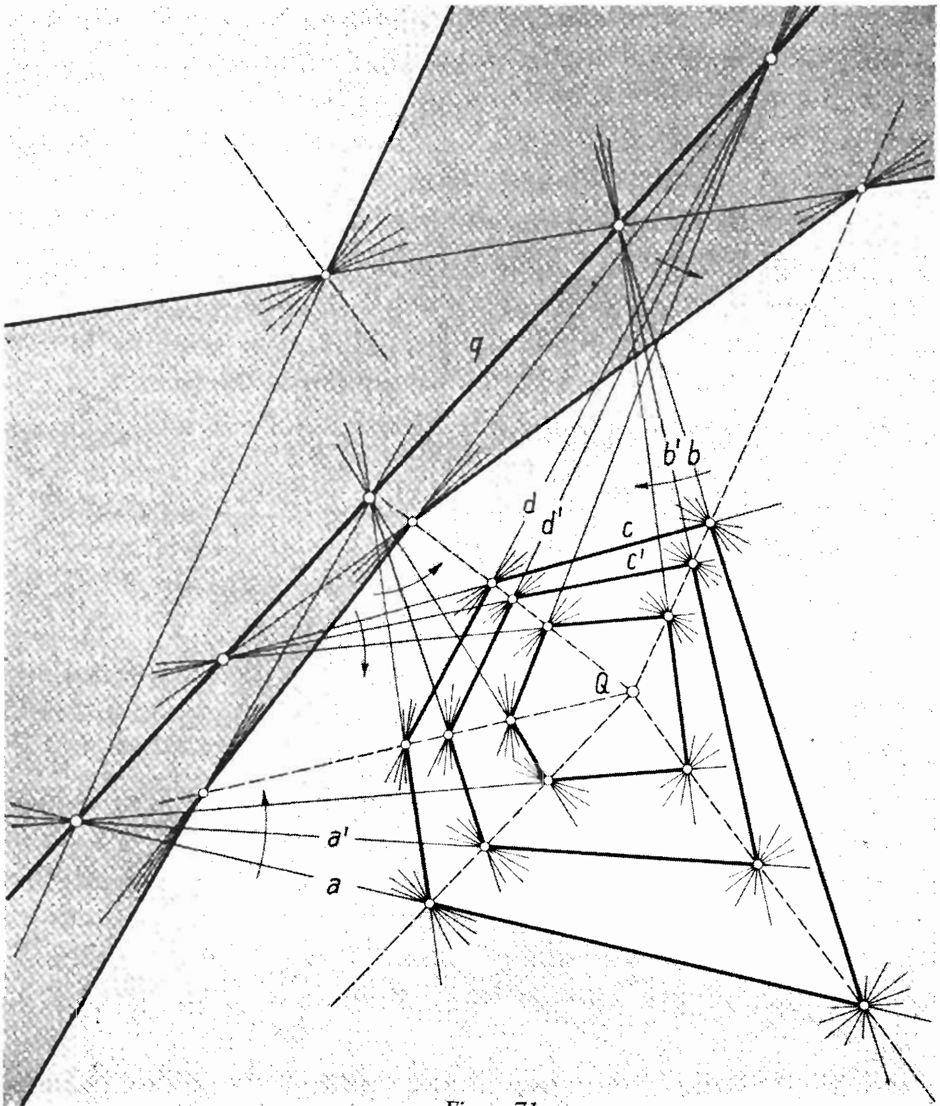


Figure 71

Suppose at a particular moment the lines emerging from q form, for example, the boundary of a five-cornered surround *abcde* of lines and q belongs to the surround's interior. Then, in the course of time, the surround's size will vary; but the surround remains five-cornered. As the movement continues, it "expands" and leaves an ever smaller neighborhood of Q uncovered. Its limiting position is reached when the vertices fall into Q and the boundary lines fall into the pencil Q.

The process, which we could call planar surrounding, brings about a metamorphosis between the line q as source and the line pencil Q as sink. If the process goes in the opposite direction, so that the lines forming a surround with-draw towards q, then we could speak of a widening. This represents a negative sucking into line q.

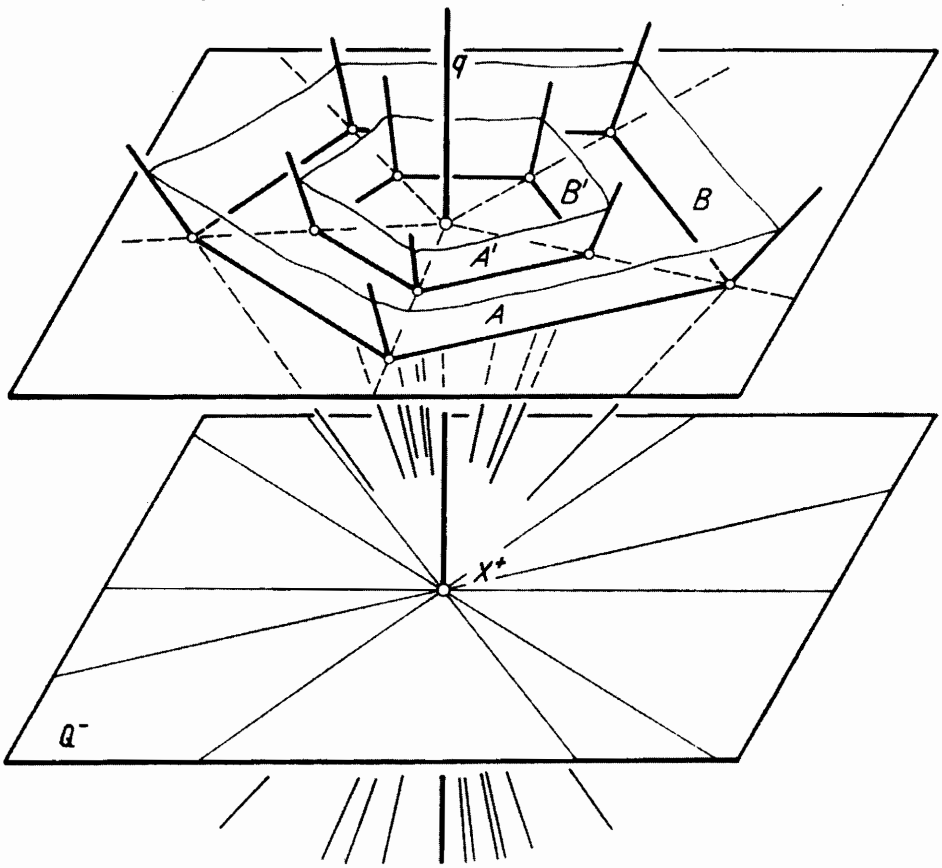


Figure 72

Looking at just the individual points and lines in Figures 70 and 71 we immediately see that stretching and compressing, surrounding and widening produce exactly the same construction. If we take the whole point field and the whole line field into consideration, however, we are dealing with processes that are qualitatively completely different. The names of the processes do not matter; perhaps better ones can be found. What does matter is being able clearly to grasp the four processes of movement themselves, processes that in reality are essentially different. With stretching (raying) and compressing (sucking), we refer to the point field as background space, while with surrounding (negative stretching) and widening (negative sucking), the line field is the background space.

We now turn to the process that is polar in space to stretching. Before, we were concerned with movement processes in a field; now we shall have processes in point geometry, in a bundle. Let there be given, in a bundle X^+, a line q as director line and a plane Q^- as source plane (Figure 72). Within the bundle we allow planes to detach themselves from Q^- as source. At the same time each plane A should rotate about a fixed bundle line in Q^- (polar to the movement of the point A in a line through Q^+). If A, B are two such planes springing from Q^-, then, to avoid total arbitrariness, the line of intersection AB should move in a fixed plane through q, describing thus a line pencil.

If the planes moving in this way are intersected with a fixed plane not belonging to the bundle, then, in the latter plane, the process described is one of surrounding, as indicated in Figure 72.

The process in the bundle can be called *enfolding* or *punctual surrounding.* For the process running in the opposite direction (also clear from Figure 72), it is natural to speak of an *unfolding* or of a *punctual widening,* which is polar to compressing.

If we want the process polar to planar surrounding, we shall need, in a bundle X^+, a source line q and a director plane Q^-. From q, lines—that is, lines of the bundle X^+— should escape from q in such a way that they move in a fixed plane of the bundle, describing thus a line pencil. We now impose the condition that if a, b are two such lines, then their connecting plane ab rotates round a fixed bundle line in Q^- so that ab runs through a plane sheaf (polar to the movement of point ab in a point range through Q in the case of planar surrounding). It is immediately evident from Figure 72 that if the moving lines are intersected with a plane, then this process appears as stretching in the plane. So, to picture this more easily, we could project stretching in a plane from a point X^+ outside this plane (forming a projection) and obtain, in this point as bundle, the process polar to planar surrounding. We call this process *spreading.* Spreading brings about a metamorphosis between a line and a line pencil which both belong to the same bundle, while planar surrounding mediates between a line and a line pencil both belonging to the same field.

The process going in the opposite direction, where the lines arise out of the plane Q^-, is called *enclosing.* This is polar to widening.

To summarize, we show how the eight processes described relate to each other. They fit into a scheme (explained in Chapter 4) that has now been used many times:

I. Planar stretching and compressing in the field X^- — **I′. Planar surrounding and widening** in the field X^-

↕ ↕

II. Enfolding and unfolding in the bundle X^+ — **II′. Spreading and enclosing** in the bundle X^+

Planar stretching and compressing ($Q^+ \longleftrightarrow q$): mediates between a point Q^+ and a point range q in $X^- = qQ^+$.

Enfolding and unfolding ($Q^- \longleftrightarrow q$): mediates between a plane Q^- and a plane sheaf q in $X^+ = qQ^-$.

Planar surrounding and widening ($q \longleftrightarrow Q^+$): mediates between a line q and a line pencil Q^+ in $X^- = qQ^+$.

Spreading and enclosing ($q \longleftrightarrow Q^-$): mediates between a line q and a line pencil Q^- in $X^+ = qQ^-$.

Having explained the simplest movement-forms in planar geometry and the polar processes in the geometry of a point, we now look at the corresponding processes in point space and plane space. Picture a given plane Q^- and a given point Q^+. Suppose points arise out of Q^+ as source and move away from Q^+ in straight lines, thus describing point ranges through Q^+. If A, B are two of these points, let their movements be coordinated in such a way that the connecting line AB meets the director plane Q^- in a fixed point (Figure 73). If A, B, C are three such points, the points of intersection of AB, BC and CA with Q^- remain fixed during the movement, that is, the plane ABC runs through a plane sheaf whose carrying line belongs to the plane Q^-.

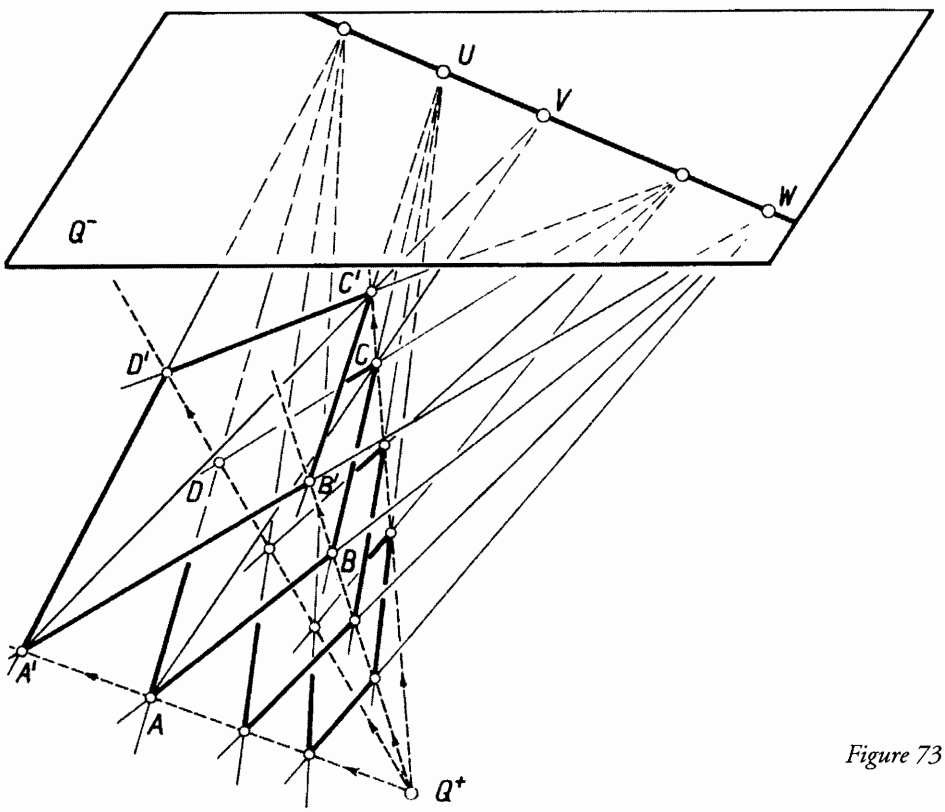


Figure 73

Suppose the points emerging from Q^+ form, at a certain moment, the figure *ABCDE* . . . , and that A' is the position A reaches at a second moment. Then, in accordance with the imposed condition, the corresponding positions B', C', D', . . . of the other points can immediately be constructed. To get say, B', AB is intersected with Q^- and the point of intersection W obtained is connected with A′; the connecting line WA' meets BQ^+ in the required point B'. C can then be determined with the help of either A, A' or B, B', and so on. That this construction determines the positions at the second moment uniquely is a consequence of Desargues' Theorem, as will be shown later.

The movement process described has the following properties:

1. All elements of the bundle Q^+, that is, its lines and planes, remain fixed.
2. All elements of the field Q^-, that is, its lines and points, remain fixed.
3. The points that move run through ranges whose carriers belong to the bundle Q^+.
4. The planes that move run through sheaves whose carriers belong to the field Q^-.
5. 6. The lines that move run through line pencils whose carrying points lie in Q^- and whose carrying planes go through Q^+.

The self-polarity of the whole process is nicely shown by these properties, the process mediating as it were in the tension created by the pair Q^+, Q^-.

If we are given Q^+ and Q^-, and two positions A^+, $A^{+\prime}$ of a point or two positions A^-, $A^{-\prime}$ of a plane, then for any figure containing A^+ or A^- we can determine, by the above construction, the corresponding figure containing $A^{+\prime}$ or $A^{-\prime}$, respectively.

If we direct our attention primarily to point space and imagine a core domain containing Q^+ in its interior, then as the movement continues the domain expands, leaving free an ever smaller domain about Q^- and eventually occupying the whole of point space. A limit is reached as the boundary of the core finally falls into the field Q^- as sink. We call the process stretching from Q^+ towards Q^-. With the flow of movement in the opposite direction it is compressing from Q^- towards Q^+. We could also speak of raying out from Q^+ and of sucking in towards Q^+.

Taking plane space as background space and observing the metamorphosis of a surround of planes to whose interior Q^- belongs, say the one surrounding the core domain we have just been considering, we could describe the situation as follows. Planes detach themselves from the source plane Q^- and rotate about fixed lines in Q^- towards Q^+, forming at each moment a surround of planes. This "expands," leaving free an ever smaller core domain around Q^+, eventually filling up the whole of plane space. A limit is reached as the surround's boundary falls into the bundle Q^+. A fitting name for the process is surrounding, and for the process going in the opposite direction, widening (negative sucking).

Stretching and *surrounding* are mutually polar in space, as are *compressing* and *widening*.

Stretching ($Q^+ \longleftrightarrow Q^-$): transition from point Q^+ to point field Q^-.

Surrounding ($Q^- \longleftrightarrow Q^+$): transition from plane Q^- to plane bundle Q^+.

Compressing ($Q^- \longleftrightarrow Q^+$): transition from point field Q^- to point Q^+.

Widening ($Q^+ \longleftrightarrow Q^-$): transition from plane bundle Q^+ to plane Q^-.

If we put a fixed plane X^- through Q^+ and see how the movements appear in this planar section we get:

from stretching, planar stretching in X^-,

from surrounding, planar surrounding in X^-,

from compressing, planar compressing in X^-,

from widening, planar widening in X^-.

The line of intersection of X^- with Q^- is the director line.

But if we take a fixed point X^+ in Q^- and consider the projection of the movements in space from this point X^+ as bundle of projection we obtain:

from stretching, spreading in X^+,

from surrounding, enfolding in X^+,

from compressing, enclosing in X^+,

from widening, unfolding in X^+.

The connecting line of X^+ with Q^+ is the director line.

Again we note that it is not the words but the processes themselves that matter. The names are introduced as a spur to clear visualization of the details of the metamorphoses in question. The fact that four types of simplest movement-form arise out of geometry itself is what is important: stretching and compressing in point space, surrounding and widening in plane space (counterspace).

The eight processes in plane and point geometry described arise necessarily by taking sections and forming projections respectively. Geometrical intuition is greatly enlivened and enriched by considerations of this kind. Equipped with this way of thinking, we can represent a variety of processes of nature with new mental images. In the world of forces too we are thereby led to look for four types which, in their different ways of acting, manifest in the four elementary metamorphoses.

EXPLANATION. Suppose we are given a plane Q^- and a point Q^+. Furthermore let $ABCDE\ldots$ represent a particular position of the metamorphosing figure under consideration. In a later phase, let point A' correspond to point A. With the specification of this single point A', the new phase is uniquely determined, as we shall now explain in detail.

The point B' corresponding to B is found as follows. The point of intersection $W = (AB, Q^-)$ is identified and B' obtained as the point of intersection $(WA', BQ+)$.

To construct C' from C with the help of A and A', we first identify $V = (AC, Q^-)$ and obtain $C' = (VA', CQ^+)$.

Now the point corresponding to point C could also be constructed with the help of B and B', that is to say by determining $U = (BC, Q^-)$ and then forming the point of intersection $C_1 = (UB', CQ^+)$.

But the two points C' and C_1 coincide, because the triangles ABC, $A'B'C'$ are in perspective with respect to Q^+ thus forming a Desargues configuration. Hence the sides BC, $B'C'$ both intersect the line VW in the same point, namely U. Therefore it comes to the same thing whether we construct C' using A and C or using B and C. By repeated application of this fact it turns out that the new phase $A'B'C'D'E'\ldots$ is indeed uniquely determined by $ABCDE\ldots$ together with A'; in other words the new phase is independent of which points are used for its construction. The whole process is regulated by Desargues' Theorem.

The movement-forms (in a field and in a bundle) we have characterized have their basis in Desargues' Theorem for a field and for a bundle, respectively.

Two phases of a figure being transformed are, by the very movement brought about between Q^+ and Q^-, in perspective. In fact they are in perspective both with respect to the source point Q^+ and the director plane Q^-, that is, the connecting lines of corresponding points go through Q^+, and the lines of intersection of corresponding planes lie in Q^-.

In a field, the figures are in perspective both with respect to the point Q^+ the line q, and in a bundle there are corresponding perspectivities with respect to the line q and the plane Q^-.

Later we shall return to these perspectivities in a wider context.

REMARK. If we survey the processes dealt with, mediating between Q^+ and Q^-, between q and Q^+, and between q and Q^-, an obvious question is: Is there not also a natural intermediary between two lines p, q? We shall delay answering this question till later.

EXERCISES

1. Draw figures corresponding to Figures 70 and 71
   a) for a seven-sided domain ABCDEFG and a seven-cornered surround *a b c d e f g* respectively;
   b) for a circular domain and a surround surrounding a circular domain respectively.

2. Draw figures corresponding to Figures 70 and 71 for the following special situations:
   a) when q is the limit line of the plane, Q an arbitrary point;
   b) when Q is a limit point, q an arbitrary line;
   c) when Q and q belong to each other;
   d) when q is the limit line of the plane and Q belongs to it.

3. Q^+ is the center of a sphere, Q^- a plane not meeting the sphere's surface. Form, for various positions of Q^-, a vivid mental picture of the metamorphosis mediated between Q^+ and Q^- with
   a) point space regarded as background space (transformation of the point-interior of the sphere; stretching and compressing);
   b) plane space regarded as background space (transformation of the plane-interior of the sphere, that is, the surround of planes surrounding the sphere; surrounding and widening).

4. Let q be the axis of a circular conical surface and Q^- a plane through its vertex X, a plane that contains no generating line of the cone. Form clear mental pictures of unfolding and enfolding (look at the punctual surround of planes of the cone) and of spreading and enclosing (look at the punctual region of lines inside the cone).

Chapter 12

The Structuring of the Field and Bundle by Four and Five Elements

In Part One we expounded in some detail the simplest structurings of space, of the field and of the bundle by two, three, and four elements. In this chapter we go further into the structuring of the field and the bundle by means of four elements and also deal with the case of five dividing elements. The forms that occur are particularly suitable as exercises for becoming familiar with projective geometry. What follows is intended as a brief outline, a series of suggestive sketches that can be worked out fully in lectures and in exercises.

The *complete planar 4-point* is a form determined by four points A, B, C, D of a plane. The points A, B, C, D, no three of which should belong to the same line, are called its vertices, the six connecting lines AB, AC, AD, BC, BD, CD determined by the vertices, its sides. The complete planar 4-point can be viewed as a degenerate tetrahedron, whose vertices are displaced in such a way that they come to lie in a plane; its edges become the 4-point's sides.

In planar geometry the figure polar to the complete 4-point is the *complete planar 4-side*.[6] Such a figure is determined by four lines a, b, c, d of a plane, no three of which belong to the same point. The four lines a, b, c, d are called its sides, their six points of intersection ab, ac, ad, bc, bd, cd, its vertices.

The two forms of point geometry that, in space, are polar to these forms are the *complete punctual 4-plane* and the *complete punctual 4-edge.* The former is determined by four planes A,B,C,D of a bundle, of which no three belong to the same line; it has six edges. The latter is given by four lines a, b, c, d of a bundle, of which no three belong to the same plane; it has six faces.

If a complete punctual 4-plane or 4-edge is intersected with a plane not belonging to the bundle in question, then a complete planar 4-side or 4-point respectively is obtained as figure of intersection. Conversely, if we form the projection of a complete planar 4-side or 4-point from a point not belonging to a plane of the 4-side or 4-point, then we obtain a complete punctual 4-plane or 4-edge respectively.

[6] Planar 4-point and planar 4-side are sometimes called *quadrangle* and *quadrilateral*, respectively.

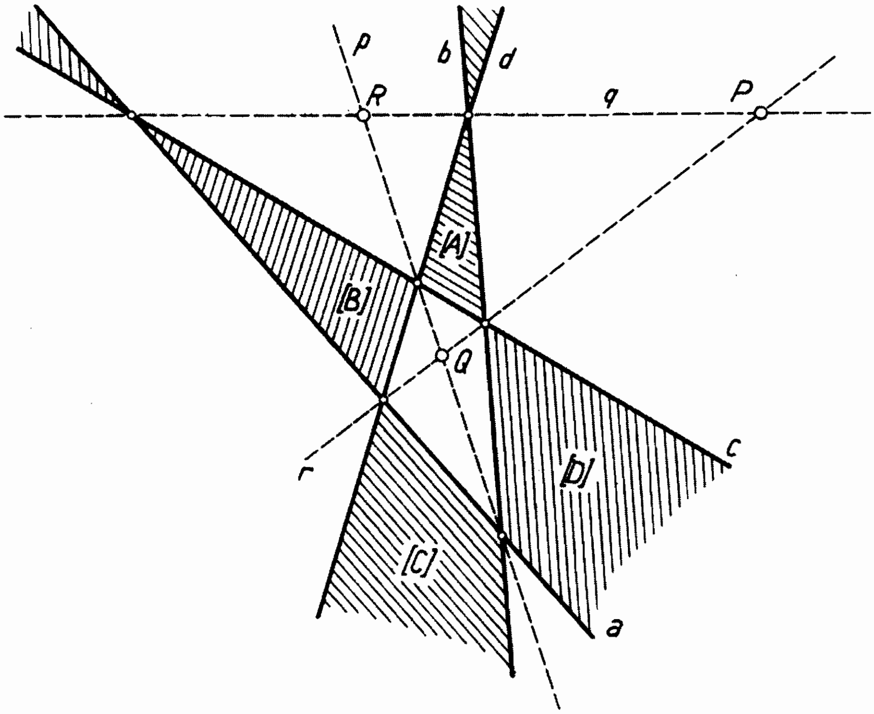


Figure 74

First we investigate the complete planar 4-side *abcd.* The three sides *a, b, c* divide the plane as point field up into four three-sided core domains. On the fourth line *d,* the three lines *a, b, c* determine three segments. Each segment divides a three-sided core of the division brought about by *a, b, c* into one three-sided and one four-sided domain. Since there are just three dividing segments, exactly one of the four cores of the former division remains untouched by *d.* The division brought about by a 4-side *abcd* thus shows four three-sided and three four-sided core domains. To name the domains in a suitable way, notice that only three of the four lines take part in the formation of a three-sided core (Figure 74). For this reason we call the core in whose boundary *a, b, c* take part [D], the core in whose boundary *b, c, d* play a part [A], and so on. The naming of the three four-sided cores follows directly.

It is a remarkable fact that in each of the seven cores there is a particular point that in the nature of things is unique.

First we consider the four-sided domains (Figure 74). Each determines two diagonals, the connecting lines of opposite vertices. But because of the relative positions of the four-sided cores there are not six but only three diagonals in all, namely the connecting lines

$$(ab,\ cd) = p \qquad (ac,\ bd) = q \qquad (ad,\ bc) = r$$

of pairs of vertices of the 4-side which are not already connected by a, b, c, d. These three lines are the extra sides of the 4-side. They form its extra 3-side. The vertices $P = qr$, $Q = rp$, $R = pq$ of this extra 3-side are the extra vertices of the complete 4-side.

Thus in each four-sided core there is a point that, by the very nature of its occurrence, obviously has a special significance. We can speak of P, Q, R as the "middle points" of these cores, which we call $[P]$, $[Q]$ and $[R]$ correspondingly.

An objection could be made at this point. Now p, q, r are three distinct lines; otherwise a, b, c, d would not form a 4-side. The points ad, bc divide their connecting line r into two segments. As the figure suggests, the points of intersection P and Q, in which r is cut by p and q respectively, do not belong to the same one of those segments. One might object, however, that there is at first no compelling reason for this. Indeed, the possibility cannot immediately be ruled out that the connecting line $(ab, cd) = p$ even goes through P, so that p, q, r belong to the same point. On the basis of A and O, however, it can be shown that p, q, r do in fact form a 3-side. That is,

Two opposite vertices of a complete 4-side, and the two extra vertices belonging to the line joining them, separate each other.

Proof. Consider the core [D] formed by a, b, c which is not met by d. As in Chapter 6, let the segments forming the boundary of [D] be called a^1, b^1, c^1 and the complementary segments a^2, b^2, c^2. So the line d meets a^2, b^2, c^2. Let the angle fields of the vertices of [D] be given appropriate signatures. Hence the point of intersection P of (bc, ad) with (ac, bd) has a signature beginning with 22; this must thus, by the argument of Chapter 6, be 221. Q and R are correspondingly given signatures 212 and 122, respectively. Hence P, Q, R are distinct and p, q, r do indeed form a 3-side. The result can now be seen from the signatures of P, Q, R.

Looking at the figure as described so far, we notice right away that more lines present themselves. The extra vertex P, for example, is not yet connected with two vertices of the 4-side (namely ab and cd); the same goes for Q and R. We draw in these six lines (Figure 75): from P to ab and cd, from Q to ac and bd, and from R to ad and bc. Even though the original 4-side can be chosen arbitrarily, the six lines mentioned have the property that three at a time belong to the same point: three of them meet each other in a point of core [A], three others in a point of core [B], and so on. These points we naturally call A, B, C and D respectively.

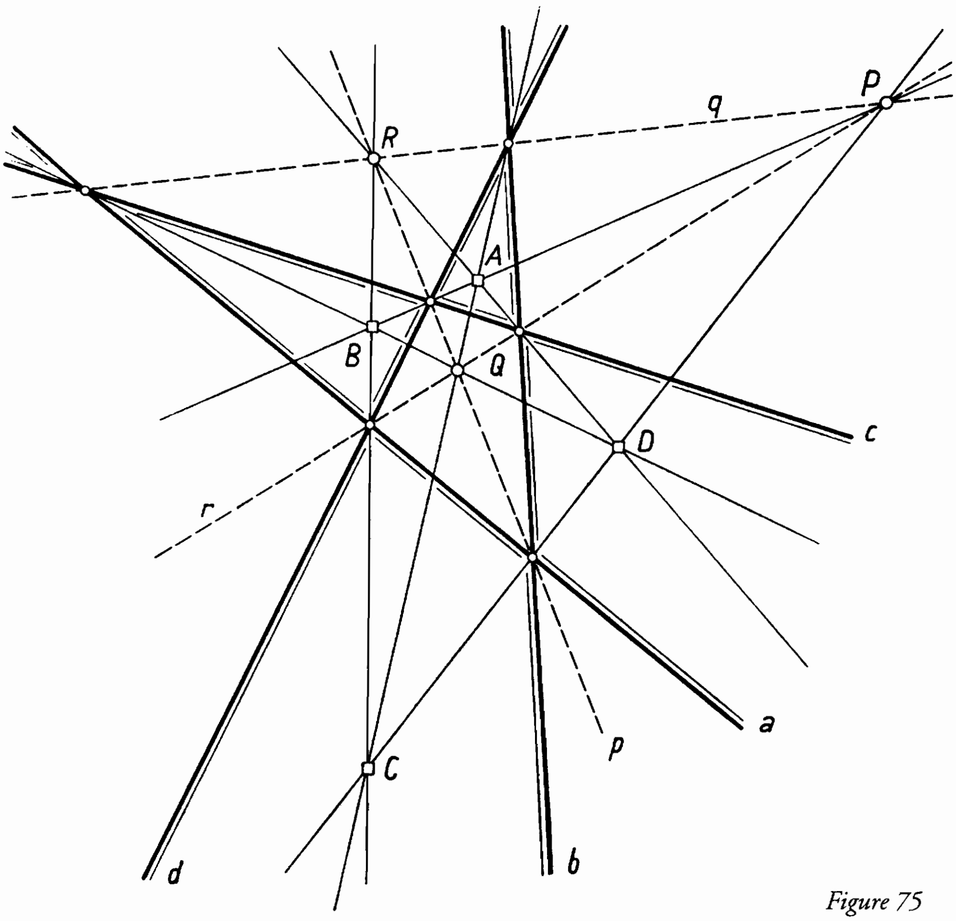


Figure 75

This last fact is a consequence of Desargues' Theorem. Take, for example, the 3-point given by [A] and look at its position in relation to the extra 3-side *pqr*. Each extra side contains exactly one vertex of [A]. Furthermore, [A] and the extra 3-side are in perspective with respect to line a:

$$\left.\begin{matrix} pqr \\ bcd \end{matrix}\right\} \ pb,\ qc,\ rd \text{ belong to } a$$

Hence the connecting lines of corresponding vertices, namely the line through $pq = R$ and bc, the line through $qr = P$ and cd, and the line through $rp = Q$ and db, belong to one point, which we call A. For $[B]$, $[C]$, $[D]$ we give outline arguments only:

$$\left.\begin{matrix} pqr \\ adc \end{matrix}\right\} \ pa,\ qd,\ rc \text{ lie on } b. \text{ So } (R,\ ad), (P,\ dc), (Q,\ ca) \text{ go through } B$$

$$\left.\begin{matrix} pqr \\ dab \end{matrix}\right\} \ pd,\ qa,\ rb \text{ lie on } c. \text{ So } (R,\ da), (P,\ ab), (Q,\ bd) \text{ go through } C$$

$$\left.\begin{matrix} pqr \\ cba \end{matrix}\right\} \ pc,\ qb,\ ra \text{ lie on } d. \text{ So } (R,\ cb), (P,\ ba), (Q,\ ac) \text{ go through } D$$

The points A, B, C, D could fittingly be called the middle points of the relevant three-sided cores.

On their own, the three extra sides p, q, r organize the point field into four three-sided domains. In each of these domains is inscribed just one of the cores $[A]$, $[B]$, $[C]$, $[D]$. Thus

A 4-side abcd determines firstly an extra 3-side qpr and secondly an accompanying 4-point ABCD. The vertices of the extra 3-side are the middle points of the three four-sided cores, and the vertices of the 4-point are the middle points of the four three-sided cores into which the point field is divided by the 4-side. Each three-sided core is inscribed in one of the four cores determined by the extra sides.

REMARK. It would be a pity if we were not able to dismiss the blandishments of mere intellectuality that these facts are self-evident. To be sure, these are elementary phenomena that can be deduced from **A** and **O** (without using continuity). But the ability to create such harmony-filled form pictures is not something given to the intellect. The latter can only link together step by step the abstract relationships. Such an image as that indicated in Figure 75 can have a meditative effect, and we can experience in it a form-creating spirit's "fresco" of fourness interwoven with threeness. Anyone who cannot (or will not) themselves work their way through to perceiving such a fundamental figure as something like a four-note chord with a related three-note chord, is simply a philistine. To them the poetry of mathematics is mere noise, while to others it can reveal a mystery.

If we look at Figure 75 more closely, we find yet more remarkable properties. The core $[A]$ belongs to exactly one of the domains determined by PQR; call it $[A]'$. This domain is itself inscribed in just one of the four cores determined by the 3-point BCD, namely the one containing A. Call this $[A]''$, so that in $[A]$, $[A]'$, $[A]''$ we have three cores of which the first is inscribed in the second and the second in the third. The three-membered chains $[B]$, $[B]'$, $[B]''$ and $[C]$, $[C]'$, $[C]''$ and $[D]$, $[D]'$, $[D]''$ of cores inscribed in each other arise corre spondingly.

The form determined by a complete planar 4-side *abcd* with its extra 3-side *pqr* together with the accompanying 4-point ABCD, we call the planar *Fundamental Harmonic Configuration.* The explanation for the name will be found in Part Three.

The planar Fundamental Harmonic Configuration consists of $4 + 3 + 6 = 13$ lines and $6 + 3 + 4 = 13$ points. As is immediately apparent, it is self-polar in the field. Through it, the point field is divided up into 3 times 8 plus 4 times 6, that is, 48 three-sided cores.

The *complete planar 4-point* also leads to the Fundamental Harmonic Configuration. Let A, B, C, D be its vertices (Figure 76). Drawing its six sides, we see that these intersect each other in three other points besides the vertices, which we label

$$P = (AB, CD), \qquad Q = (AC, BD), \qquad R = (AD, BC)$$

and call the extra vertices of the 4-point. These form the extra 3-point and its sides $p = QR$, $q = RP$, $r = PQ$ are the extra sides of the 4-point. In the case of the 4-side, we proved that its extra sides cannot belong to the same point. Similarly for the 4-point, it is clear that its extra vertices P, Q, R do not lie in the same line. In fact

Two opposite sides of a complete 4-point, and the two extra sides belonging to the point of intersection of those opposite sides, separate each other.

For the proof, we just need to apply correctly the thinking of the corresponding proof on page 111.

The configuration formed by the complete 4-point and its extra three-point possesses, as well as the 4 + 3 = 7 points already mentioned, a further six points of intersection, namely 2 on each of its extra sides. Sets of three of these six points belong to a line as a result of Desargues' Theorem. For example:

$\left.\begin{matrix} PQR \\ BCD \end{matrix}\right\}$ in perspective with respect to A, since PB, QC, RD belong to A

Hence the points of intersection of corresponding sides, namely (r, BC), (p, CD) and $(q,$ c$DB)$ lie in a line, which we call a.

The following are similarly in perspective:

$\left.\begin{matrix} PQR \\ ADC \end{matrix}\right\}$ with respect to B,

$\left.\begin{matrix} PQR \\ DAB \end{matrix}\right\}$ with respect to C,

$\left.\begin{matrix} PQR \\ CBA \end{matrix}\right\}$ with respect to D

This gives four lines a, b, c, d. These form the 4-point's accompanying 4-side. Once again we have produced the Fundamental Harmonic Configuration.

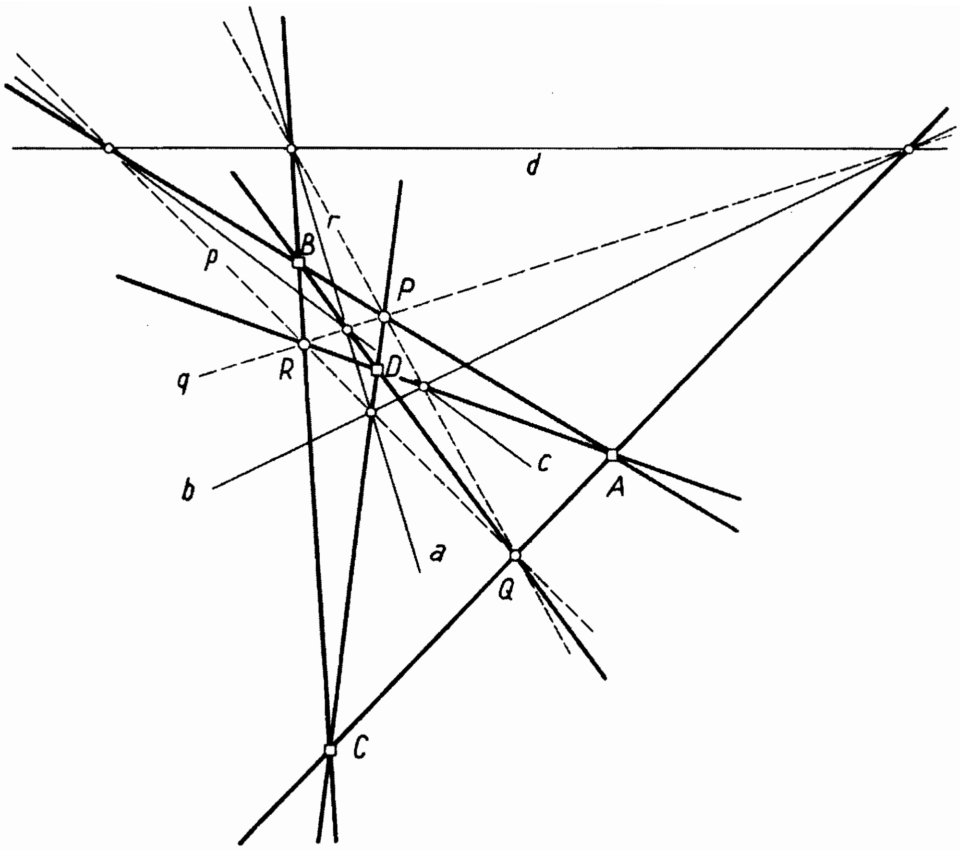


Figure 76

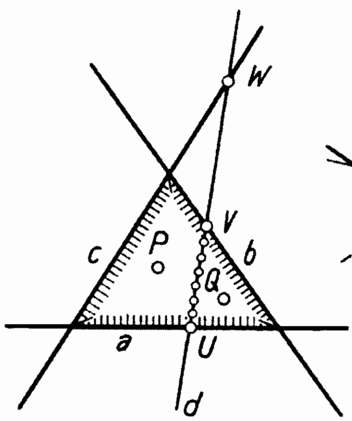
Corresponding to the dividing up of the point field by the 4-side we have the structuring of the line field by the 4-point. The former was conveniently done by considering the dividing up by three lines to give four cores, and examining the behaviour of the fourth line. This divides three of the cores into two pieces each. In order similarly to look at the structuring of the line field, we must consider the structuring of the latter by three points *A*, *B*, *C* into four surrounds (Figures 46 to 49) and see how, through the addition of a fourth point *D*, three of these surrounds are divided up into two surrounds each.

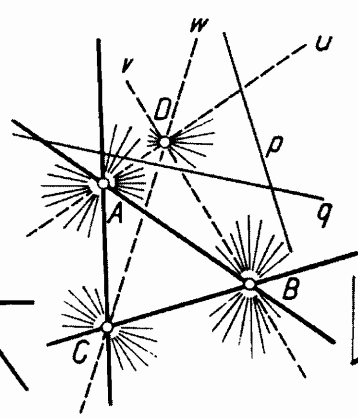
The division of a three-sided core *abc* by a line *d* is indicated in Figure 77. Figure 78 shows the corresponding division of a three-cornered surround *ABC* by the point *D*. Omitting detailed explanations, we place together some of the correspondences:

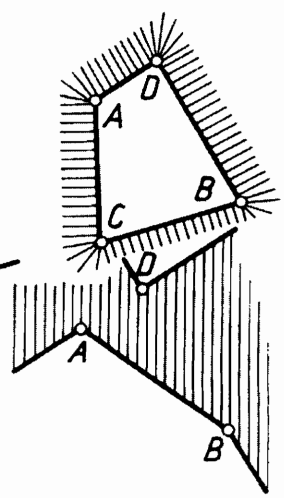
Point field	*Line field*
$U = ad,\quad V = bd,\quad W = cd$	$u = AD,\quad v = BD,\quad w = CD$
Line segments bounding the core. U and V belong to the boundary, W is an exterior point.	Angle fields bounding the surround. u and v belong to the boundary, w is an exterior line.
The segment (U, V) consisting of points of the core.	The angle field (u, v) consisting of lines of the surround.
(U, V) divides the core.	$u, v)$ divides the surround.
P and Q do not belong to the same part of the core because P cannot be brought to Q along any point path (moving point) inside the core without crossing the dividing segment.	p and q do not belong to the same part of the surround because p cannot be brought to q along any line path (moving line) inside the surround without crossing the dividing angle field.
In Figure 77, P belongs to the four-sided sub-core.	In Figure 78, p belongs to the four-cornered sub-surround.

In Figure 79 the two sub-surrounds are shown separately.

Remark. Much can be learned from this little example. Once again it highlights our one-sided mind-set. While every detail of the division of the core is easily grasped, the situation with the division of the surround must be made clear step by step. And yet there is no intrinsic reason to prefer one process to the other. A predominantly line-conscious (or plane-conscious) being would find the process of dividing the core more complicated than the division of the surround, which from this being's point of view would be immediately clear. What "onesided point-consciousness" describes is clear in this context. We should thus express nothing but the facts, lest we acquire a preference for something which. is not essential to them.







Figures 77, 78 and 79

Once the process of dividing a surround is made clear, it is no longer hard to work out the structuring of the line field by the 4-point, and finally to survey the planar Fundamental Harmonic Configuration also in the line field. The lines *p, q, r* are the "middle lines" of four-cornered surround regions, and the lines *a, b, c, d* are the "middle lines" of three-cornered surrounds.

The punctual Fundamental Harmonic Configuration can be developed correspondingly, starting from the complete 4-edge or 4-plane. This is referred to in the exercises.

Before expounding the situation with five dividing elements, we first explain a general property of the division of the field. Three lines divide the point field up into four three-sided cores. For this division—let it be called {3}—the number V of vertices is three, the number F of faces (cores) is four, and the number E of edges (i.e. segments) bounding the open spaces is six.

If a fourth line is added, it is divided by the first three into three segments. Each of these three segments divides up just one of the faces of {3} into two pieces. So the number F increases by 3 and the number of vertices also increases by 3, by the addition of the three points of intersection of the fourth line with the first three. But the number of edges E is increased by 2 times 3 equals 6 by the addition of the three segments on the fourth line and, on each of the first three lines, by just one edge being divided up into two edges by the fourth line. Thus for the {4} division by four lines we get

$$V = 3 + 3 = 6, \quad F = 4 + 3 = 7 \quad \text{and} \quad E = 6 + 6 = 12$$

If a fifth line is added to {4}, this is divided into four segments by the first four. Each of these four segments divides just one of the cores in {4} into two pieces. Thus the number F is increased by 4. With the addition of the four points of intersection of the fifth line with the first four, the number V of vertices also increases by 4. The number E of edges, on the other hand, increases by 2 times 4 equals 8 with the addition of the four segments on the fifth line and the division into two of exactly one edge on each of the first four lines. Hence for the {5} division by five lines it turns out that

$$V = 6 + 4 = 10, \qquad F = 7 + 4 = 11, \qquad E = 12 + 8 = 20$$

Reasoning thus, we can see how the numbers V, F, E increase:

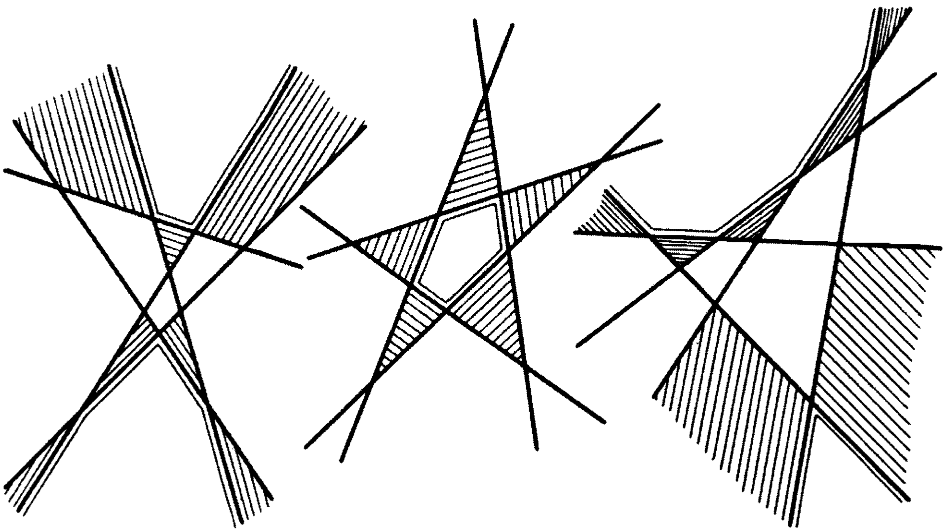
	V	F	E
{3} division by three lines:	3	4	6
A fourth line increases this by	3	3	6
{4} division by four lines:	6	7	12
A fifth line increases this by	4	4	8
{5} division by five lines:	10	11	20
A sixth line increases this by	5	5	10
{6} division by six lines:	15	16	30
A seventh line increases this by	6	6	12
{7} division by seven lines:	21	22	42
. . . .			

Hence for each division $V + F - E = 1$.

The {5} division produced by five lines shows a particularly beautiful symmetry. The complete planar 5-side given by five lines *a, b, c, d, e,* of which no three belong to the same point, has ten vertices, namely *ab, ac, ad, ae, bc, bd, be, cd, ce, de.* As a matter of fact,

Five lines divide the point field up into one five-sided, five four-sided and five three-sided cores, the last forming a connected ring round the five-sided core.

The five lines can be chosen quite arbitrarily, and the type of division that manifests will always be the same. Figures 80 to 82 show three examples. This is worthy of note since already with six lines at least two types of division arise.



Figures 80, 81 and 82

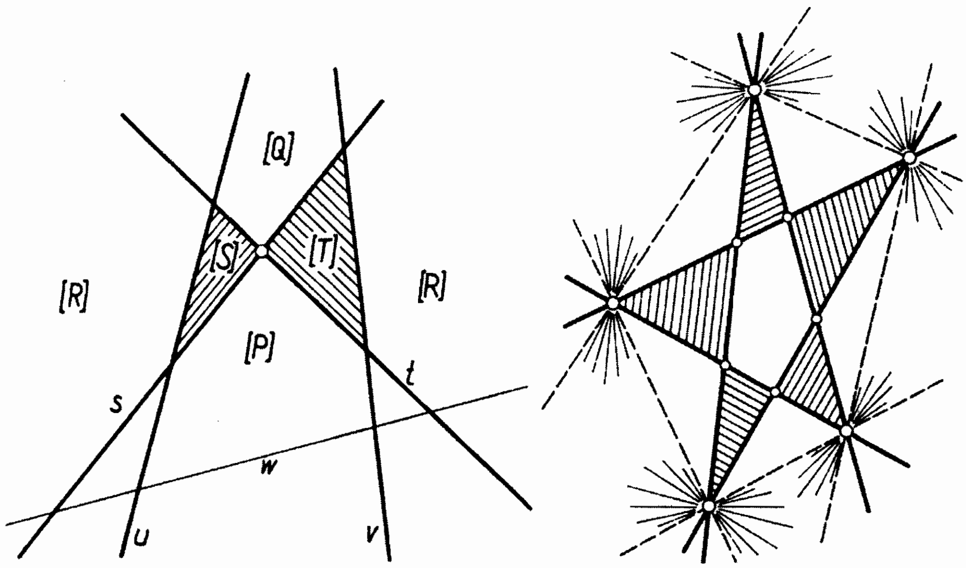
If we go along the boundary of a five-sided core in {5}, in one or the other sense the five lines are brought into a cyclic order as a result. Given five arbitrary lines of which no three belong to the same point, we can thus always label them *a, b, c, d, e* in such a way that (*a, b, c, d, e*) gives this cycle. Since {5} contains only one five-sided core, this cycle is uniquely determined.

Five lines uniquely determine a cycle (a, b, c, d, e).

The proof, that is, the derivation from **A** and **O**, is not as simple as one might expect. Of the various possible ways of showing it, we choose one that offers an insight into how the {5} division comes into being.

Proof. First we prove that the {5} division has at least two three-sided cores that have a common vertex and belong to the same angle field.

To see this, we start from a property of the {4} division. As the transition from the {3} division by three lines to the {4} division by the introduction of a fourth line immediately shows, {4} has the following properties. Firstly, in each edge a three-sided core meets a four-sided core; secondly, any two of the four three-sided cores have a common vertex and belong to the same angle field.



Figures 83 and 84

We now see what can happen with the introduction of a fifth line. This is divided by the first four into four new edges. Because of the first property mentioned above, two consecutive edges of the four divide one three-sided and one four-sided core of {4}. Thus the four edges on the fifth line divide successively a three-, a four-, a three- and a four-sided core of {4}. Therefore two three-sided cores are untouched by the fifth line. This shows that {5} contains at least two three-sided cores, that, by the second property of {4} mentioned above, necessarily have a common vertex and belong to the same angle field.

We now consider a {5} division produced by five arbitrary lines no three of which belong to the same point. We assume, as we may by what was just demonstrated, that there are two three-sided cores possessing the property mentioned. The four lines to which the boundary of these two cores belongs we label as follows (Figure 83): the two lines belonging to the vertex are called *s* and *t*, the other two *u* and *v*. Furthermore, let [*S*] be the core bounded by *s, t, u* and [*T*] the core bounded by *s, t, v*. In the {4} division by *s, t, u, v*, as well as the two three-sided cores, there are also two four-sided cores which come together in the vertex *st*; we call these [*P*] and [*Q*]; the third four-sided core in {4} we label [*R*]. We now have to see how the fifth line—call it *w*—can lie.

w intersects *u* and *v* and thus has a segment within the angle field (*u*, *v*) containing [*P*], [*Q*], [*S*] and [*T*]. Since *w* does not meet the last two, this segment cannot contain points from [*P*] and points from [*Q*]; thus it divides either [*P*] or [*Q*] necessarily into a five-sided and a three-sided domain, since the boundary segments of [*P*] and [*Q*] lying on *s* and *t* are not met. Thus we have shown that {5} contains a five-sided core.

It is easy to see that there is not another five-sided core. After all, one such could exist only by a division of [*R*]. But the opposite edges of [*R*] lying on *u* and *v* cannot be met by *w*. So *w* must intersect the other two opposite edges of [*R*] belonging respectively to *s* and *t*. This means that [*R*] is divided up by *w* into two four-sided cores.

{4} also contains, besides [*P*], [Q], [*R*], [*S*], [*T*], two other three-sided cores, which, because they border on [*R*], are each divided by *w* into one three-sided and one four-sided domain.

Thus {5} consists of one five-sided, five three-sided and five four-sided cores. The lines *s*, *t*, *u*, *v*, *w* can therefore be labelled *a*, *b*, *c*, *d*, *e* in such a way that (*a*, *b*, *c*, *d*, *e*) represents the cycle determined by the five-sided domain.

Correspondingly, for the complete planar 5-point with its ten sides, the following is true:

Five points divide the line field into one five-cornered, five four-cornered and five three-cornered surrounds, the last forming a connected ring round the fivecornered surround.

Here we must interpret the word "round" appropriately for the line field. The complete 4-point and the complete 4-side appear in the Fundamental Harmonic Configuration in a well-defined mutual equilibrium. The 5-side and the 5-point also stand in a harmonious relationship with each other, but this is of an essentially different nature. This is expressed in Figure 84:

Five of the ten vertices of a complete planar 5-side belong to the boundary of the five-sided core. The other five form a complete 5-point, five sides of which coincide with the sides of the 5-side, the other five sides of which belong to the boundary of the 5-point's five-cornered surround.

Thus the complete planar 5-side gives rise to an endless sequence of five-sided cores, in which each lies completely within its successor and each is framed by a ring of five three-sided cores. A 5-point gives a corresponding sequence of surrounds.

EXERCISES

1. Study the planar Fundamental Harmonic Configuration for the case when
   a) the lines *a, b, c, d* form a rectangle;
   b) the lines *a, b, c* form an equilateral triangle and *d* is the limit line.

2. Construct the planar Fundamental Harmonic Configuration
   a) out of *PQR* and *A*; *b*) out of *pqr* and *a*.

3. In a drawing, emphasize appropriately the twelve domains [*A*], [*A*]′, [*A*]″, [*B*], etc. of the Fundamental Harmonic Configuration, which fit into each other in threes.

4. A {3} structuring of the line field by *A, B, C* is given. Take a fourth point *D* and carry out in detail the transition from {3} to {4}.

5. What meaning do the numbers *V, F, E* introduced in the division of the point field have for the corresponding structuring of the line field?

6. Show that there are at least two types of {6} division.
   (To do this consider a {5} division and give a sixth line different positions in relation to the five-sided core in {5}.)

7. The Fundamental Harmonic Configuration consists of a complete 4-plane and a complete 4-edge, with a common extra 3-plane (3-edge). It contains planes and 13 lines. It is obtained by forming the projection of a planar Fundamental Harmonic Configuration. It arises in the cube and octahedron in a particularly regular form (Figure 85). Studying this in detail is recommended. Each of the 13 lines is perpendicular to its corresponding plane. The four cross lines of the cube form the complete 4-edge. The four planes through the middle point *M* of the cube perpendicular to the cross lines form the complete 4-plane. (Each such face goes through six edge middle points.) The six cross planes form the six faces of the complete 4-edge. The middle lines of the cube (connecting lines of *M* with the middle points of faces) form the extra 3-edge, and the middle planes of the cube determined by pairs of middle lines, the extra 3-plane. Finally, the connecting lines of the middle points of opposite edges of the cube are the six edges of the complete 4-plane. If the figure is intersected with any plane not belonging to bundle *M*, for example a face of the cube or of the octahedron, a planar Fundamental Harmonic Configuration is obtained.

8. Make the following clear to yourself. The complete planar 5-point gives rise to an endless sequence of five-cornered surrounds, each completely within the next and each supported by a ring of five three-cornered surrounds.

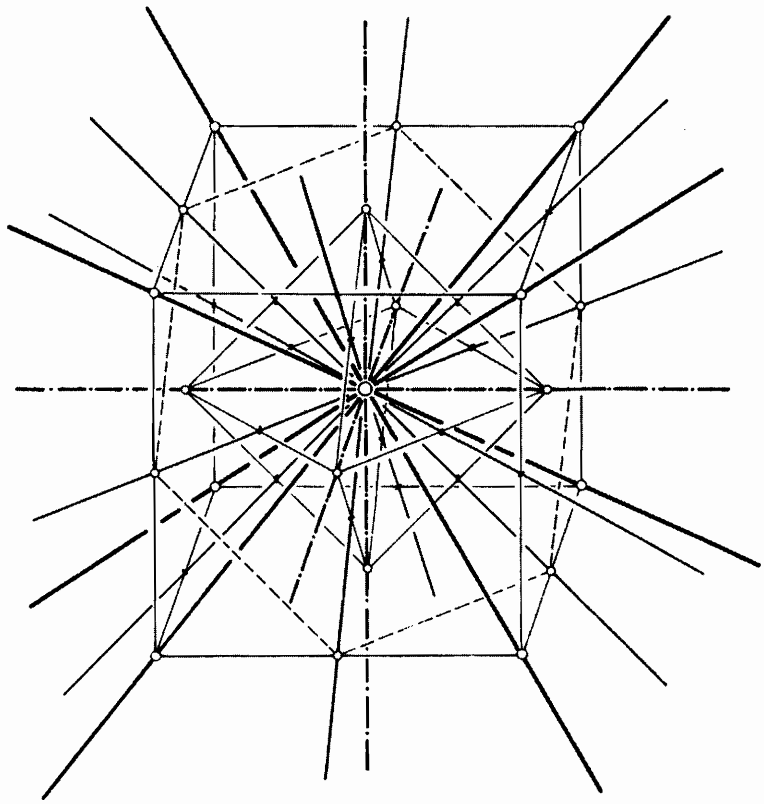


Figure 85

Chapter 13

Two Basic Exercises for Understanding Counterspace

In this chapter we present two exercises that can lead to a clearer picture of counterspace. This will not be a question of describing a theory so much as the constructing of certain forms which will serve to broaden a point-biased mindset. Working through these exercises gives us the means to see diverse spatial forms in the realm of nature in a new way.

We restrict ourselves first to the geometry of the field. We begin with a simple figure which, though apparently rather insignificant, is particularly well suited to the exercise mentioned. The latter consists in developing the corresponding "counter-figure:" the figure that, in the field, is polar to the figure we started with.

In a field, suppose two points A and B are given (Figure 86), and let the connecting line AB be called c. In each line pencil A and B we choose a sense of running through the lines, and in each we assume that there is an arbitrary sequence of lines;[7] let them be numbered in the chosen running-through sense. Let the sequence of lines in pencil A be 1, 2, 3, 4, . . . and the sequence in pencil B be $1'$, $2'$, $3'$, $4'$, . . . If we look at this figure against the background of the point field, then the lines of the pencils are seen to produce a net of four-sided core domains. The meshes of this net are contained in a large mesh whose boundary segments belong to the first and last lines of the chosen sequences (in Figure 86 this large mesh is bounded by 1, $1'$, 5, $5'$). The mesh structure is immediately clear, but we shall nevertheless indicate exactly how the interior of a mesh can be characterized. Any two lines of the pencil A other than c, together with any two lines of the pencil B other than c, constitute a complete 4-side (for example the 4-side determined by 2, 3 and $3'$, $4'$, which is emphasized in Figure 86). A and B are two of the six vertices of the 4-side, c is an extra side. The point of intersection of the two other extra sides is the "middle point" of one of the three four-sided core domains created by the 4-side. This domain is the mesh formed by the four lines in question that we have in mind. Each 4-side formed by a pair of lines from each of the pencils A and B determines *a* unique interior mesh with respect to c. Each point P of such a mesh is characterized by the fact that it is separated from each point Q on c, other than A and B, by the two mesh sides of pencil A and by the two mesh sides of pencil B.

[7] These are thought of as finite sequences to begin with, but the statements can be suitably extended to infinite sequences. Ed.

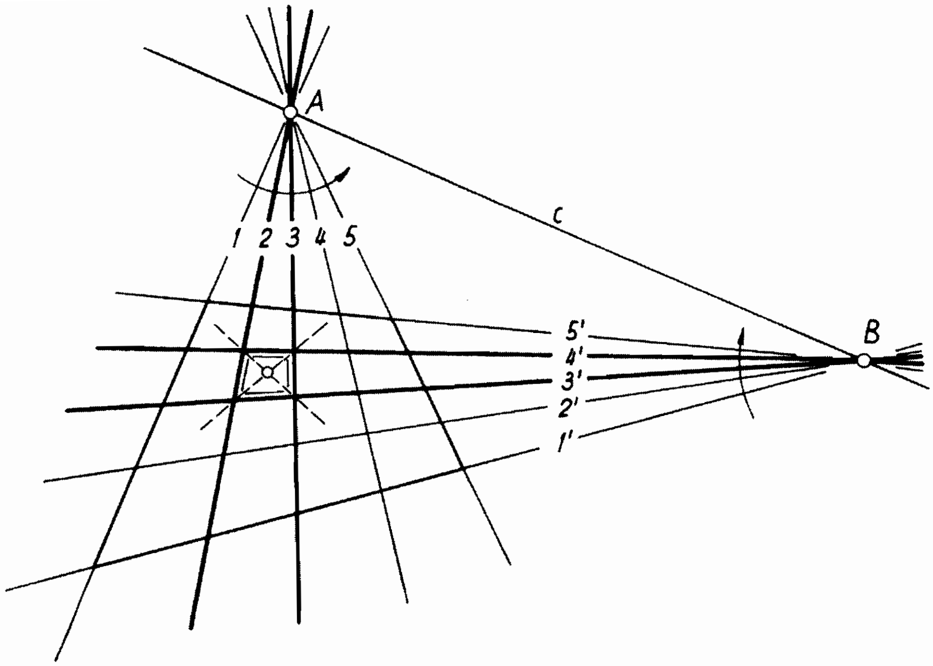


Figure 86

We now develop the polar figure in the field, the counter-figure. Suppose *a* and *b* are two lines and let *C* be their point of intersection. In each point range *a* and *b*, a sense of running through the points is chosen, and in each a sequence of points, 1, 2, 3, 4, . . . and 1′, 2′, 3′, 4′, . . . respectively, is numbered in the running-through sense in question (Figure 87).

In Figure 86 the two sets of five pencil lines produce, besides *A* and *B*, twenty-five more points of intersection, the vertices of the meshes. Correspondingly, in Figure 87 we have to draw, besides the lines *a* and *b*, twenty-five more lines connecting the chosen points. This produces a tangle not easy to comprehend at first.

Just as before we embedded the figure in the point field, so now we shall consider the counter-figure against the background of the line field. To the point-meshes of the former correspond surround regions of the latter. At first, faced with the question of how these surround regions come about, our power to visualize is somewhat at a loss. Here we are helped by the characterization given above of a mesh's interior point, the content of which we polarize. If we take any two points of range *a* and any two points of range *b* (in Figure 87 the points 2, 3 and 3′, 4′ are chosen as an example), then these determine a complete 4-point of which *a* and *b* represent two of the six sides and *C* an extra vertex. The connecting line of the other two extra vertices is the "middle line" of one of the three four-cornered surrounds which is created in the line field by the 4-point. This surround (shown dark grey in Figure 87) must properly be described as the 4-point's interior with respect to *C*.

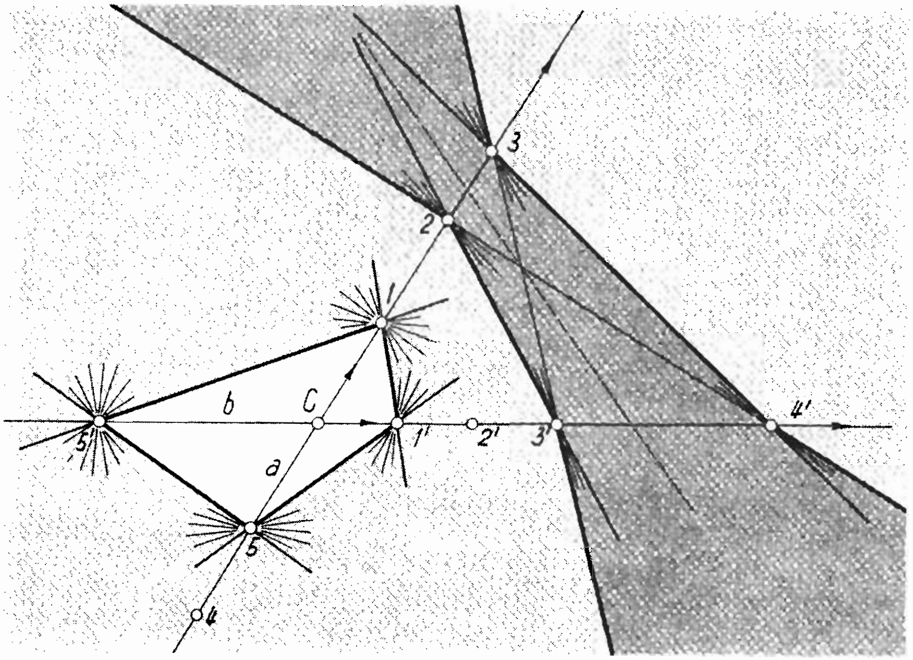


Figure 87

Each line p of this interior region is characterized by the fact that it is separated from every line q through C, other than a and b, by the two surround vertices belonging to range a, as well as by the two surround vertices belonging to range b. The interior lines are thus easy to determine.

The single line-meshes are all contained in the surround formed by the first and last points of the chosen sequences (in Figure 87 by 1, 1′, 5, 5′). Figure 88 shows the same as Figure 87 but with a different positioning of the points in relation to the limit line of the plane. (Both figures show the boundary angle fields of the surrounds concerned.) Whether the limit line belongs to the interior or exterior of the surround region concerned depends on the positioning of the surround vertices.

It is not difficult now to form a clear picture of the individual line-meshes.

We can say in an obvious sense that the original figure, Figure 86, shows points that, with regard to the forming of point-meshes, cannot be reached, namely the points on line c. A point of c cannot be caught in a mesh by the given construction. Polar to that, the counter-figure exhibits unreachable lines, namely the lines through C, which by the construction can never become interior lines of a line-mesh.

The figure in the point field and its counter-figure in the line field are mathematically equivalent in every respect. If we consider them purely in terms of the formal relationships they show, then they do not differ at all. To each relationship in the one figure, for example "$AB = c$," "point of intersection 23′," etc., there corresponds precisely one relationship in the other: "$ab = C$," "connecting line

23′," etc. Yet for our consciousness, figure and counter-figure represent two completely opposite forms, the thing to be stressed being that one form can be grasped immediately while the other can only be comprehended with some effort.

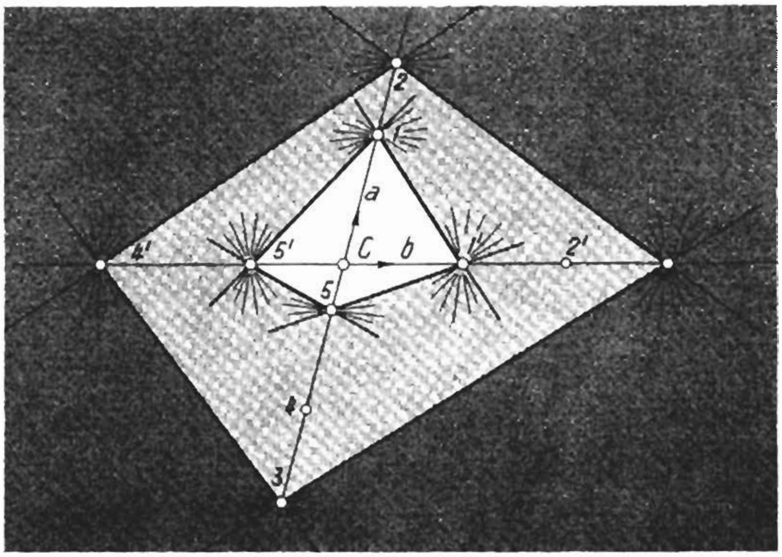


Figure 88

The figure in the point field with the point-meshes is applicable in the most diverse branches of natural science, while the corresponding form in the counter-field has not really been considered seriously until now. The following question alone should give cause for reflection: Does nature really avail herself only of the one form and leave the other, mathematically equivalent one unused? Is the truth perhaps that the natural scientist of today looks at phenomena through monochrome spectacles? If we take account only of central forces, that is, forces for which points of application can be given, then the point field is the space to use. The counter-figure, on the other hand, calls for the investigation of forces for which one cannot speak of a point of application. In the plant world we see forces at work—in the shooting of seeds, in bud formation, in the unfolding of the leaf—that are not merely central.

REMARK. The objection that the figure in the point field is simpler than the counter-figure in the line field is evidently not justified. We have two people discuss this, *X* having developed a bias towards point-consciousness, his partner *Y* a preference for line-consciousness:

X: The lines in one of your surrounds produce a muddle. Compared to that, the points of my point-mesh are neatly separated from each other.

Y: I don't see that. To me, the points inside one of your meshes are confusing and hard to grasp. After all, any two of them have a line in common, and the order shown by these connecting lines hardly appears simple to me!

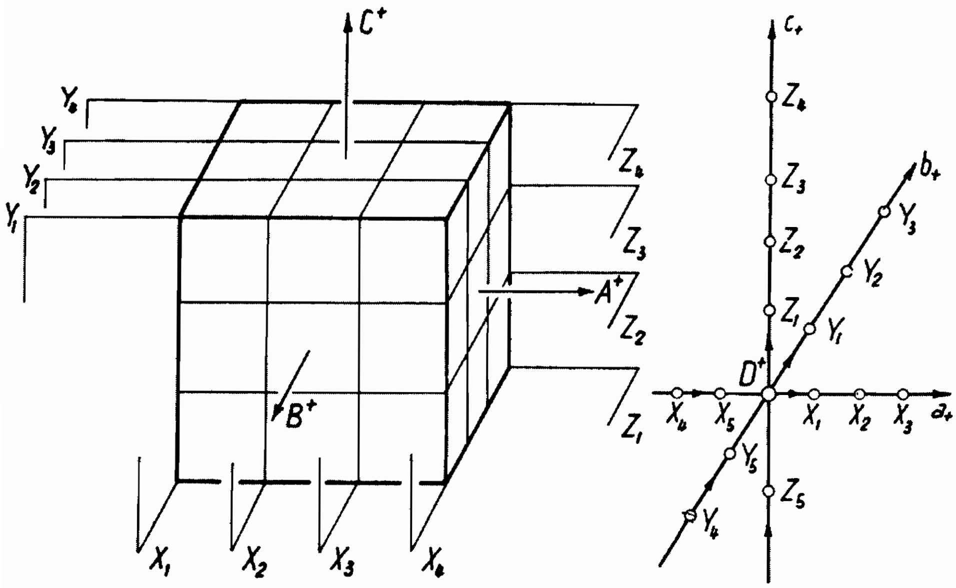
X: What you say seems to be far more true of the lines of your surrounds.

These have multifarious intersections, and their mutual overlapping makes for obscurity.

Y: Can't you see it's just the opposite? It is your structure that lacks clarity, because of the connecting lines of what you call the 'neatly separated points' of your mesh!

And so on.

Now that we have understood, using mesh formation, how things stand in field and counter-field, we go over to the development of an exercise that does the corresponding thing for space and counterspace.



Figures 89 and 90

To this end we recall the formation of the hexahedron (Figure 16). In a plane D^-, through each of three lines a_-, b_-, c_- with no point common to all three, we laid two planes. The three pairs of planes formed a hexahedron. Now we imagine that in the plane sheaf a_- there are not just two planes—apart from the plane D^-— but more planes X_1, X_2, X_3, X_4 . . .; likewise in the sheaf b_- there are planes Y_1, Y_2, Y_3, Y_4 . . . ; and in sheaf c_- there are planes Z_1, Z_2, Z_3, Z_4 . . . Suppose the planes are enumerated in their natural order. The enumeration thus fixes a sense of running through the planes in each of the three sheaves. These planes obviously create a number of hexahedral cells that constitute a subdivision of a large hexahedral cell. The latter is determined by the first and last planes of the sequences X_1, X_2, X_3, . . ., Y_1, Y_2, . . ., Z_1, Z_2, . . . If each sequence consists of three, four, five, . . . planes then the large hexahedron is divided up into 8, 27, 64, . . . cells, respectively. Three pairs of consecutive planes form a cell. Four edges of such a cell run through each of $A^+ = b_-c_-$, $B^+ = c_-a_-$ and $C^+ = a_-b_-$.

All this can easily be seen from, for example, a regular hexahedron (Figure 89). In this case the lines a_-, b_-, c_- carrying the sequences of planes lie in the limit plane.

The interior of a cell can be described as follows. For example, suppose the cell is the one formed by the pairs X_1, X_2 and Y_3, Y_4 and Z_2, Z_3. Then each point P inside the cell is separated from each point Q of the plane $D^- = a_-b_-c_-$, except for the points of the lines a_-, b_-, c_-, by all three pairs X_1, X_2 and Y_3, Y_4 and Z_2, Z_3. The point $X_1Y_3Z_2$ is a vertex, the point $X_2Y_4Z_3$, its opposite vertex.

Notice how easily we can grasp this way of partitioning point space. Among other things, this is of course connected with the fact that we continually come across this kind of cell formation in daily life.

The second exercise mentioned consists in being able to understand the polar formation in counterspace equally clearly. From Chapter 5 we know that this is going to involve a number of connected octahedral surrounds of planes. In the case of the hexahedral cells, the plane $D^- = a_-b_-c_-$ played a special role as bearer of unreachable points. In the same way, in the polar form a point D^+ takes on a special role as bearer of unreachable planes.

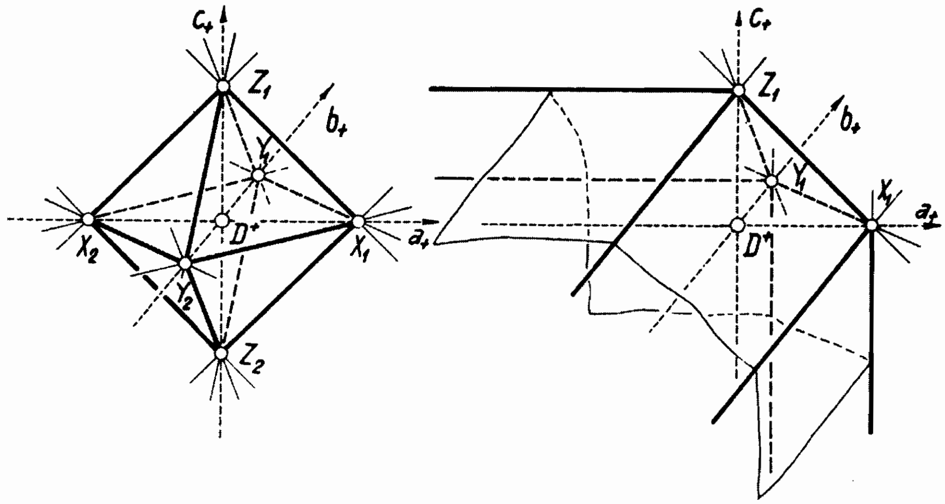
With the hexahedral division we started off with three lines a_-, b_-, c_- of a plane D^-. This time we shall take as a basis three lines a_+, b_+, c_+ of a point D^+, lines that have no plane in common. Before, we chose a sequence of planes in each of the sheaves a_-, b_-, c_-. Now, polar to this, we have to choose in the point ranges a_+, b_+, c_+ naturally ordered sequences of points X_1, X_2, X_3, . . . and Y_1, Y_2, Y_3, . . . and Z_1, Z_2, Z_3, . . . respectively (Figure 90). Choosing sequences of, say, four points each, this determines 27 octahedral surrounds of planes, that, polar to the cells, are joined in the simplest way possible and together fill a large octahedral surround. Faced with the task of seeing all this clearly before us, we feel even more helpless than in the case of plane geometry. But with the right procedure, the difficulty can be overcome.

A cell was determined by three pairs of consecutive planes. Polar to this an octahedral surround is given by three pairs of consecutive points, these points constituting the six vertices of the surround. As an example, we take the points X_1,X_2 and Y_1, Y_2 and Z_1, Z_2, say. The plane $X_1Y_1Z_1$ represents one face of the octahedron, the plane $X_2Y_2Z_2$ the opposite face (Figure 91). All eight faces are given by the planes

$X_1Y_1Z_1$,	$X_2Y_1Z_1$,	$X_1Y_2Z_1$,	$X_1Y_1Z_2$,
$X_2Y_2Z_2$,	$X_1Y_2Z_2$,	$X_2Y_1Z_2$,	$X_2Y_2Z_1$,

where a plane of the first row and the one below it in the second row are opposite faces. The three-sided point domains bounding the octahedron must be chosen so as to create an octahedral core *in point space* with D^+ in its interior.

Figures 91, 92 and 93 show three different situations in relation to the limit plane of space, which for our mental picturing is a unique plane. Suppose we set off from D^+ and go either via X_1 to X_2, or via Y_1 to Y_2, or via Z_1 to Z_2. In the case of Figure 91, we would cross the limit plane; in Figure 92, X_2, Y_2, Z_2 are themselves limit points and the limit plane forms a face of the octahedron; in Figure 93, starting from D^+, the path in the given sense does not go through the limit plane. One should be certain about the octahedral cores in each case. Two three-sided planar boundary domains meet in each edge; for example, domains in the planes $X_1Y_1Z_1$ and $X_1Y_1Z_2$ in the edge X_1Y_1, domains in the planes $X_2Y_2Z_2$ and $X_2Y_2Z_1$ in the edge X_2Y_2, and so on. With practice, the relevant octahedral cores are quickly recognized. The sought-after octahedral surround is the region of all the planes that do not meet that core; it is the surround flowing around it.



Figures 91 and 92

To obtain all the planes belonging to the interior of the octahedral surround, we can proceed as with the above-described cell formation in the polar situation as follows. We choose a point on a_+ between X_1 and X_2 with respect to D^+, a point on b_+ between Y_1 and Y_2 and a point on c_+ between Z_1 and Z_2, the concept "between" always being in relation to the point D^+. The plane determined by three such points is an interior plane. Each interior plane P is separated from each plane Q through D^+ not belonging to any of the lines a_+, b_+, c_+, by the pair X_1, X_2, the pair Y_1, Y_2, and the pair Z_1, Z_2. The totality of planes P with this property forms the interior of the octahedral surround, which envelops the above-described octahedral core with D^+ hidden in its interior.

In Figure 91 the limit plane is an interior plane of the surround; in Figure 92 the limit plane of space is a boundary element of the surround; in Figure 93 the limit plane does not belong to the surround's interior.

Once we have clearly understood the formation of the surround produced by the three pairs of points X_1, X_2, Y_1, Y_2, and Z_1, Z_2, it is not difficult to bring to consciousness the surround determined by any three pairs of points of the sequences on a_+, b_+, c_+, as well as the positions of the individual surrounds in relation to each other.

However far we go with the cell formation using the sheaves a_-, b_-, c_-, there will always be a neighborhood of the plane $D^- = a_- b_- c_-$ left uncovered. Polar to this, however far the forming of surrounds using the ranges a_+, b_+, c_+ is allowed to continue, a neighborhood of the point $D^+ = a_+ b_+ c_+$ will always exist as virgin territory.

What we said about the corresponding construction in the line field holds true to an even greater degree for the formation of surrounds in plane space. When one thinks of the extraordinary importance of surround formation in the organic realm, the fact that such forms are given—even if to begin with only in geometric pictures—in the very first principles of space can be an enormous comfort. To be able actually to use this surround formation requires a thoroughgoing expansion of consciousness. Thanks to normal school education we know how to use point space in our mental pictures. In the same way we shall have to learn, through a new schooling, to implement counterspace as well.

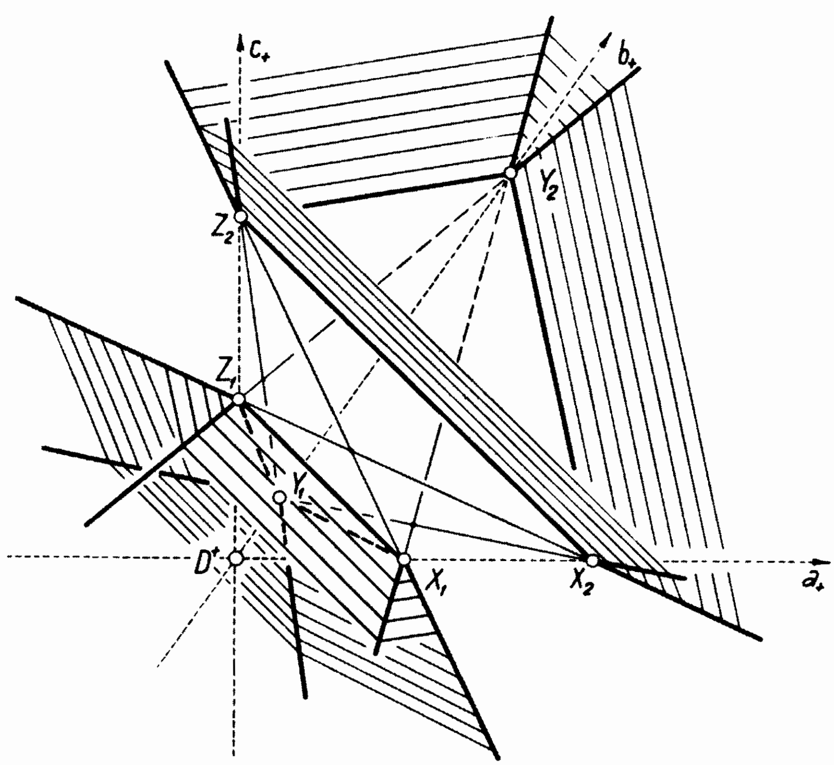


Figure 93

REMARK. In Figures 86 and 89 we have an immediate sense for whether the subdividing into meshes and sub-cells respectively is done more or less equally. We can also acquire the corresponding feeling for measurement in counterspace.

If, in pencils A and B of Figure 86, scales are introduced that assign to each line of the pencil an ordinal, then we have a coordinate frame in which each point of the field not lying on c is determined as the point of intersection of a line in A with a line in B. If scales are fixed in the plane sheaves carried by a_-, b_-, c_- (see text relating to Figure 89), then we obtain a coordinate frame in which each point of space not lying in D^- is determined as the point of intersection of three planes, namely a plane from each of the sheaves a_-, b_-, c_-.

Polar to this, by introducing suitable scales on the ranges a and b (Figures 87 and 88) or on the ranges a_+, b_+, c_+ (Figures 90 to 93), we obtain a coordinate frame for the line field or for plane space, respectively.

In Chapter 21 we shall see how scales in keeping with the essence of spatial relationships can be introduced into the basic forms.

EXERCISES

1. Develop Figures 86 and 87 in detail with four lines through each of A and B and four points on each of a and b, respectively. The nine resulting point-meshes/line-meshes, as well as their boundary segments/angle-fields, should be represented in a suitable way.

2. Develop the forms of plane and point geometry polar to Figures 86 and 87. (This can be done directly by forming the appropriate projections.)

3. The formation of eight cells determined by three sets of three planes X_1, X_2, X_3 and Y_1, Y_2, Y_3 and Z_1, Z_2, Z_3 in sheaves a_-, b_-, and c_- respectively is easy to see. Develop the counter-form consisting of eight octahedral surrounds.

4. The planes X_1, X_3 and Y_1, Y_3 and Z_1, Z_3 in the previous exercise form the hexahedron that is made up of the eight cells. Now let the planes X_2, Y_2, Z_2 run through this hexahedron's middle point M^+. Construct the polar form in counterspace as well. (The simplest way to see this is to use a regular octahedron.)

Chapter 14

The Six-structuring of Space

Five lines of a plane, no three of which go through the same point, always structure the plane as point field in the same way, no matter how the lines are chosen; this was shown in Chapter 12. They determine a cyclic order that becomes evident in the boundary of the ever-present five-sided core. A ring of five three-sided cores surrounds this core.

The dividing up of point space by five planes has already been outlined. We now take a further step and study the structuring of space by six planes, no three of which belong to the same line, and no four of which belong to the same point—in other words, the complete spatial 6-plane. This form is extremely interesting and full of impressive features. To understand this form by realizing it in clear mental images is the aim we set ourselves. Such a 6-plane always produces the same structuring of point space, no matter how the planes are chosen. The six planes evince a particular cyclic ordering, shown by a closed ring of six tetrahedral cores. Two neighboring cores of the ring "peak" each other: that is, they have in common a vertex and the edge lines radiating from it. The structuring always contains two six-faced principal cores with purely four-sided boundary domains, and it also contains two principal points with special properties. This is just a first indication of the six-structuring's particularities.

The six planes, which we shall call 1, 2, 3, 4, 5, 6 for short, have 15 lines of intersection:

12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56

These are the edges of the complete 6-plane. Its twenty vertices are the points of intersection of the six planes taken three at a time, as follows:

123, 124, 125, 126, 134, ... , 456

The 6-plane structures point space into 26 cores, as we shall see.

To construct the form, start by drawing a tetrahedron. Then intersect the tetrahedron with a plane; this gives us a Desargues' Configuration (Figure 60). Now intersect the tetrahedron with a second plane, producing a second Desargues' Configuration. Lastly, determine the line in which these two planes (which intersect the original tetrahedron) intersect each other. The drawing is easier than one would at first imagine.

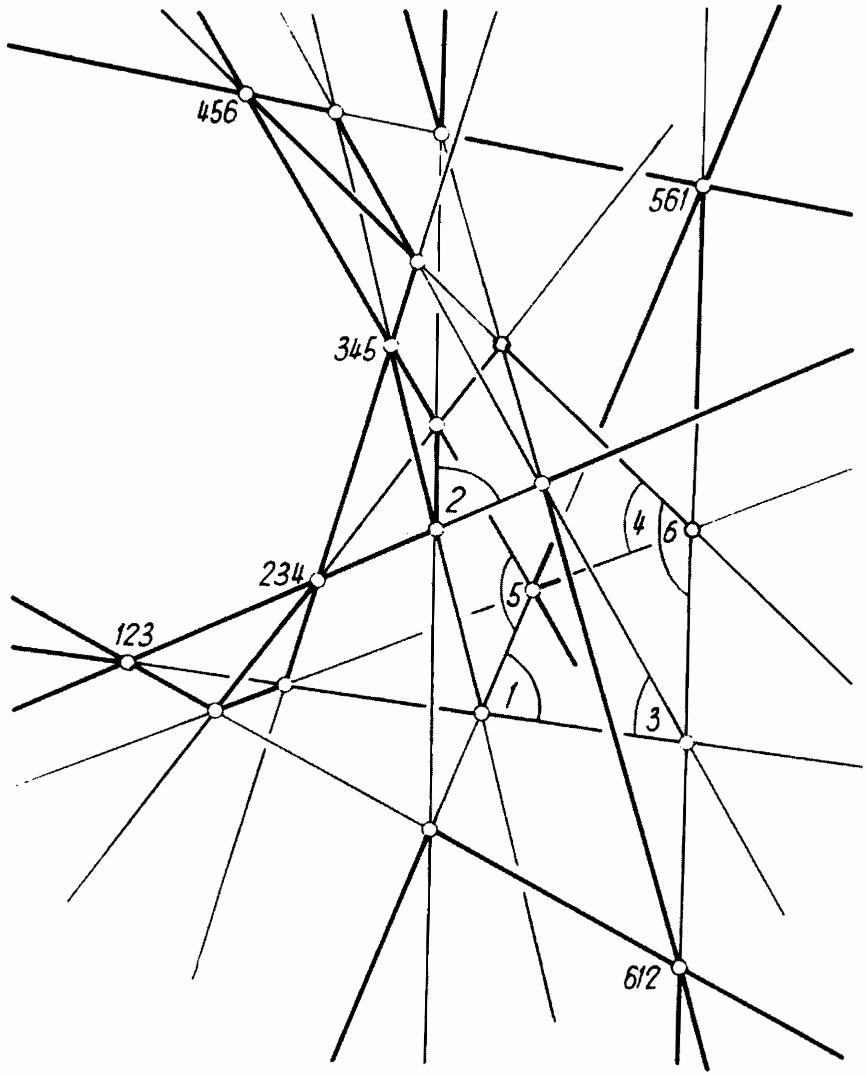
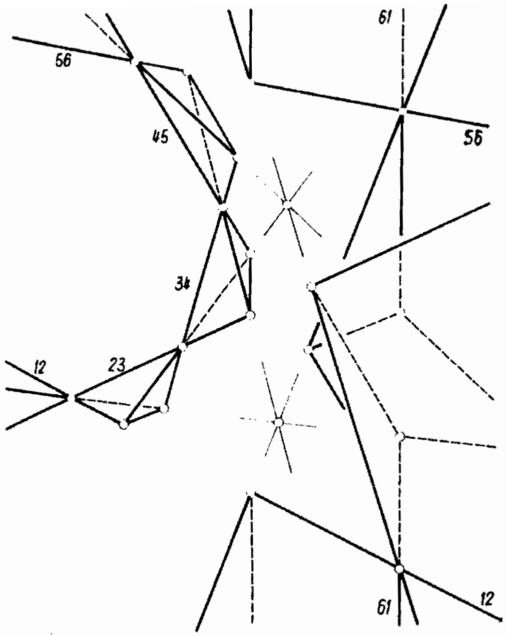
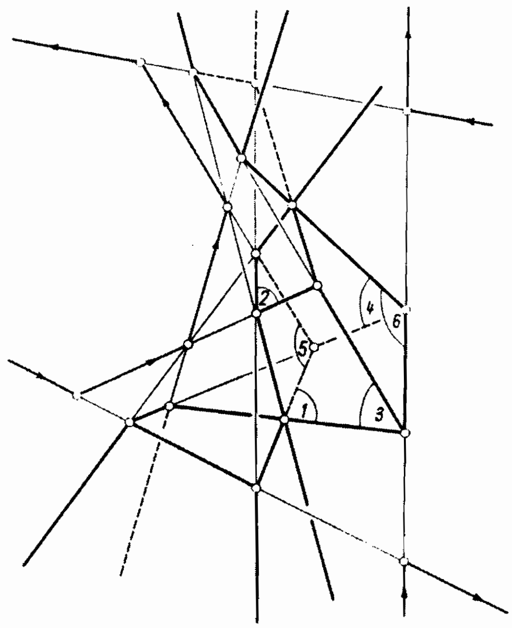


Figure 94

In Figure 94, consider the tetrahedron with apex 345 and base—thought of as horizontal—to the left of and adjoining the number 1. This tetrahedron has been intersected with plane 2 and with plane 6.

In Figure 97, choose the tetrahedron with an edge running from left to right horizontally at the top and its opposite edge below going back and to the right from vertex 612 at the front. This has been intersected with the two planes 3 and 4.





Figures 95 and 96

The cycle of six planes is found as follows. First look for the tetrahedral cores: there are six of them, and they successively "peak" each other. Then consider the six tetrahedron edges that connect pairs of the peak vertices in which two tetrahedral cores meet. These edges form a closed path called *the principal path.* Label the lines containing the edges of the principal path 12, 23, 34, 45, 56, 61, starting arbitrarily, and the vertices of the path 123 (namely the intersection of 12 and 23), 234 (the intersection of 23 with 34), 345, 456, 561, 612. The planes

1=(61,12),	2=(12,23),	3=(23,34),
4=(34,45),	5=(45,56),	6=(56,61),

taken in the sequence 123456 give the required cycle (123456).

In Figure 94, three of the six tetrahedral cores lie entirely in the finite. One core extends from vertex 123 (bottom left) leftwards over the limit plane and from the right back to a three-sided domain to be found in plane 6. And one core extends from point 456 as apex (top left) to a three-sided domain in plane 1. Finally, a tetrahedral core extends, from the edge at the top right, upwards over the limit plane and from below back to the edge on 12. Figure 95 shows, on a smaller scale, the ring of tetrahedra in Figure 94.

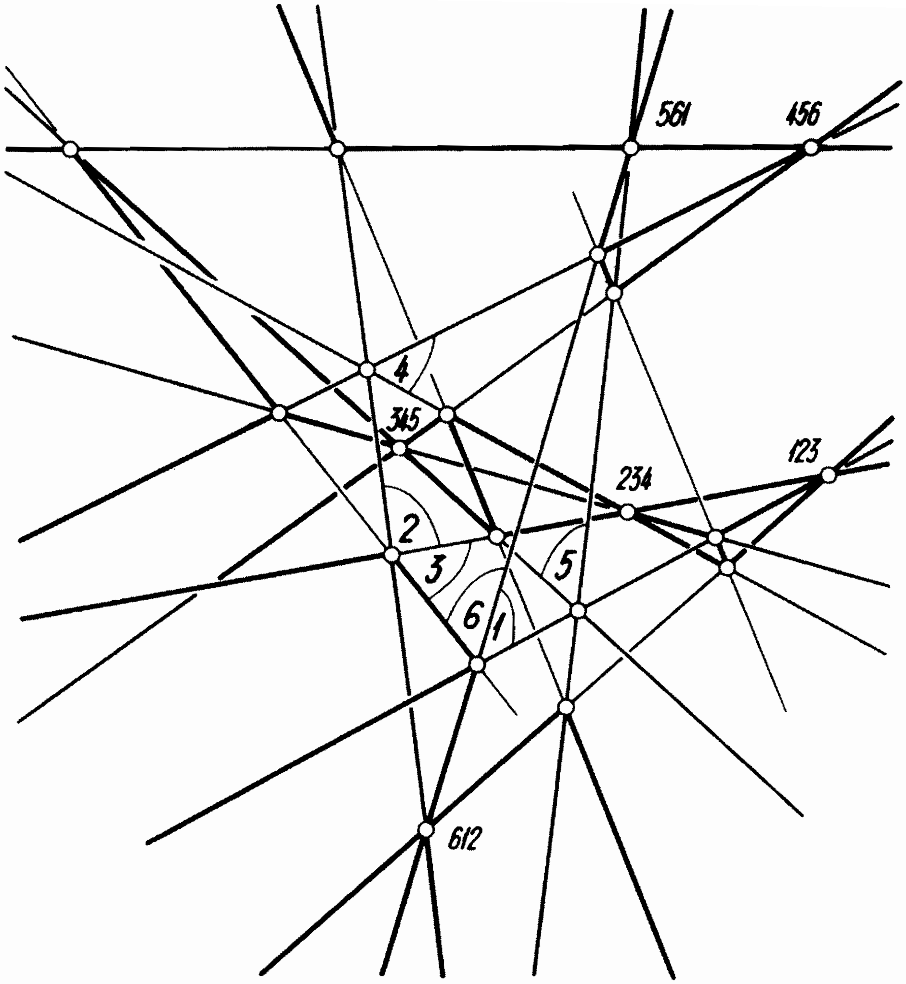


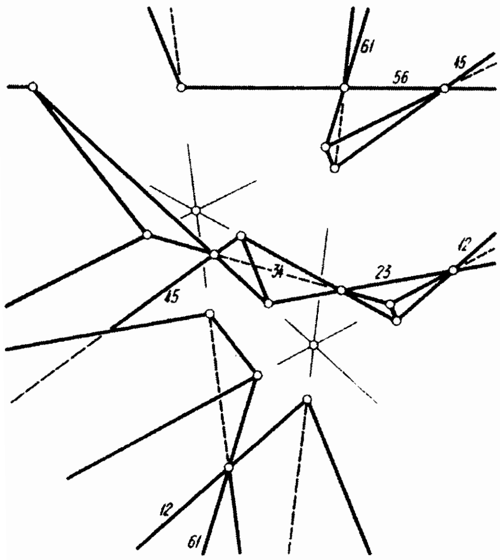
Figure 97

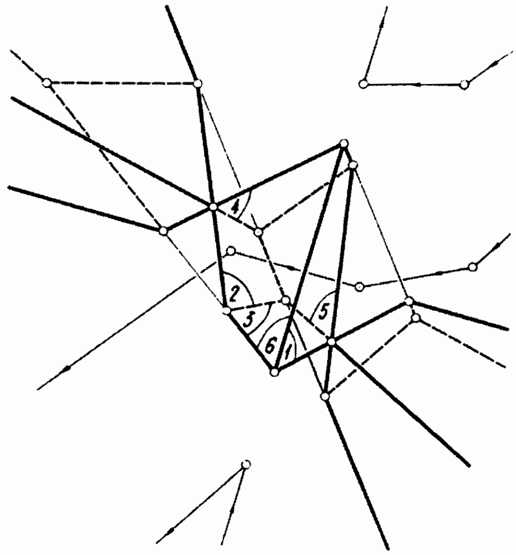
In Figure 97, too, three of the six tetrahedral cores are completely in the finite. Figure 98 shows the ring reduced in size.

Of the twenty vertices of the complete 6-plane, eighteen appear as vertices of the six tetrahedral cores. The other two, the principal points of the 6-plane, are the points of intersection 135 and 246. The complete 6-plane thus has two vertices that are qualitatively dfferent from the other eighteen.

The principal points are opposite vertices of two six-faced cores with foursided boundary domains; these two cores we call the principal cores. Planes 1 and 2 contain opposite faces of one of the principal cores, as do planes 3 and 4, and 5 and 6. The second principal core, linked with this one, has opposite faces in planes 2 and 3, 4 and 5, 6 and 1. The remarkable positions in relation to each other of the two principal cores deserve scrutiny.

In Figure 96, the principal cores of Figure 94 are shown reduced in size; the same is done in Figure 99 for the principal cores of Figure 97.





Figures 98 and 99

The opposite faces of one of the principal cores intersect in lines 12, 34, 56; the opposite faces of the other intersect in lines 23, 45, 61.

Of the twenty vertices of the complete 6-plane, fourteen (twice eight minus two) are claimed as vertices of the two principal cores. The other six are precisely the vertices of the principal path.

Once the cycle of six planes has been found, everything else appears in the most beautiful order. For example, successive tetrahedral cores are formed from the planes

1234, 2345, 3456, 4561, 5612, 6123

On each line of the configuration there are four points. The cycle (123456) gives them in their natural order. For example, the four points on 12 arise in their natural order as the intersections of 12 with the planes 3, 4, 5, 6, respectively; similarly, the four points on 24 are the intersections of 24 with the planes 1, 3, 5, 6.

In the configuration there are six planar structurings, each composed of five lines. That is, each of the six planes is intersected by the other five in five lines, which determine, in the plane in question, a five-sided domain with a ring of five three-sided domains.

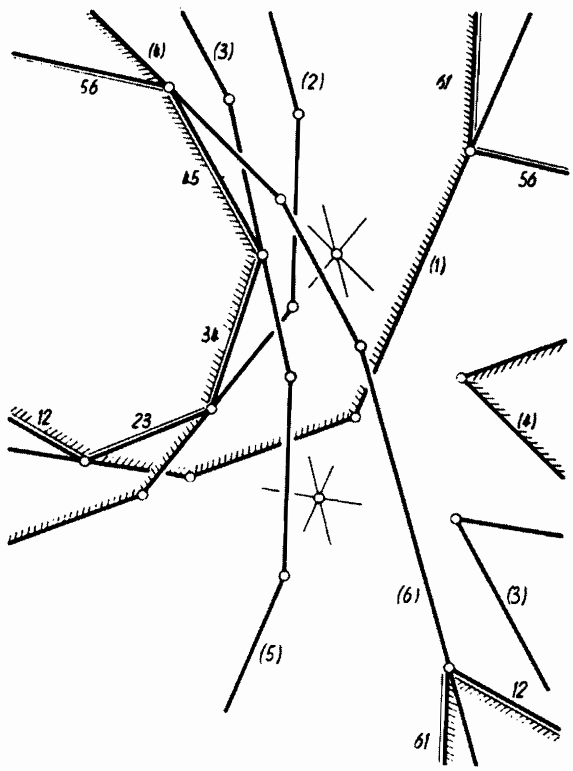


Figure 100

The five-sided domain in plane 1 is obtained as follows: We take the lines 12, 13, 14, 15, 16 in that order, giving the cycle (12, 13, 14, 15, 16) of the five lines; from it the corresponding domain is easily determined. Similarly, we have, for example, in plane 3 the cycle (31, 32, 34, 35, 36), in plane 4 the cycle (41, 42, 43, 45, 46), and so on. To simplify identification, Figure 100 shows, reduced in size, the six five-sided domains (1), (2), . . . , in planes 1, 2, . . . respectively, of the configuration of Figure 94 (domains (1) and (4) are slightly emphasized). These are joined together in a characteristic way, in that pairs of them have an edge of the principal path in common: the five-sided domains (1) and (2) have the path edge on line 12 in common, (2) and (3) the edge on 23, and so on. Each such five-sided domain is the common boundary of two six-faced cores. If we differentiate between front and back of these domains, then the number of five-sided domains is twelve.

Figure 101 represents a case in which both principal points 135 and 246 belong to the limit plane; it is therefore a matter of the interpenetration of two triangular prisms' faces. In the case of Figure 101 five of the tetrahedral cores are entirely in the finite. Figure 102 shows, reduced in size, the six five-sided domains of the configuration of Figure 101.

In order to be able to state in a concise way which planes bound a core, as well as the nature of the boundary domains, we introduce the characteristic of a core. This consists of a sequence of six numbers: the first relates to plane 1, the second to plane 2, the third to plane 3, and so on. The number itself is the number of segments bounding the domain lying in the plane in question. Thus the characteristic (330033) represents the tetrahedral core which is bounded by each of the planes 1, 2, 5, 6 in a three-sided domain, and in whose formation planes 3 and 4 are not involved. The characteristic (553443) represents a six-faced core involving all six planes. Planes 1 and 2 each bound the core in a five-sided domain, planes 3 and 6 each in a three-sided, and planes 4 and 5 each in a four-sided domain.

The two principal cores have the same characteristic, namely (444444). You should ascertain that one principal core can be seen as an interpenetrating system of the three tetrahedra 1234, 3456, and 5612, and the other as an interpenetrating system of 2345, 4561, and 6123.

The other 24 cores are uniquely determined by their characteristics. First there are the six tetrahedral cores

(333300), (033330), (003333), (300333), (330033), (333003)

Then there are twelve five-faced cores bounded by two three-sided and three four-sided domains, namely

(334440), (444330), (044433), (033444), (403344), (304443),
(330444), (440334), (444033), (433044), (443304), (344403).

Finally there are six six-faced cores bounded by two three-sided, two four-sided and two five-sided domains, namely

(553443), (355344), (435534), (443553), (344355), (534435)

For example, in Figure 101 the cores (435534) and (443553) are not difficult to recognize.

Of special interest is the question of how the hexahedron, whose structure we studied earlier (pages 45, 46, and 84), comes about as a degenerate form of the general complete 6-plane. We reach a deeper understanding of the cube, and in general of the Fundamental Structure, if we can see how the cube structure is connected with the general 6-plane. To this end we show, with the help of the figures, how to effect the transition to the special 6-plane, to the hexahedron.

In the latter, the lines of intersection of opposite faces form a 3-side in a plane. If 1 and 2, 3 and 4, 5 ,and 6 are pairs of opposite faces of a hexahedron (for example A^- and A_1^-, B^- and B_1^-, C^- and C_1^- in Figure 16) then the lines of intersection 12, 34, 56 all belong to one plane, whereas in a general 6-plane they are skew. Each set of four planes 1, 2, 3, 4 and 3, 4, 5, 6 and 5, 6, 1, 2 goes through a point (namely C^+, A^+, and B^+ respectively). Hence the tetrahedral cores produced by 1234, 3456, and 5612 must all have shrunk to a point.

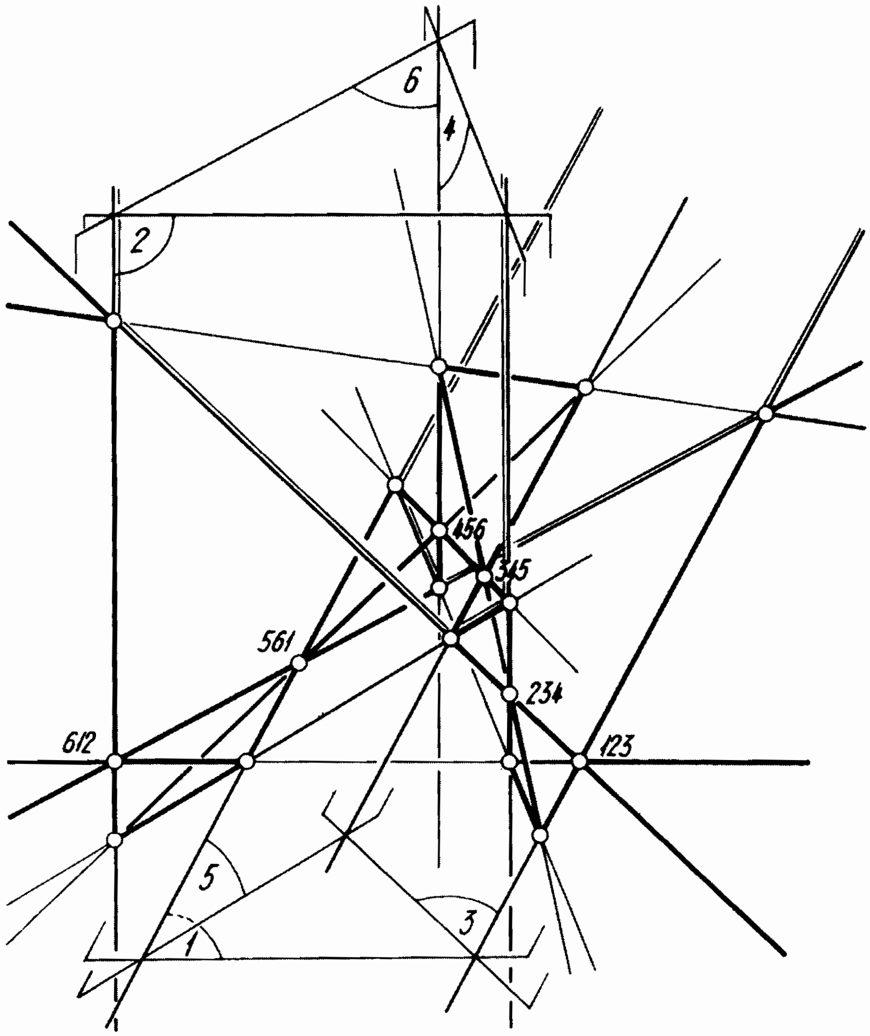


Figure 101

The core determined by 2345 of interest here is the tetrahedral domain adjoining the cube's edge on 25, with opposite edge on 34 = $C^{+}A^{+}$. Another of the cores in question, produced by 4561, shares an edge with the cube on 41, while its opposite edge belongs to line 56 = $A^{+}B^{+}$. The cube's edge on 63 and 12 = $B^{+}C^{+}$ have a corresponding significance for 6123.

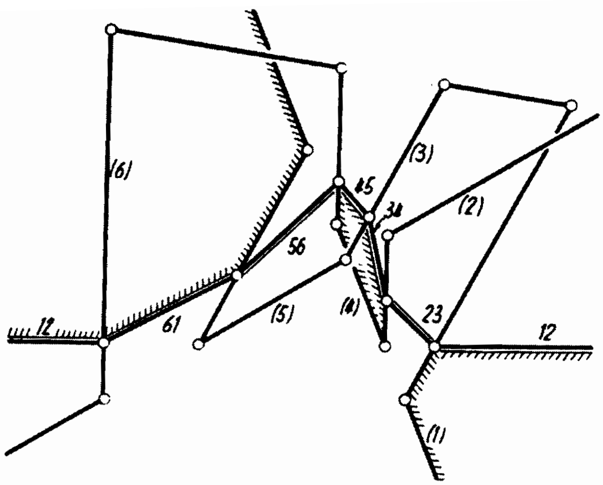


Figure 102

The second principal core turns into the six-faced core bounded by six three-sided planar domains that is attached to vertices 135 and 246 of the hexahedron. This transformed principal core is joined by three more sixfaced cores of the same type and on an equal footing with it; they are connected to the other three pairs of opposite vertices. In order for the ordinary hexahedron, and with it the Fundamental Structure, to materialize from the general 6-plane, the duality of the two principal cores must be abolished.

First Remark. We have outlined only the simplest properties of the six-structuring here. Many remarkable things could still be said. Incidentally, the complete 6-plane configuration is connected with the possible forms of the so-called cubic space curves and the cubic developables.

Second Remark. The structuring of point space by the general 6-plane was described in this chapter in a direct, pictorial way. The actual proof (that is, from **A** and **O**) that six generally positioned planes always produce it, is rather laborious. We can bring something surprising to full consciousness here. In themselves, **A** and **O** are undoubtedly simple and to begin with uninteresting. Yet these simple axioms give rise to the 6-plane having the remarkable properties we have described. It is extraordinarily important actually to experience this contradiction —call it a tension if you will. A first consequence is to realize that **A** and **O** obviously contain much more than one had at first suspected.

With seven planes, different types of structuring are possible. Finding the number of different types is a difficult problem. As for the types of cores in the general case of any number of dividing planes, apparently only the following is known so far: in all divisions of space created by n planes (n greater than 3) there exist at least n tetrahedral cores.

The number V_n of vertices, the number C_n of cores, the number F_n of face portions bounding the cores and the number S_n of segments bounding the cores are easy to state. In fact

$$V_n = \tfrac{1}{6}n(n-1)(n-2), \qquad S_n = \tfrac{1}{2}n(n-1)(n-2),$$
$$F_n = \tfrac{1}{2}n(n-1)(n-2) + n, \qquad C_n = \tfrac{1}{6}n(n-1)(n-2) + n$$

On the other hand, to mathematical thinking, access to the different qualities of the various structurings is largely closed even today.

THIRD REMARK. We have only described one aspect of six-structuring. The polar aspect is the structuring of plane space by six points. To understand the structuring of space into 26 surrounds of planes, by "polarizing" what has been described in this chapter, is an interesting though not particularly easy exercise.

EXERCISE

Take any tetrahedron, cut it with two planes neither of which goes through the tetrahedron's vertices, and determine the cycle of the six planes. Drawing the ring of tetrahedral cores, the two principal cores, the characteristic positions of the six six-faced cores, each containing two five-sided planar cores in its boundary—these all provide rich material for various sorts of exercises. A true picture of the 6-plane is obtained only when these drawings are carried out for various different relative positionings of the six dividing planes.

Chapter 15

The Simplest Curved Surface, which is Saddle-shaped Everywhere

In the preceding pages we have looked mainly at figures built up either from lines or segments or from planes or sections of planes bounded by straight lines. How do we go from the realm of straight lines and planes over to the world of curved forms? Is there a path leading naturally from one to the other? We can indeed indicate such a path: it is given by the characteristic way in which line space, that is, the totality of all lines, is related to point and to plane space. We should say from the start that the effort the beginner may have to apply to understanding the following constructions yields a rich reward! One can come to a vivid experience of how a curved surface incarnates, as it were, in line space. Furthermore, the simplest planar curved forms will be produced as a result.

We start with a problem that was solved in earlier exercises. Suppose three pairwise skew lines a, b, c are given as *director lines*. We can then construct the meeting lines of a, b, c, that is, lines that intersect all three given lines a, b, c.

To do this we regard one of the three lines, for example b (Figure 103), as the carrier of a plane sheaf. Let B^- be a plane of this sheaf. We bring B^- to intersect lines a and c. If A and C are the points of intersection then the line $x = AC$ is a meeting line. The reason why AC intersects b, in B^+ say, is that AC and b both lie in the plane B^-. If we let B^- run through the plane sheaf (b), then x runs through the totality of the meeting lines of a, b, c. (Figure 103 shows a second position of B^-.)

We could also choose B^+ on b, form the planes aB^+ and cB^+ and determine their line of intersection; this too is a meeting line x.

One and only one meeting line lies in each plane B^- of the sheaf (b); one and only one meeting line goes through each point B^+ of the range (b).

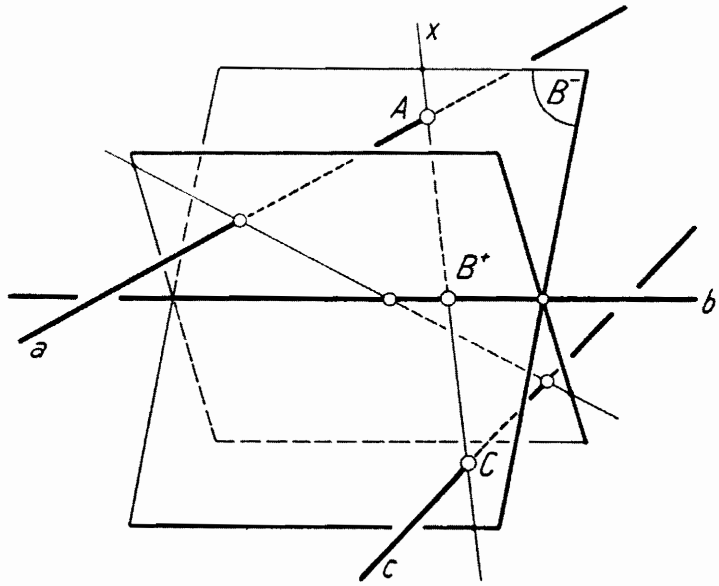


Figure 103

Since we could use line *a* or line *c* instead of *b* in the construction, it follows that

In each plane of each director line there is exactly one meeting line; through each point of each director line there is exactly one meeting line.

Another important property is immediately clear. If A_1, A_2, A_3, . . . are various positions of A and C_1, C_2, C_3, . . . the corresponding positions of C, then the order relationships of the points A_1, A_2, A_3, . . . agree with those of the corresponding points C_1, C_2, C_3, . . . , since the sequences $A_1 A_2 A_3$ . . . and $C_1 C_2 C_3$. . . arise as intersections of the same sequence of planes B. In particular, if (A_1 A_2 A_3 . . .) gives the natural ordering, then so does ($C_1 C_2 C_3$ . . .) . What is valid for ranges (A) and (C) is also true for ranges (A) and (B) , and for (B) and (C). In short:

Through the meeting lines, the point ranges on the director lines are related to each other in a way that preserves order.

The case when one of the director lines, *b* say, is a limit line of space is particularly easy to see. In this case the planes of sheaf (*b*) are parallel. In Figure 104 think of (*b*) as the sheaf of horizontal planes, *a* as a vertical line and *c* as a line running from left above to right below. If we choose equally spaced horizontal planes through *b*, then these cut both *a* and *c* in a row of equally spaced points. (The concept "equally spaced" is used merely for purposes of illustration here.)

In Figure 104 it can be seen that the meeting lines *x* of the three director lines *a, b, c* generate a *curved surface*, so that our initial question is answered: A bridge can be built from the domain of the linear and the planar to the world of curved forms.

We now show how to draw the picture of any number of meeting lines. At the same time, we give one of the most important constructions of the so-called conic sections.

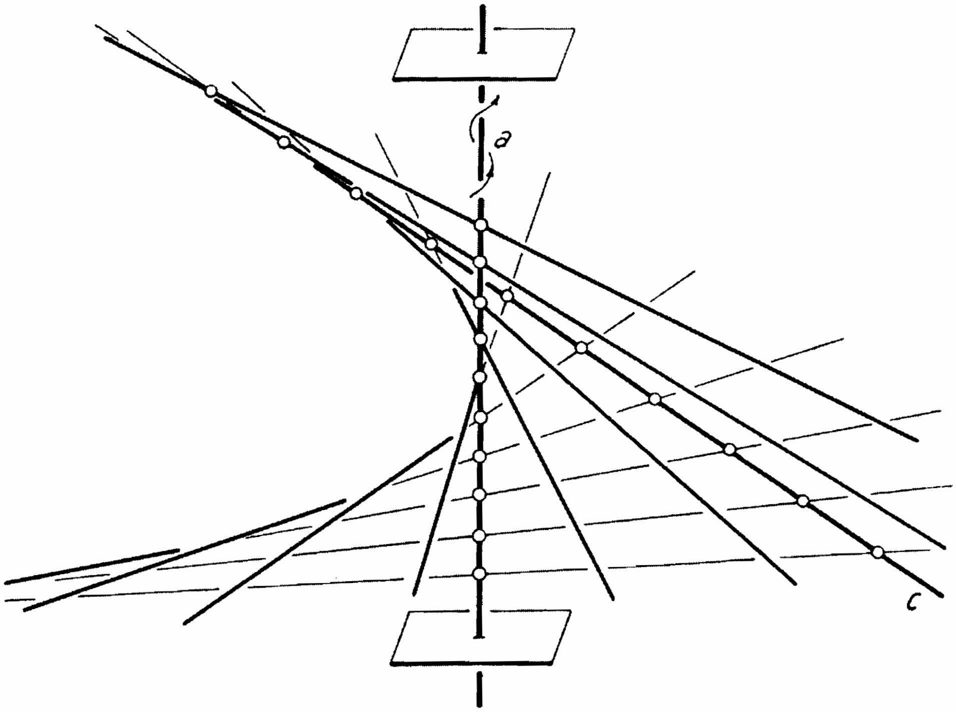


Figure 104

Suppose, in a plane, that the pictures of three director lines *a*, *b*, *c*, and the pictures of two lines *x*, called *u* and *v*, are given. In fact, we can choose five arbitrary lines in the plane for this, provided no three belong to a point, as we shall see. To link a particular mental picture with Figure 105, imagine a as a vertical line. Suppose the director line b runs from below left upwards and to the right, nearer to the observer than *a*. Still further in front and nearer to the observer runs director line *c*, again upwards and to the right, climbing slightly towards the back but less steeply than *b*. To add the picture of a third meeting line *w* to the two, *u* and *v*, we already have, we need only to draw three auxiliary lines. This can be done with ruler alone, that is, without compasses. To explain the construction we imagine the three director lines *abc* and the three meeting lines *uvw* in the sequence

$$aubvcw$$

and name the points of intersection as follows:

$$au = 1,\ \ ub = 2,\ \ bv = 3,\ \ vc = 4,\ \ cw = 5,\ \ wa = 6$$

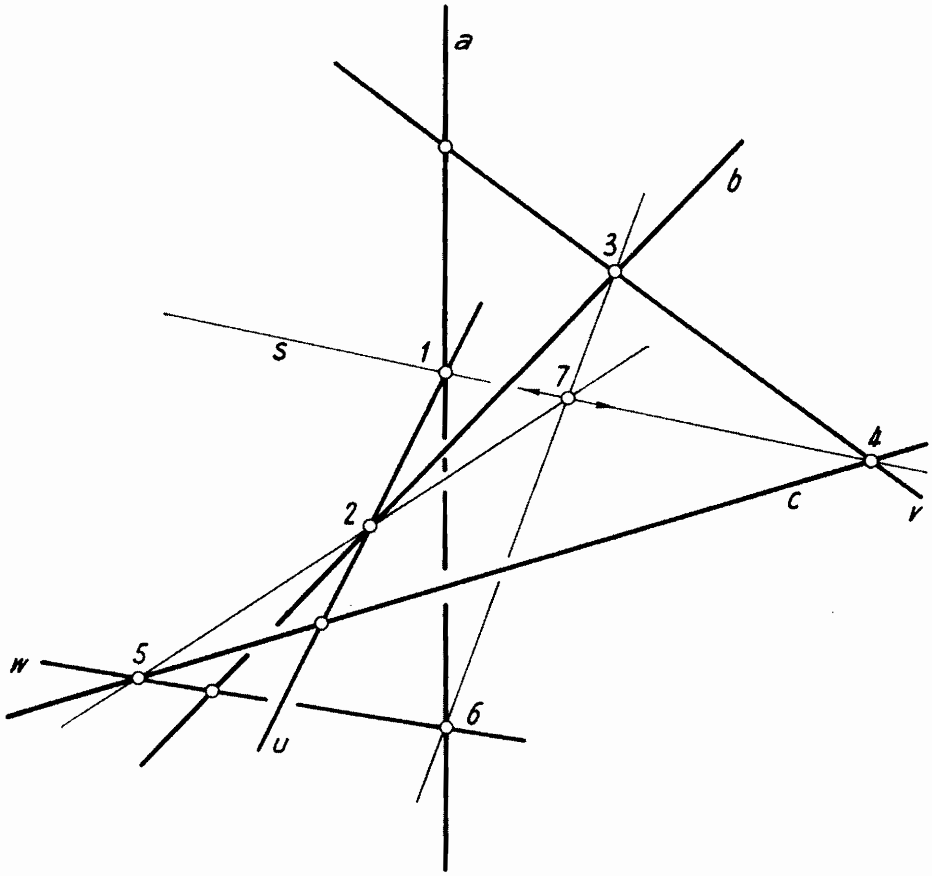


Figure 105

Of these points of intersection, only 1, 2, 3, 4 are given. On the connecting line s = 14 we now choose an arbitrary point 7, then intersect line 27 with c and line 37 with a. The line connecting the resulting points of intersection 5 (of 27 with c) and 6 (of 37 with a) is a meeting line w.

To prove this, we have to show that the line 56 constructed in the drawing represents a line that actually meets—in space—the director line b. This is easy to see since line 27 clearly belongs to both plane $b7$ and plane $c7$; 27 is thus the line of intersection of these planes, and in particular meets line c. Furthermore, line 37 lies in plane $b7$ as well as in plane $a7$; 37 is thus the line of intersection of these planes and meets line a. These points of intersection, labelled 5 and 6 in the figure, both lie in plane $b7$. Their connecting line w = 56 thus meets the director line b, and so w is indeed a meeting line.

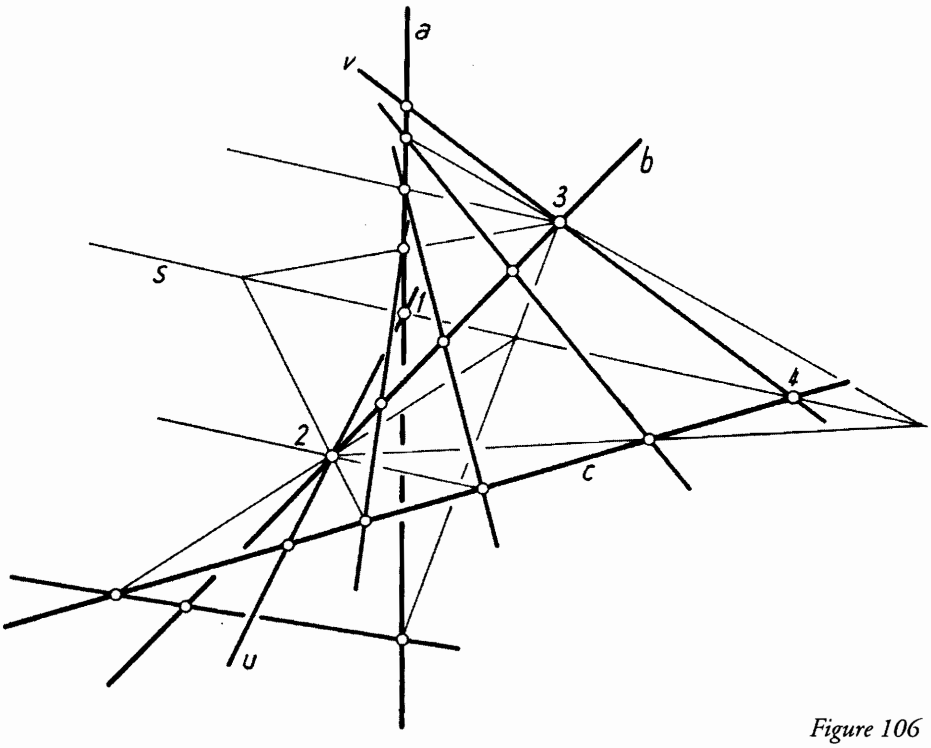


Figure 106

By letting the auxiliary point 7 run along the whole of line *s*, we can represent all the meeting lines *x* of *a, b, c.* In Figure 106 a number of these lines have been constructed. The curved surface they generate is clearly seen. In view of the elementary nature of its begetting, this surface may be described as the *prototype* of the curved surface. Mathematicians call it a *hyperboloid of one sheet*; in a special case, when one of the director lines belongs to the limit plane (Figure 104), it is called a *hyperbolic paraboloid.*

The lines in a plane representing the meeting lines constitute the *envelope of a curve,* namely the set of tangents of a so-called conic section.

The construction of Figure 105 could also be considered from another point of view. From the three director lines *a, b, c* and the two meeting lines *u, v,* we constructed a third meeting line *w.* We call the resulting form a web and denote it (*abc, uvw*). The six lines of a web (*abc, uvw*) form a skew 6-side of a special kind. Suppose we take *a, b, c* as first, third and fifth lines and *u, v, w* as second, fourth and sixth lines respectively, the skew 6-side we have in mind being *aubvcw.* Then *au* = 1, *ub* = 2, *bv* = 3, *vc* = 4, *cw* = 5, *wa* = 6 are its six successive vertices, and 14, 25, 36 are its diagonals. By the construction, the following holds true.

The three diagonals of the 6-side aubvcw in the web (abc, uvw) go through one point.

Irrespective of our earlier reasoning, the proof follows immediately from the concept of the web (*abc, uvw*), as follows.

The diagonal 14 is the line of intersection of the planes *av* and *cu*;
the diagonal 25 is the line of intersection of the planes *cu* and *bw*;
the diagonal 36 is the line of intersection of the planes *bw* and *av*.

14 and 25 lie in plane *cu* and thus meet. 25 and 36 lie in plane *bw* and thus intersect each other. 36 and 14 belong to plane *av* and therefore intersect. Since, however, the three diagonals do not lie in the same plane, they must, by Proposition 24, belong to a bundle.

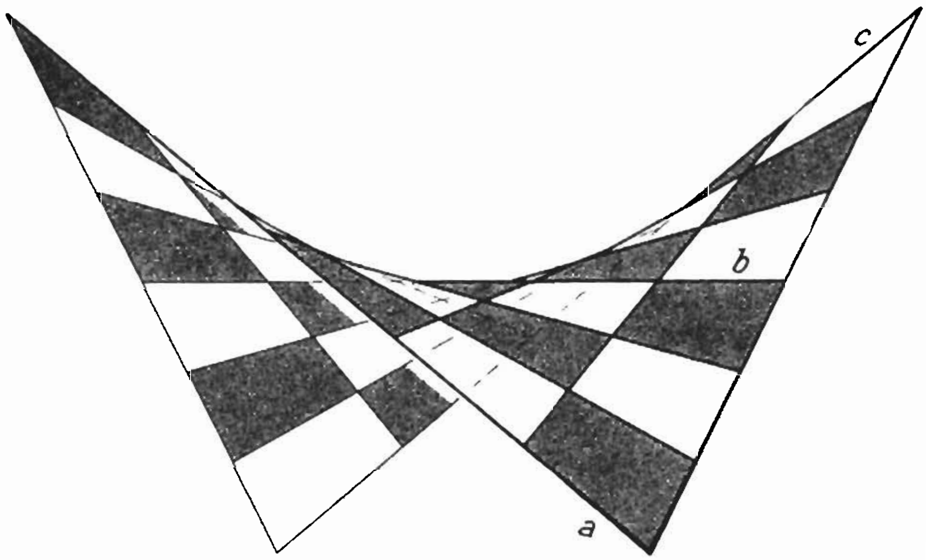


Figure 107

The web (*abc, uvw*) leads to an essential question. We started with *a, b, c* as director lines and constructed the three meeting lines *u, v, w*. Now *u, v, w* are also three pairwise skew lines, and *a, b, c* are three meeting lines of *u, v, w*. The construction of Figure 105 could of course be used to determine further meeting lines, apart from *a, b, c* of *u, v, w*.

The web (*abc, uvw*) thus allows us to construct two curved surfaces *F* and *G*, namely surface *F* with director lines *a, b, c* and surface *G* with director lines *u, v, w*. The question arises: How are *F* and *G* related to each other? Certainly the initial web (*abc, uvw*) belongs to both surfaces. Immersing ourselves in the conditions in force, so to speak, we arrive at the conjecture that *F* and *G* coincide. Intuition tells us that it could not be otherwise. This is indeed the case. This fact is one of the most important propositions of geometry. We call it

The Web Proposition: Let F be the surface generated by the meeting lines of the director lines a, b, c ; let G be the surface formed by the meeting lines of the director lines u, v, w. If (abc, uvw) is a web, then F and G are identical.

For the proof, which we give later, we shall have to appeal fundamentally to the phenomenon of continuity.

Our surface therefore carries two families of generating lines: first the family (x) of meeting lines of the directors a, b, c and secondly the family (y) of meeting lines of the directors u, v, w. Each line of the family (x) meets each line of the family (y), while two lines of the same family are always skew.

If five generally positioned lines are drawn in a plane, and any three of them are called a, b, c and the other two u, v, then we can always regard this as the picture of three pairwise skew lines with two meeting lines. By constructing more lines we obtain pictures like the ones shown in Figures 107, 108, 109 and 110. The figures, which can be made much more beautiful using colors, show how the constructed lines can be drawn so that the surfaces are clearly seen. (In figures like 108 and 109, the surface meshes at the front could be colored light red, shaded according to the distance from the observer, and what is visible of those at the back could be blue, shaded similarly.) Any arbitrary piece of such a surface is *saddle-shaped.*

Notice in Figures 108 to 110 the characteristic way in which point space is effectively divided into two parts by the surface; both parts are ring-shaped and they embrace each other.

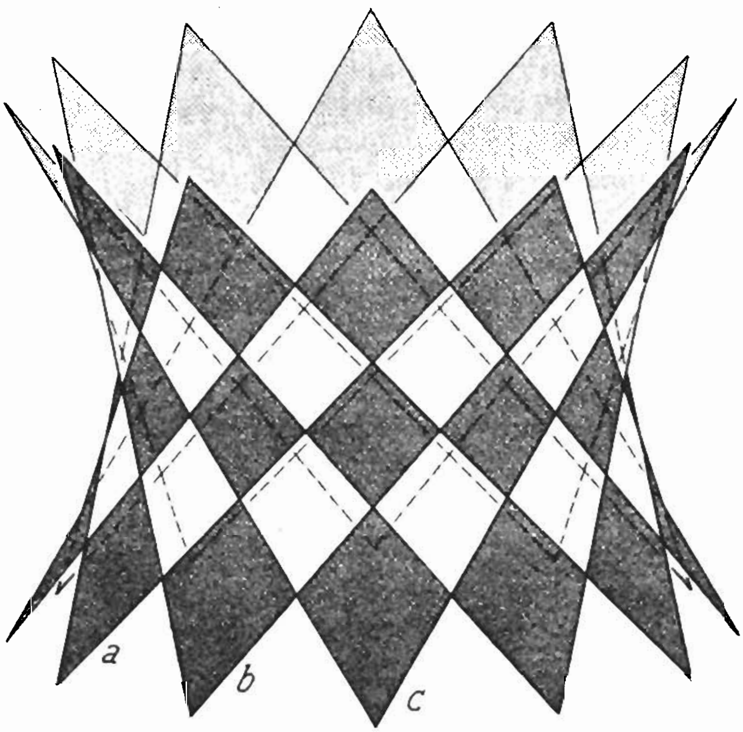


Figure 108

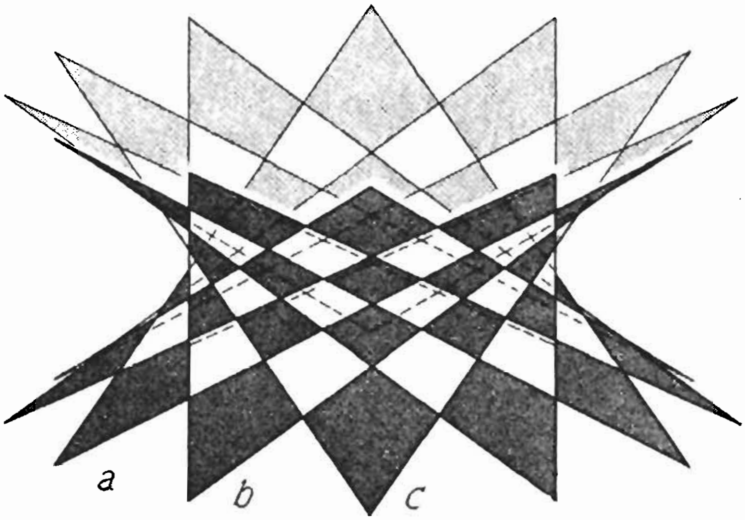


Figure 109

REMARK. We have been interpreting various figures in planes as pictures of forms in space. What are we taking for granted in the pictorial representation? The unstated requirements are:

1. A point in space should be represented by a point of the plane.
2. A line in space should be represented by a line of the plane or, in a specialcase, a point of it.
3. If a point and a line belong to each other, then the same should hold true for their representations.

These assumptions are fulfilled if we "project" the spatial form from a point O onto the plane in question, that is, if a point A in space is represented by the point of intersection of the "line of sight" AO with the plane.

With this interpretation, the curve envelope in each of the Figures 107 to 110 is the section of the cone envelope that can be sent from O to touch the surface. The fact that the position of the viewpoint O is not determined uniquely by the drawing in the plane, means that the observer can associate quite different three-dimensional mental pictures with one and the same planar figure. This suggests the possibility of being able to have spatial forms vary in a mental picture.

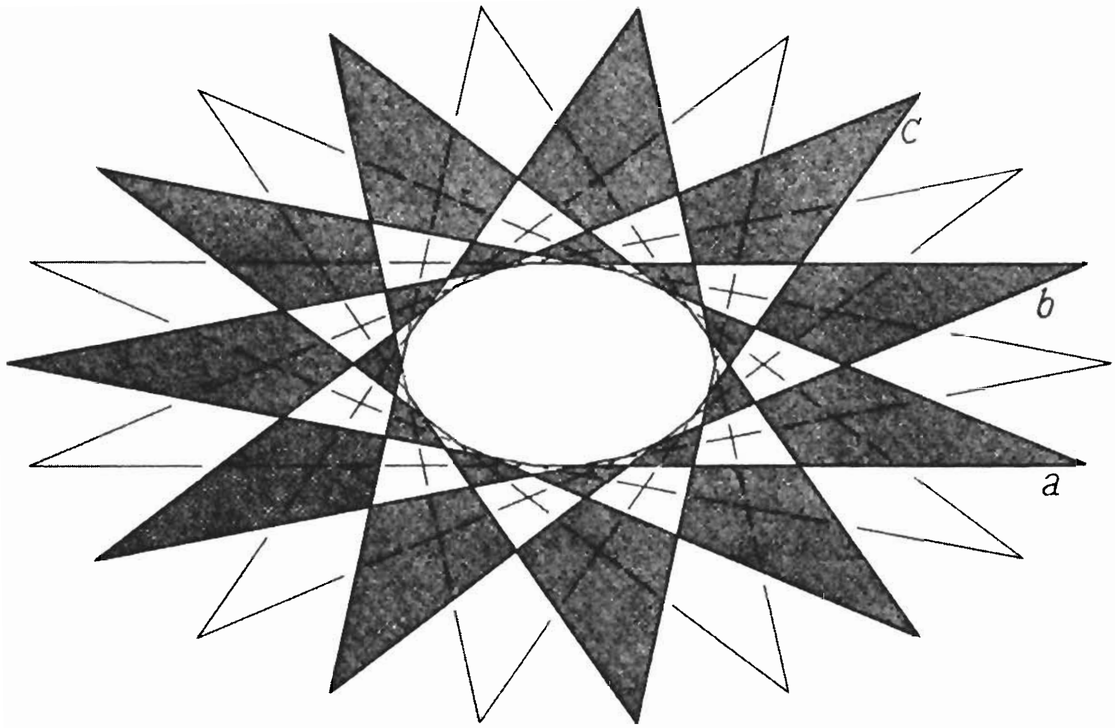


Figure 110

EXERCISES

1. In a plane choose five lines a, b, c, u, v of which no three go through the same point b and draw, using the construction developed in Figure 105, a large number of lines x. Do this for various positions of the five starting lines. Each time, associate the drawing with several different mental pictures by varying the positioning of the corresponding lines in space.

2. From the construction in Figure 105 arises the following result. If the vertices 5, 6, 7 of a moving 3-point move on fixed lines (5 on c, 6 on a, 7 on s, say) in such a way that the two sides 57 and 67 run through fixed points (57 through 2 and 67 through 3, say) then the third side 56 runs through the set of meeting lines of the directors $a = 61$, $b = 23$, and $c = 45$. Moreover, s is a line that meets the two skew lines a and c.

3. Pictures of parts of various hyperboloids and hyperbolic paraboloids can be drawn according to the construction developed in Figure 105 using appropriate color effects. Notice how the same figure can be interpreted as various surfaces.

Chapter 16

Curves and Envelopes of Curves

In this chapter we show some important properties of planar curves and planar envelopes of curves. This will not be a question of developing a complete theory of curves and curve envelopes, but merely of describing some facts which, as experience shows, make stimulating exercise material.

A planar arc of a curve is a set of points showing the same order relationships as the points of a segment: We can run through the arc in two opposite senses; if attention is drawn to any number of points of the arc, then these can be given in their natural order within the arc (in one or the other sense); the set of all points of the arc is gap-free.

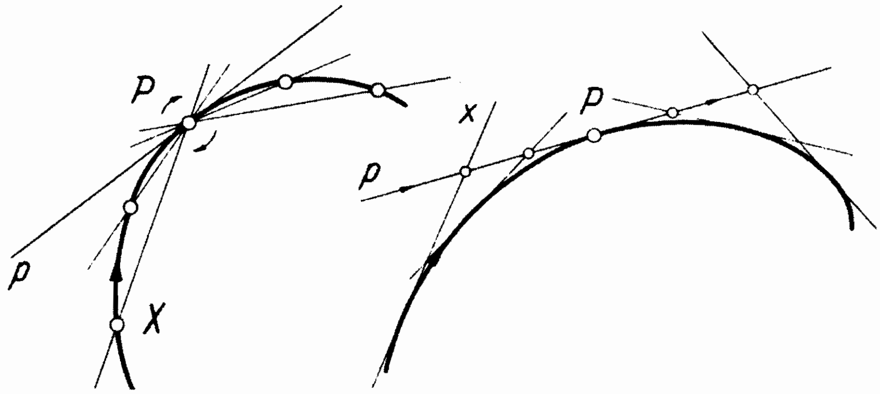
Of course, the ordering properties outlined are far from being sufficient to characterize the form "arc of a curve" as we visualize it. Above all, we associate with this form the concept of the direction that the arc has at each of its points P. This is given by the *tangent* p in P.

The arc we have in mind is thus not only an ordered set of points but also a set of lines, such that each point P has assigned to it a line p (of this set) containing the point P.

Between the set of points P and the set of tangents p of an arc there exist two relationships, which are mutually polar in the geometry of the field:

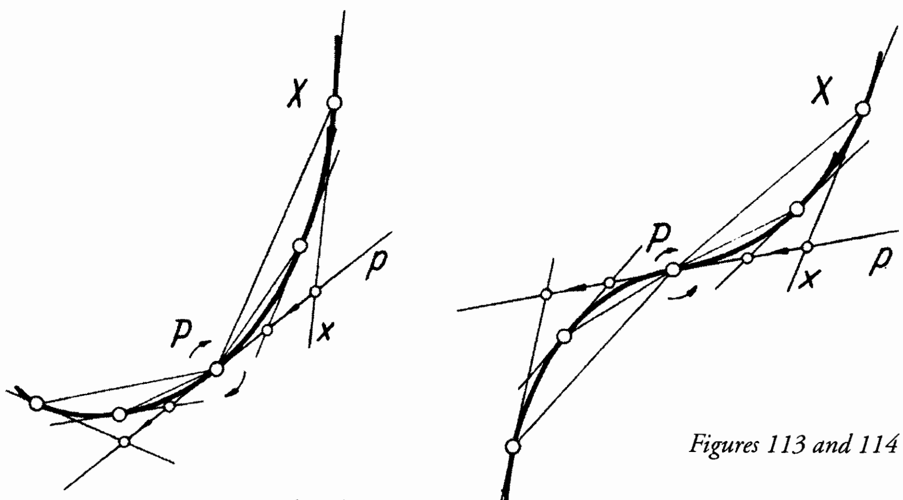
Let P be a fixed point of the arc and suppose X runs through the points of the arc towards P from one side or the other. Then the connecting line PX tends towards a definite limiting position that is the same in both cases, namely the line of contact (tangent) p of P (Figure 111).	Let p be a fixed tangent of the arc and suppose x runs through the tangents of the arc towards p from one side or the other. Then the point of intersection px tends towards a definite limiting position that is the same in both cases, namely the point of contact P of p (Figure 112).

The curve envelope composed of the lines of contact creates the arc as form of its points of contact. We say that a point P of the arc and its tangent p constitute the element $P \cdot p$ of the arc.



Figures 111 and 112

Suppose a line moves in a line pencil in some unspecified way. Then we say the position p of the line has characteristic +1 or –1 according to whether, in passing through the position, the sense of the movement remains the same or changes. Similarly, if a point moves arbitrarily in a point range, then we assign the characteristic +1 or –1 to the position P of the point according to whether, in passing through this position, the sense of the movement stays the same or changes.

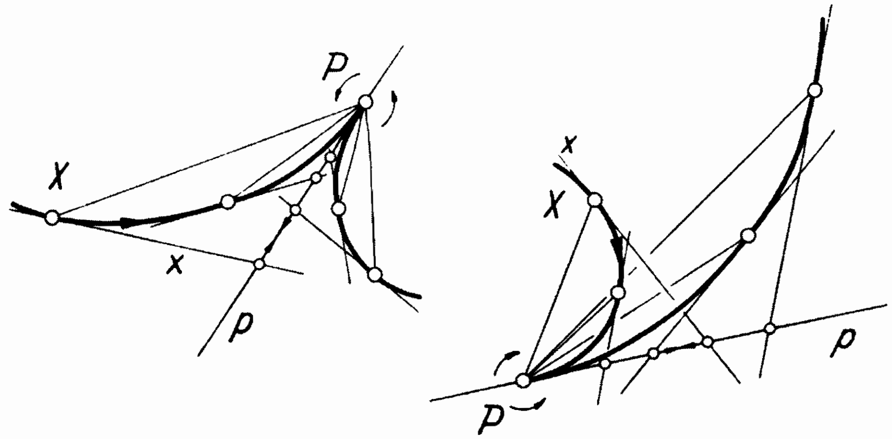


Figures 113 and 114

We now imagine the element $X \cdot x$ running through all the elements of an arc in their natural order and investigate what characterizes its passage through a fixed element $P \cdot p$. In a suffciently small neighborhood of the fixed element $P \cdot p$, we would normally expect the following:

If the moving element $X \cdot x$ passes through the element $P \cdot p$, thought of as fixed, then the connecting line PX rotates in the pencil (P), passing over p without any change in the sense of the rotation, and the point of intersection px runs along the range (p), passing over P without any change in the sense of the motion (Figure 113).

For certain special elements of the arc it is possible that, in the passage through $P \cdot p$, either: the sense of the rotation of PX changes while the sense of the motion of px remains the same (Figure 114), or: the sense of the rotation of PX remains the same while that of the motion of px changes (Figure 115), or finally: both the sense of rotation of PX and that of the motion of px changes (Figure 116).



Figures 115 and 116

In the normal case, we call the element $P \cdot p$ regular, in the other cases singular. A singular element we call either an inflexion (Figure 114), a thorn cusp (Figure 115) or a beak cusp (Figure 116).

It is useful to assign a characteristic (k, l) to each element $P \cdot p$ of the arc. Let $k = +1$ mean that the sense of the movement of the point px does not change in passing through P, let $k = -1$ indicate a change of sense. Correspondingly $l = +1$ or -1 according to whether the sense of rotation of the line PX remains the same or changes in passing through p. Thus a regular element has characteristic $(k, l) = (+1, +1)$, for an inflexion $(k, l) = (+1, -1)$, for a thorn cusp $(k, l) = (-1, +1)$ while for a beak cusp $(k, l) = (-1, -1)$.

In the geometry of the field, inflexion and thorn cusp are polar and the beak cusp is self-polar.

As well as the singular elements mentioned, an arc can also exhibit the following singularities:

Double points, or more generally three-fold, four-fold, five-fold, . . . points; these are points through which the arc passes more than once (Figure 117).

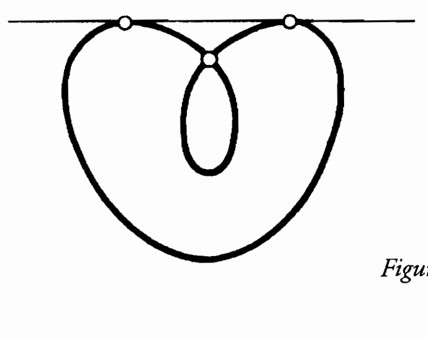


Figure 117

Double tangents, or more generally three-fold, four-fold, five-fold, . . . tangents; these are lines which, if we run through the curve envelope, we meet more than once (Figure 117).

In the first instance, we look at arcs consisting purely of regular elements and having neither multiple points nor multiple tangents. Let the first and last elements of the arc be $A \cdot a$ and $B \cdot b$ respectively. (Naturally these boundary elements occupy a special position.)

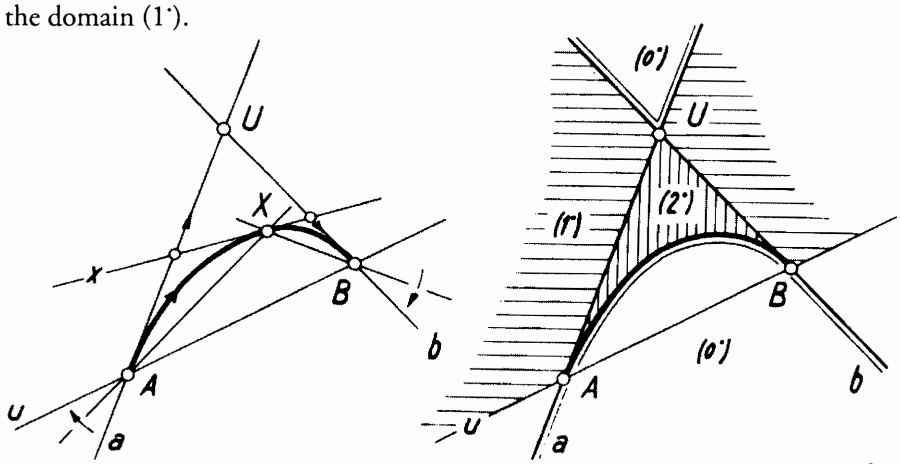
We call a singularity-free arc *simple* if a contains no point of the arc apart from A, and furthermore no tangent goes through A other than a (Figure 118). U is what we call the point of intersection ab of the end-tangents, and u the connecting line AB of the end-points.

Let $X \cdot x$ be any element of the arc. Consider that domain, of the four three-sided core domains determined by the lines a, b, u, that contains X, and call it $[abu]$. Correspondingly, let $[ABU]$ mean that region, of the four three-cornered surround regions determined by the points A, B, U, that contains x. Thus all arc points X other than A and B lie in $[abu]$, and all arc tangents x other than a and b belong to $[ABU]$.

If $X \cdot x$ runs through all elements of the arc starting from $A \cdot a$ (Figure 118), then, without any change in the sense of the motion,

> the point ax moves from A to U in the segment in a bounding $[abu]$,
> the line AX moves from a to u in the angle field in A bounding $[ABU]$,
> the point bx moves from U to B in the segment in b bounding $[abu]$,
> the line BX moves from u to b in the angle field in B bounding $[ABU]$.

The angle field of U containing $[abu]$ is—as point domain—divided into two domains by the arc as point-form, two domains which are joined along the arc: one domain (2˙) each of whose points sends just two tangents to the arc, and one domain (0˙) whose points send no tangents to it (Figure 119). (0˙) could be called the interior domain determined by the arc. Each of the points of the lines of the complementary angle field in U sends just one tangent to the arc; they form the domain (1˙).

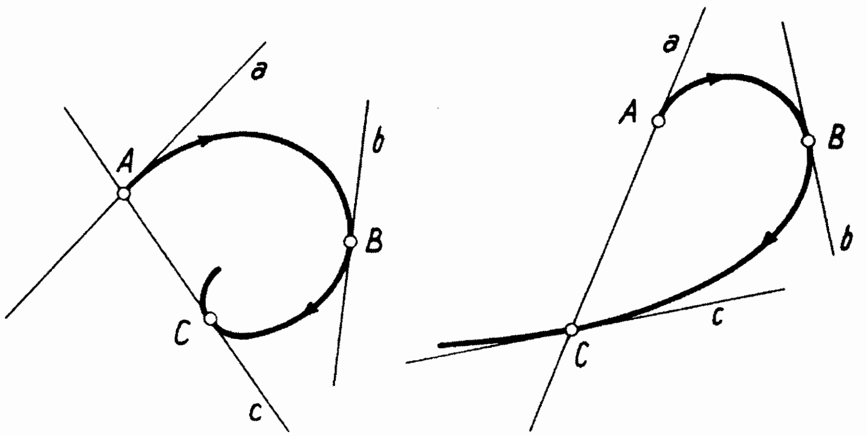


Figures 118 and 119

The segment of u whose points contain the lines of $[ABU]$ is—as region of lines—structured into two regions by the arc as line-form, two regions which are joined along the arc. That is, a region (2′) each of whose lines has just two points in common with the arc, and a region (0′) whose lines have no points in common with it. (0′) is the interior region determined by the arc. The lines of the points of the complementary segment have exactly one point in common with the arc; they form the region (1′).

Through the arc's deviation from the straight line, a core domain (0) is isolated and at the same time a surround region (0′) is won. Consider precisely, in comparison, the "righteousness" of the line's conduct in point and line field.

Whenever we extend a simple arc in such a way that no singularities of any kind appear, we nevertheless reach an element at which the arc run through ceases to be simple. This can happen in two ways: an element $C \cdot c$ is reached for which either the line c contains the starting point A (Figure 120) or the point C belongs to the starting line a (Figure 121).



Figures 120 and 121

In the first case, a spiral path results that is run through "inwards," in the second case, a spiral path results leading "outwards."

As soon as either one of the two cases mentioned has occurred, we have a singularity-free arc whose end-elements are qualitatively different; it may be called a *spiral arc.* It is a self-polar form. It is clear which end-element we call inner and which outer.
The following is true of any spiral arc:

The tangent of the outer endpoint belongs to no other point of the arc.

The point of contact of the inner end-tangent belongs to no other tangent of the arc.

(1.) *The number of tangents going through the outer end-point is equal to the number of arc points lying on the inner end-tangent.*

Imagine a spiral arc continued both inwards and outwards without limit; in this case we speak of the spiral arc being open on both sides, since no end element is available to close it. The following important proposition holds for such an arc (Figure 122):

(2.) *Every open spiral arc has both a surround region and a core domain that the arc snuggles against (osculates). The surround can degenerate into a line, the core into a point.*

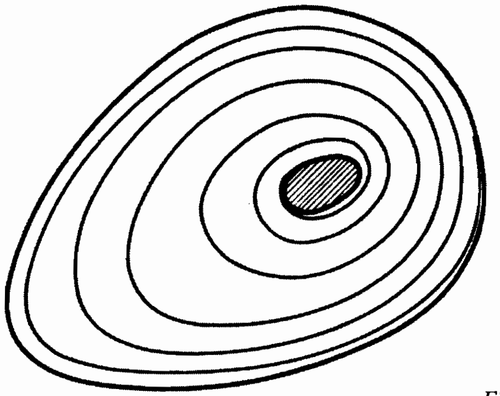


Figure 122

Therefore, according to the definitions of surround region and core domain.

In the plane of any open spiral arc there is at least one line that contains no point of the arc and at least one point that contains no tangent of the arc. (The logarithmic spiral shows that this proposition cannot be strengthened.) This implies that if we want to go along a path in a field and avoid any singularity, we shall necessarily have either to snuggle against a surround region or to embrace a core domain. If we want to evade the latter necessities, yet not stand still, then we must make up our minds to form singularities: inflexions, double points, etc.

REMARK. The propositions just stated are also a concern of the fine arts. They should not be withheld from young people, because a vivid comprehension of these curves prepares the ground for an understanding of spiritual truths. They indicate in particular how, for the eurythmist, geometry has the ability to bring to consciousness what is essential.

We now consider arcs in which singularities are permitted. A *closed arc* is one which runs back into itself. Here the word *curve* shall always mean a closed arc with a *finite number* of singular elements.

We let the element $X \cdot x$ run through a curve. Its characteristic at a given place we denote (k, l). The curve bears a certain relationship to the point and line fields. To enable us to see this, we choose an arbitrary point S and an arbitrary line s in the field. The following general law then holds (Figure 123 makes this clear):

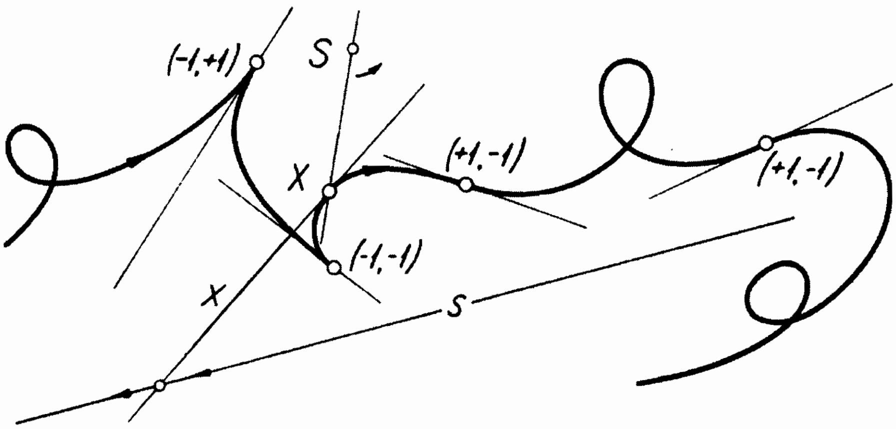


Figure 123

(3.) As X moves, the connecting line SX moving in the pencil S has characteristic
k whenever x does not go through S,
l if X coincides with S, $-k \cdot l$
if x goes through S, unless X coincides with S.

(3.) As x moves, the point of intersection sx moving in the range s
has characteristic
l whenever X does not lie on s,
k if x coincides with s, $-l \cdot k$
if X lies on s, unless x coincides with s.

REMARK. That so many properties of the movement along the curve are expressed in so concise a form is astonishing. These propositions were discovered by Karl Georg Christian von Staudt (1796 – 1867), the brilliant researcher of modern geometry. Mathematicians should note that these propositions can serve as the basis for a systematic theory.

As may be checked in some examples, the appearance of any singularity has the following effect: A piece of arc, no matter how small, containing the singularity has *more than two* points in common with certain lines and sends *more than two* tangents through certain points. From this it follows that an arc,and in particular a curve (that is, *a* closed arc), that has no more than two points in common with any line, cannot possess any singularity. The same is true of arcs which send no more than two tangents through any point.

Suppose a curve is given. If, in the field of the curve, there exists a line that possesses n points in common with the curve, but there is no line meeting the curve in more than n points, then we say that the curve is of n-th order.

If, in the plane of the curve, there exists a point through which the curve sends m tangents, but there is no point containing more than m tangents, then the curve is said to be of m-th *class*.

A second-order curve is an *oval.* It is at the same time of second class. Conversely, any curve without singularities is of second order and of second class, that is, an oval.

REMARK. It may not be superfluous to mention that the concepts of order and class used here are substantially more general than the usual concepts of algebraic order and algebraic class.

There arises the question, "What forms can a third order curve show?" The curve should have at most three points in common with any line and this number must be attained for at least one line. The search for forms determined by "threeness" can be made into a stimulating and exciting exercise. Careful investigation yields forms of four sorts, namely:

a) curves with three inflexions (Figures 124, 125, 126);

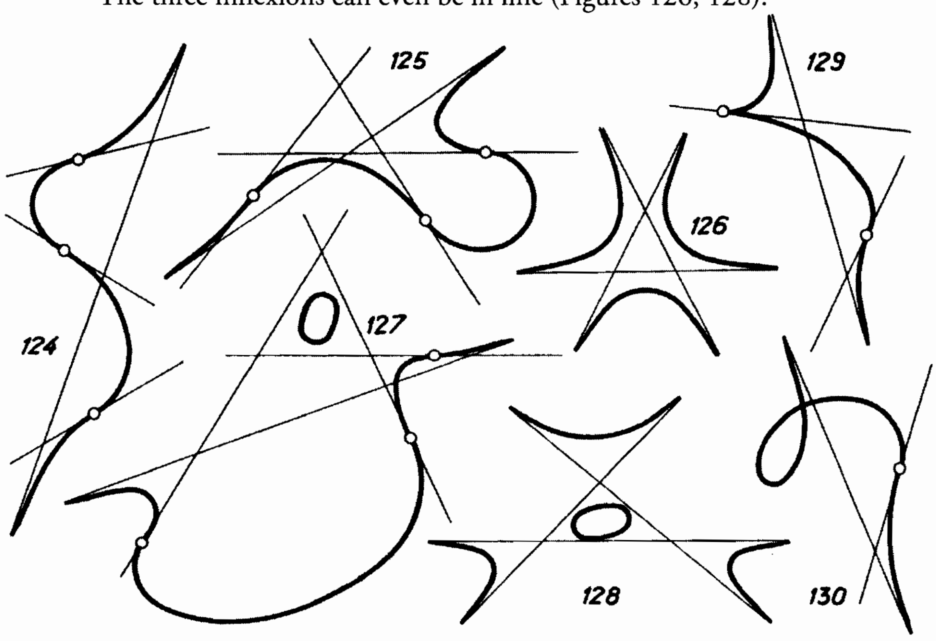
b) two-branched forms (consisting of two sub-curves), one branch having three inflexions and the other being an oval (Figures 127, 128);

c) curves with one inflexion and one thorn cusp (Figure 129);

d) curves with one inflexion and one double point (Figure 130).

Whether a form with three inflexions admits a supplementary oval in such a way that the order of the two-branched form is still three, depends on how the curve stands in relation to the 3-side of inflexional tangents, as can be gathered from the figures: of the four cores produced by the three inflexional tangents, there is, in case b), one whose points do not send any tangents to the sub-curve with the three inflexions.

The three inflexions can even be in line (Figures 126, 128).

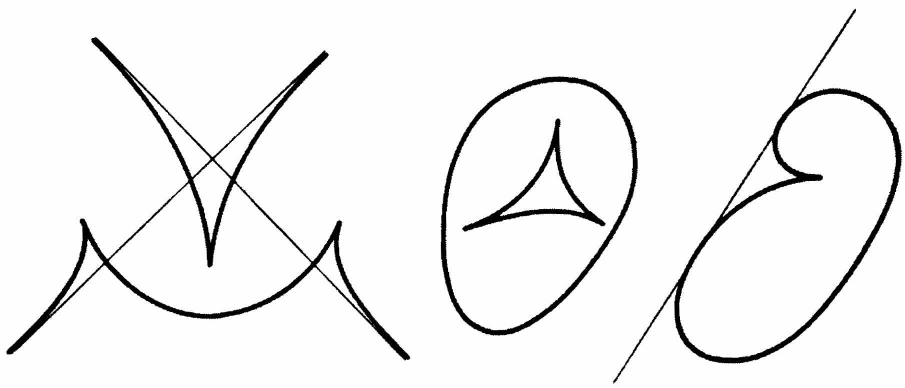


Figures 124 to 130

Polar to the third order curves are the third class curves, which send at most three tangents through each point of the plane. Here, too, the quest for possible forms can be made into an interesting exercise. The four kinds are:

a) curves with three thorn cusps (Figure 131);

b) two-branched forms (consisting of two sub-curves), one branch having three thorn cusps and the other being an oval (Figures 132);



Figures 131, 132 and 133

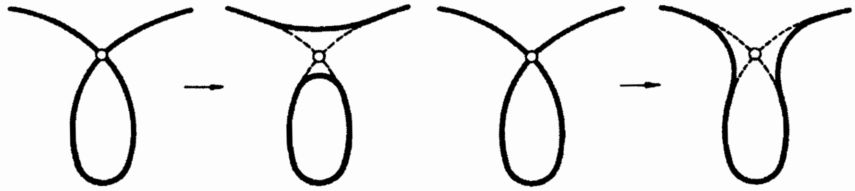
c) curves with one thorn cusp and one inflexion (Figure 129);

d) curves with one thorn cusp and one double tangent (Figure 133).

In case b), unlike case a), there is among the four three-cornered surround regions produced by the cusps, one whose lines do not meet the sub-curve with the three thorn cusps.

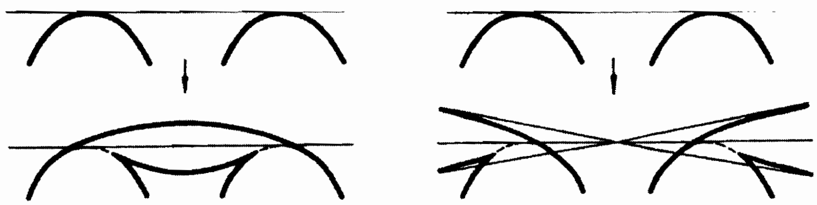
For each form there exists a counter-form that is polar to it as regards its order, class, and singularities.

For a given curve form in process of transformation, it is then an especially interesting exercise to understand clearly the polar transformation of the counterform. An example of this is the "dissolving" of a double point, which can happen in two ways. One way is to create two inflexions and separate off a sub-curve (Figure 134). In the second way (Figure 135) again two inflexions are produced but without detaching a sub-curve.



Figures 134 and 135

Polar to this, there are two ways of dissolving a double tangent. Figure 136 shows the transformation polar to that of Figure 134; the processes in Figures 135 and 137 are also mutually polar.



Figures 136 and 137

For a thorough treatment of curves, we need to investigate each form and counter-form in relation to the whole point and line field.

REMARK. The proofs from **A** and **O** of many of the propositions elucidated in this chapter are not entirely straightforward. An axiomatic treatment can be found in the author's little book *Einführung in die freie Geometrie ebener Kurven* (Birkhäuser Verlag, Basel, 1952).

The curves of fourth order and their counter-forms of fourth class already show such a multiplicity that surveying them is not easy. As to the forms created by "fiveness," to date a complete summary appears to be impossible.

EXERCISES

1. Show clearly the domains $(0^{\cdot})$, $(1^{\cdot})$, $(2^{\cdot})$ and regions $(0')$, $(1')$, $(2')$ for a given simple arc.

2. Check Proposition (1) about spiral arcs in several examples.

3. Proposition (2) about open spiral arcs is suitable for meditation.

4. Check the general laws (3) about moving along plane curves by drawing several figures with various reference points *S* and reference lines *s*.

5. Make the following facts clear by determining the regions $(0')$, $(1')$, etc. and domains $(0^{\cdot})$, $(1^{\cdot})$, etc.,

   A third-order curve consists in case c) of two simple arcs and in cases a), b) (without the oval) and d) of three simple arcs. In cases a) and b) with oval it has class six, in c) class three, and in d) class four.
   Polar to this, a third class curve consists in case c) of two simple arcs and in cases a) and b) (without the oval) and d) of three simple arcs. In cases a) and b) with oval it has order six, in c) order three, and in d) order four.

6. Determine the counter-forms for each of the curve forms of Figures 138 a–f and consider them in relation to the point and line fields. To do this, find their singularities and produce the form with polar singularities.

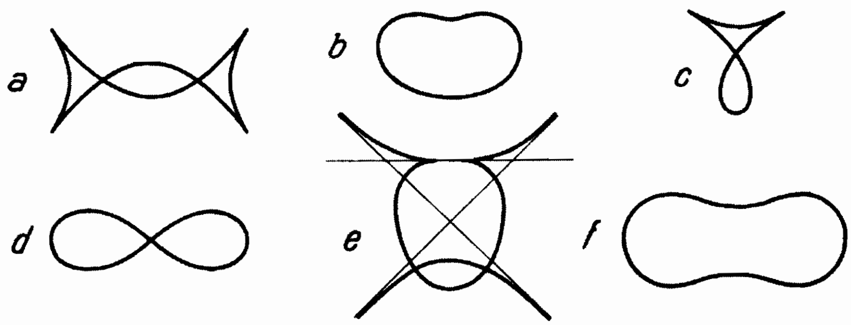


Figure 138

7. Try to form some fourth-order curves and their counter-forms of the fourth class.

8. Represent the transformation process polar to that indicated in Figures 139 a–g (a difficult exercise for the beginner).

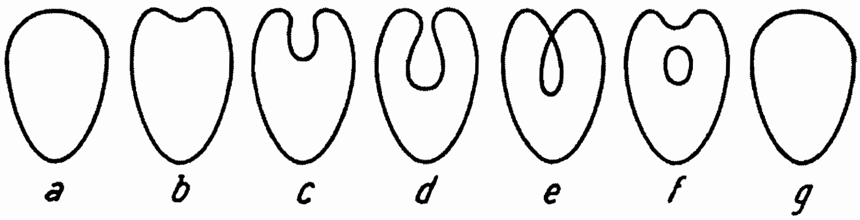


Figure 139

Remark. The book mentioned in the previous Remark contains a large number of exercises in the field dealt with here.

Chapter 17

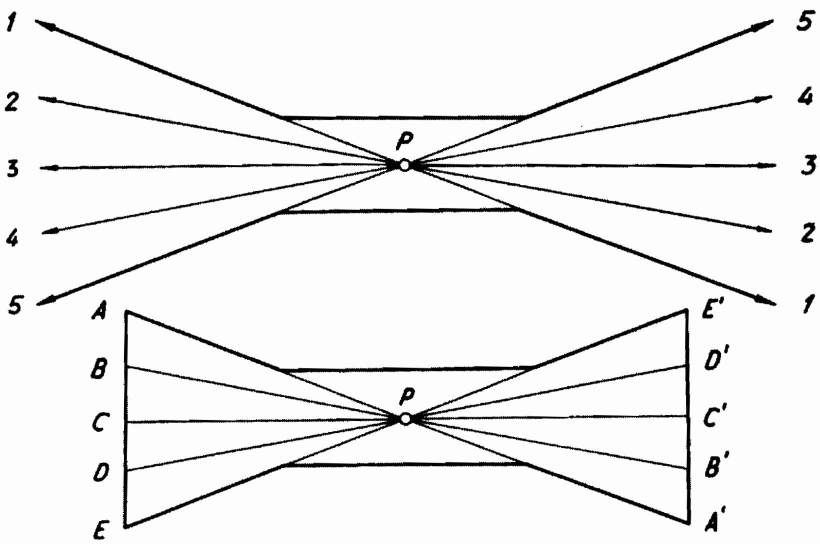
The Structure of the Plane

At many places in this book the reader will have come up against a particular type of difficulty, that of visualizing forms extending over the limit line of a plane or over the limit plane of space. Consider for example Figure 74, in which the core domain [C] extends over the limit line of the plane of the 4-side. If one walks, say, along the boundary segment lying in d, in the figure *to the top* and holds one's left arm outstretched all the while, then the latter points to the interior of [C]. But coming back—after crossing the limit line (from below in the figure)—one's left arm seems no longer to be pointing towards the interior of [C].

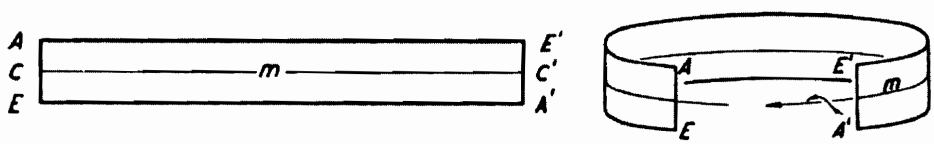
Another example: in Figure 35 (page 65) a planar core is drawn that extends over the limit line. The boundary curve comes close to two lines; consider the one that is almost horizontal. The core's boundary runs to the left above this line but comes back from the right *below* it. The boundary curve appears to have crossed from one side of this line to the other (Figures 124, 125, 129 contain something similar). There is apparently an inflexion. On the other hand Figures 126 and 128 ought to have inflexions, but after going over the limit line the curve appears to come back on the same side of the inflexional tangent.

This chapter will help to clarify these and like phenomena. We start with an angle field in a pencil (P), which we furnish with a pair of three-sided domains in the neighborhood of P (Figure 140) in order to make the angle field into a planar "strip" that is nowhere narrowed down to a point. Let 1 and 5 be the points of intersection of the lines bounding the angle field with the limit line of the plane of the strip; let 2, 3 and 4 be the limit points of three other lines of the pencil (P). As mentioned before, we have no direct means of capturing in mental pictures the limit line or its points 1, 2, 3, 4, 5. On the other hand, it is possible to visualize a *model* of the planar strip we are considering, a model reproducing not all, obviously, but certain properties of the strip. We want the model to represent the *connectedness* of the strip. With this aim, we cut straight across the strip on both sides of P. Let A, B, C, D, E and A', B', C', D', E' be the points of intersection of the respective cuts with the five lines $P1$, $P2$, etc. (Figure 141).

To obtain the desired model from the piece of strip we have cut out, we must join the boundaries AE and $A'E'$ together in such a way that A and A', B and B', etc., coincide. To this end we first of all turn the piece of strip into a rectangle (Figure 142) and bend it round as in Figure 143 so that it can be closed in the required way. To do this we could hold the end AE fixed and rotate the other end through 180° in one or the other sense around the middle line m, as indicated in the figure. If the end $A'E'$, as it approaches AE, rotates in the sense of a right-handed screw, we say that the strip has been twisted through +180°. A twist through −180° means the corresponding left-handed screw. In both cases a closed band is produced (Figures 144 and 145). Figures 146 and 147 show the same bands folded flat. These forms are named after August Ferdinand Möbius (1790 – 1868), he being the first to recognize their significance.



Figures 140 and 141



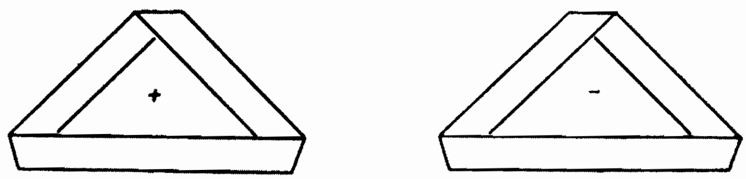
Figures 142 and 143

Had we closed the rectangular strip *without* a twist (or after a twist through an even multiple of 180°), then a band with two boundaries would result. It is immediately noticeable that the Möbius band (Figures 144 to 147) has only *one* boundary. A non-twisted closed band has two sides, an inner and an outer, which we could paint with different colors. With the Möbius band, if we begin coloring it anywhere and continue painting all the way round, then, when we come back to the place at which we started, we are momentarily disconcerted to find that *the whole band is colored.* The Möbius band does not have two sides: it is a one-sided surface. If we cut through the band across the middle line m we obtain once more a surface for which we can distinguish a front and a back.

Figures 144 and 145

The one-sidedness of the surface and the fact that there is just one closed boundary curve are both easily explained by the twist executed before the joining of the ends of the strip.

The Möbius band gives us a model of the kind of connectedness belonging to our original strip (Figure 141).



Figures 146 and 147

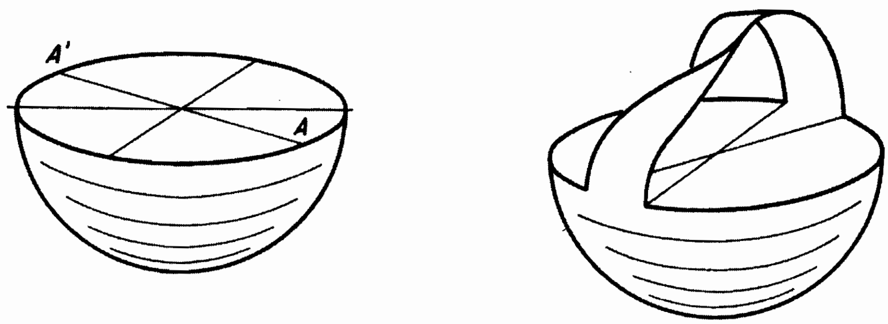
If we picture a painter painting the strip in Figure 140 red, who starts at *P* and works towards the right, to be precise, on the side of the page facing the reader—the hairs of the brush thus pointing away from the reader—and if we imagine the coloring of the strip continued over the limit line, then the painter will come back from the left, only now the paint-brush hairs will point towards the reader, since the brush will be painting the back of the page. To color the strip completely, he must cross over the limit line for a second time. The piece of strip on the page is two-sided; yet with regard to the strip as a whole, we cannot speak of a front and a back.

The answer to the question posed in the the beginning is now clear: The man's left arm also points to the interior of [*C*] when he comes back (from below), since he is now walking on what—with respect to a finite part of the plane —is its under-side.

The plane as a whole is a one-sided surface. That is, we cannot distinguish a front of the plane from a back, since this would require the crossing of a boundary line dividing one side from the other. When we remove the limit line of the plane, the complete plane (that is, the projective plane) becomes the socalled Euclidean plane. And the "slit" arising from the removal of the limit line allows us to distinguish between front and back (or top and under-side) of the plane.

The following is also made immediately comprehensible by reason of the "onesidedness" of the plane. Suppose we go along the boundary of a three-sided core domain extending over the limit line, in a certain sense (Figure 148). We place a disc in the domain near its boundary (bottom of Figure 148) and provide the disc with a sense of rotation *agreeing with that of the domain.* If we move the disc in the domain over the limit line, then its sense of rotation appears to have changed (top of Figure 148). With respect to a finite piece of the plane, as represented for example by this page of the book, we have gone, assuming the disc was originally on the front of the plane, from the latter over the limit line to the back of the plane, so that the sense of rotation seen from the front appears in reverse.

One can invent various kinds of models embodying the onesidedness of the plane. A simple one is obtained from a hemisphere as follows. Diametrically opposite points of the hemisphere's boundary, for example A and A' in Figure 149, are to be thought of as identical. We would thus have to join up the boundary circle in such a way that two such points coincide, which cannot of course be done without the surface intersecting itself. On the other hand, we can represent a closed strip as in Figure 150. A model of the whole plane would be obtained by increasing the width of the strip until the ends of the strip included the whole of the boundary circle of the hemispherical part of the surface, which, however, is impossible without self-intersection of the surface.



Figures 149 and 150

As a result of our considerations, we have established that certain rather odd phenomena are explained by the plane's characteristic one-sidedness and that the plane is not such a simple form as we tend to assume.

The properties of the plane, and of certain planar strips, we have dealt with are facts belonging to the realm of *topology*, as it is called today. This is one of the most important branches of modern mathematics. The study of knots and loops, the concern of the first two exercises below, also belongs to topology.

EXERCISES

1. Oscar Simony (1852 – 1915), in some remarkable, little-known treatises on knots and loops, has given some simple experiments that the reader is recommended to carry out. The regularity of these phenomena is endlessly interesting, though to imbue this regularity with concepts is far from easy. The simplest experiments are given here, the original work being rather inaccessible.

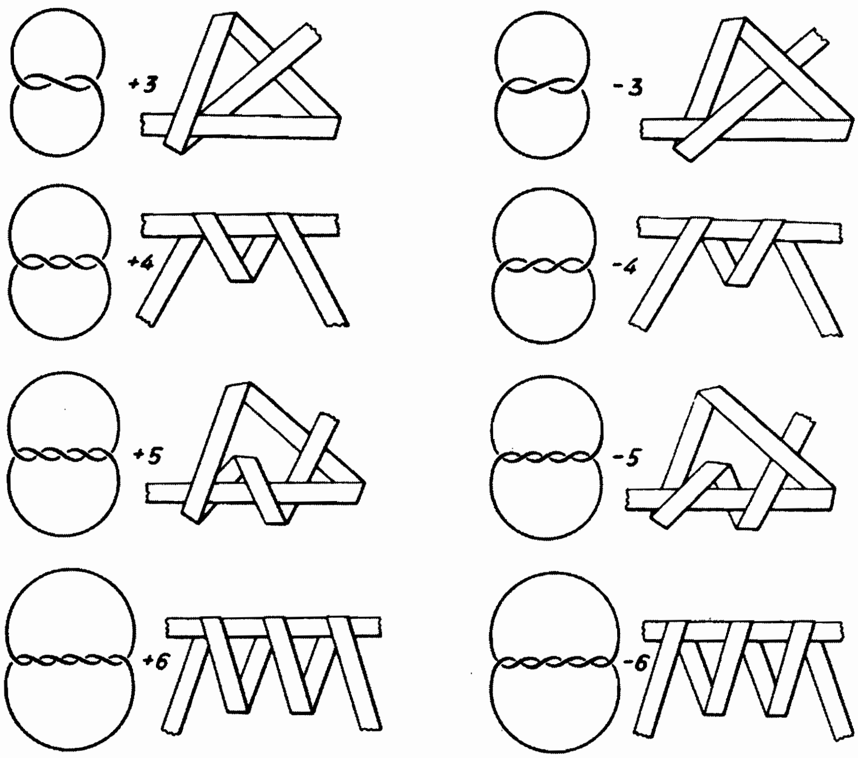


Figure 151

Take a rectangular strip of length ℓ and width w, twist it through n times 180° (n = +1, +2, +3, . . . also n = –1, –2, –3, . . .) then join it to form a closed band (Figure 143). Now cut the band along its middle line. Different kinds of forms are created by the cut, according to the number of twists made before sticking the ends together.

Verify the following propositions (Figure 151 shows, schematically, some examples):

For odd n, that is, n = ±1, ±3, ±5, . . ., the cut along the middle line produces a single closed band of length 2 ℓ and width $\frac{1}{2}w$, with a knot. The knot takes the form of a positive or negative screw according to whether n is positive or negative; the respective knots cannot be transformed into each other without cutting the band. The new band is itself twisted; to be precise, for $n = \pm 1, \pm 3, \pm 5, \dots$ it is twisted through ± 4 times 180°, ± 8 times 180°, ± 12 times 180°, . . . respectively. For even n, that is, $n = \pm 2, \pm 4, \dots$, cutting along the middle line produces two closed bands, each of length ℓ and width $\frac{1}{2}w$, which are intertwined. We can characterize the intertwining by letting one band hang from an untwisted portion of the other band and stating how many times the former is looped round the latter. The looping-round is single, double, triple, . . . for $n = \pm 2, \pm 4, \pm 6, \dots$, respectively, and has a positive or negative sense according to whether n is positive or negative. (Only for $n = +2$ and $n = -2$ can the forms produced be transformed into each other without tearing a band.) In each case, the two intertwined bands each individually show the same twisting as the original band.

2. If the cut in Exercise 1 is made not along the middle line but at a distance d from it, something more complicated happens (Figure 152). If n is odd, such a cut produces two closed bands, one having width $2d$ and length ℓ the other width $\frac{1}{2}(w - 2d)$ and length 2ℓ . The two bands are intertwined, and furthermore the longer band is tied to the shorter with a knot. Figure 152 shows the cases $n = \pm 3$ and $n = \pm 5$. If n is even, then two bands of the same length ℓ are created that are intertwined but not knotted; one has width $\frac{1}{2}(w - 2d)$, the other has width $\frac{1}{2}(w + 2d)$.

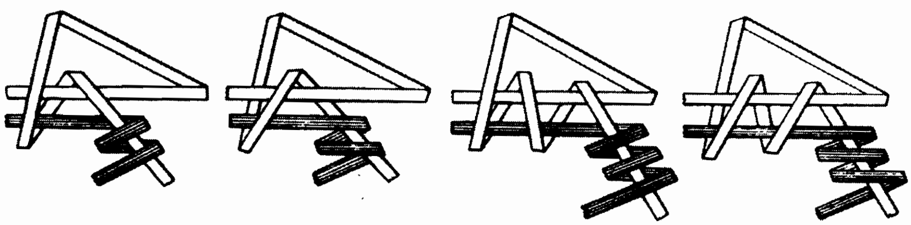


Figure 152

3. The one-sidedness of the plane can—at least in a certain sense—be grasped almost directly. Imagine oneself in the interior of a large sphere. Let the outside be painted red, the inside blue. Now suppose the sphere expands from the center towards the limit plane of space, so far that eventually it presses close up against (osculates) the limit plane, and with that relinquishes its "sphericalness." Its red outer side then covers the limit plane completely. From the center we see only the blue inner side. The limit plane's one-sidedness becomes obvious. Using the metamorphoses applied in Chapter 11, we could take, instead of the limit plane, any other plane of space.

4. A question whose answer is not immediately forthcoming is: Is a closed planar strip extending over the limit line of its plane (for example, as in Figure 140) twisted positively or negatively?

5. Imagine in a horizontal plane a line g and the pencil of parallel lines at right-angles to g. x runs through the pencil, starting from the position a (Figure 153). Imagine the line x partly provided with tassels. In the initial position the tassels are to the right of g on the upper side (with respect to a finite piece of the plane) and are dragged along as x moves. Determine when x has run through the pencil completely. Reflect similarly on a line running through an arbitrary pencil.

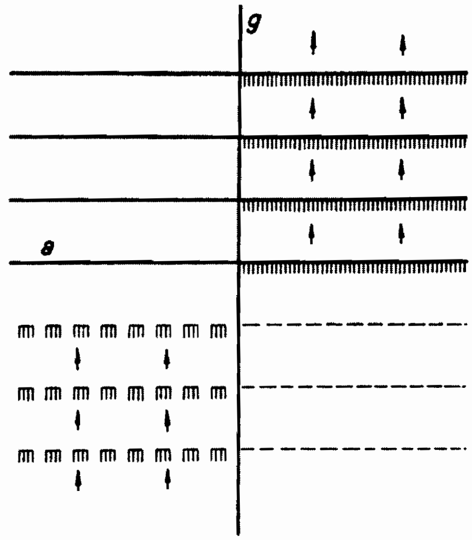


Figure 153

PART THREE: THEORY (FIRST ORDER)

Chapter 18

Harmonic Fours

In this chapter we develop a concept that turns out to be fundamental for measurement in space and counterspace. We start by defining when four points of a line form a harmonic set, also called a *harmonic four.*

We picture a line g and a point A belonging to g. This form—g and A on g—gives no occasion to highlight another point on the line or even in the space outside the line. The same is true if, on g, two points A and B are given. We could, it is true, put a plane through g and choose, in the plane, a line p through A and a line q through B. This produces the point of intersection pq but the resulting 3-side gpq is complete in itself and offers no opportunity for further construction.

But if we take *three* points A, B, C on g, then from these points, by various constructions, other points on g can be determined. We picture through g a plane X, and in this plane choose a line p through A, a line q through B and a line r through C (Figure 154). We assume that these three lines differ from g and are not members of the same pencil. They thus form a 3-side pqr with vertices $P = qr$, $Q = rp$, $R = pq$, say.

Examining this figure, we realize immediately the possibility of producing from it further points on g. First we form the lines

$$u = AP, \qquad v = BQ, \qquad w = CR$$

Let the vertices of the 3-side uvw be

$$U = vw, \qquad V = wu, \qquad W = uv$$

The two 3-sides pqr and uvw are, by their very construction, in perspective with respect to g, since $pu = A$, $qv = B$, $rw = C$ lie on g. Hence, by Desargues' Theorem, the lines connecting corresponding vertices, namely

$$PU = a, \qquad QV = b, \qquad RW = c$$

go through a point G. These lines produce, with g, the points of intersection

$$ag = A_1, \qquad bg = B_1, \qquad cg = C_1$$

By means of the construction, with three given points A, B, C of a line g there are associated in a harmonious way three further points A_1, B_1, C_1 on g.

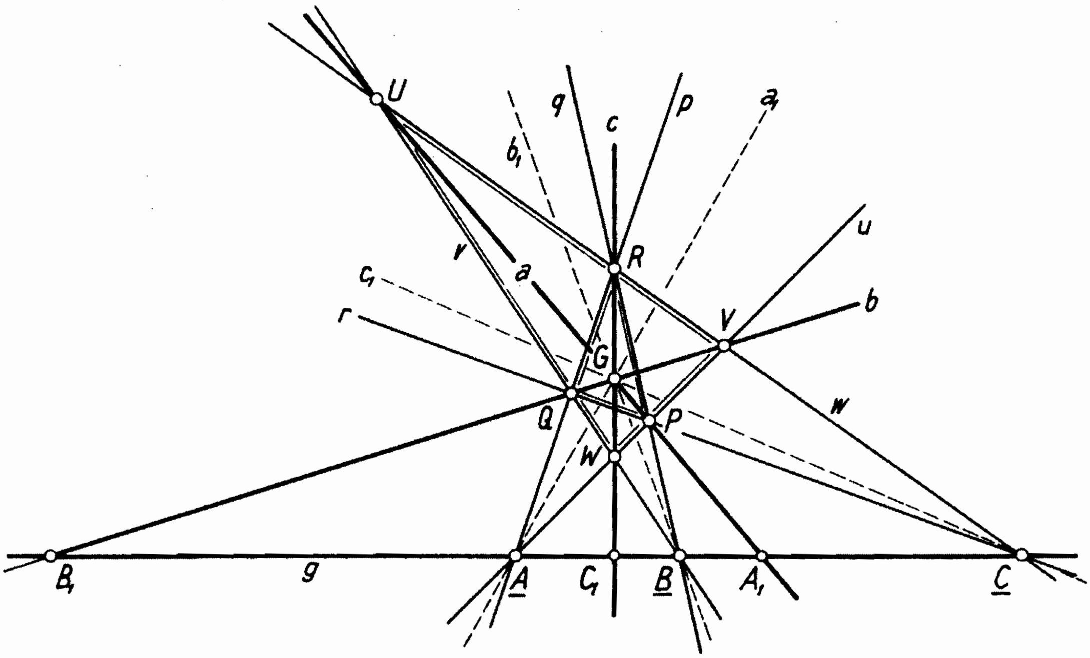


Figure 154

REMARK. Figure 154 is overburdened with notation. One should try to free oneself from the notation and grasp the form itself. In a written explanation that ought to be as concise as possible both in its wording and the number of figures, some notation has to be used.

By means of the construction that is polar in the geometry of the field, with three lines *a, b, c* of a point *G* are associated in a harmonious way three further lines a_1, b_1, c_1 in *G*.

Choose on *a* a point *U*, on *b* a point *V*, on *c* a point *W* (Figure 154) in such a way that *U, V, W* do not belong to the same point range. Let the sides of the 3-point *UVW* be labelled $u = VW$, $v = WU$, $w = UV$. Now the points of intersection

$$P = au, \qquad Q = bv, \qquad R = cw$$

present themselves. The two 3-points *UVW* and *PQR* are, by construction, in perspective with respect to *G*. Hence, by Desargues' Theorem, the points of intersection

$$up = A, \qquad vq = B, \qquad wr = C$$

lie on a line g. With G, these points provide the required lines

$$AG = a_1, \qquad BG = b_1, \qquad CG = c_1$$

(Note that, in using the same figure for the polarization, the two 3-points UVW and PQR exchange roles.)

We have related to one triple ABC in g, another triple $A_1B_1C_1$ in g; similarly to one triple abc in G, another $a_1b_1c_1$ in G. On the face of it, this relationship is infected with arbitrariness, in that the auxiliary 3-side pqr (or the auxiliary 3-point UVW) can be chosen in infinitely many ways. Yet the following is a fundamental fact:

The points A_1, B_1, C_1 associated harmoniously by the given construction with the points A, B, C, are uniquely determined by the latter: any auxiliary 3-side pqr leads to the same points A_1, B_1, C_1.

Proof. Suppose the auxiliary 3-side pqr produces the points A_1, B_1, C_1. What is to be shown is that any other 3-side $p'q'r'$ yields the same points.

To begin with we consider the complete 4-point $PQRW$ in Figure 154 on its own (Figure 155). Suppose $p'q'r'$ leads to the corresponding 4-point $P'Q'R'W'$. By construction, the two 3-points PQR, $P'Q'R'$ are in perspective with respect to g, and thus (by Desargues' Theorem) with respect to a point as well: that is, PP', QQ', RR' go through a point L.

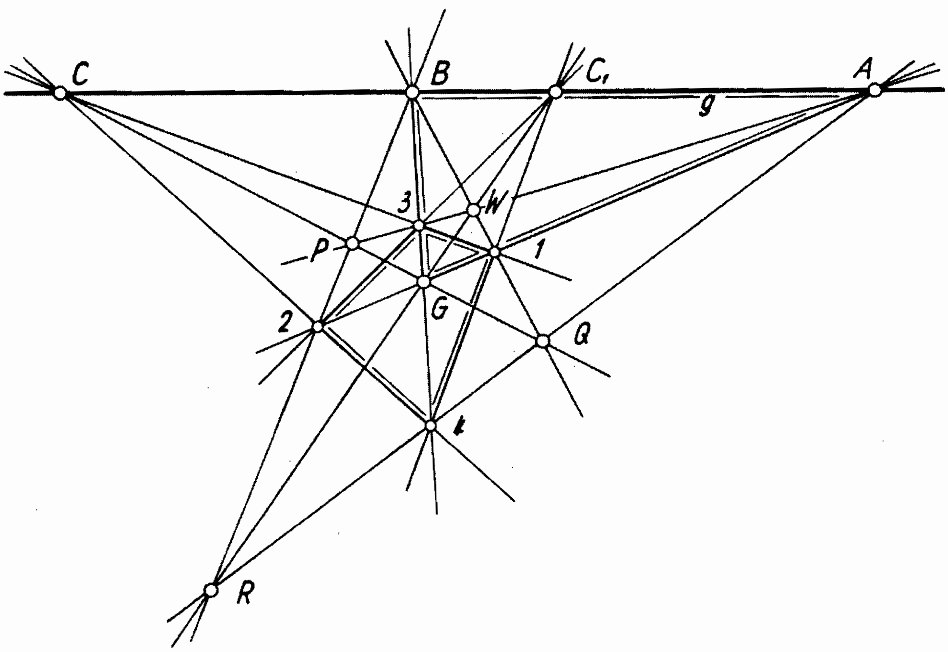


Figure 155

Again by construction, the two 3-points PQW, $P'Q'W$ are in perspective with respect to g, thus PP', QQ', WW' also belong to a point, that is, WW' goes through L as well.

Hence the two 3-points PRW, $P'R'W'$ are in perspective with respect to L. Therefore the points of intersection $(PR, P'R') = B$, $(PW, P'W') = A$ and $(RW, R'W')$ of corresponding sides lie in a line. Thus $R'W'$ goes through C_1.

Thus any 3-side pqr we use leads to the same point C_1. By the same reasoning applied to the 4-points $PQRU$ and $PQRV$, we obtain the result that A_1 and B_1 are also uniquely determined by A, B, C.

The six points A, B, C, A_1, B_1, C_1 (something similar is true of the six lines a, b, c, a_1, b_1, c_1) can be combined in three sets of four points or "fours," the three initial points appearing in each such four:

$$A, B, C, C_1 \text{ and } B, C, A, A_1 \text{ and } C, A, B, B_1$$

The positioning of the four points of such a set, for example A, B, C, C_1, is characterized by the following property (Figure 155):

There exists a complete 4-point of which two opposite sides go through A, two other opposite sides through B, a fifth side through C and the sixth through C_1.

For the construction of C_1 from A, B, C, we need no more than a complete 4-point $PQRW$ as auxiliary figure.

For reasons which will become apparent, such a set of four points is called harmonic. The harmonic four A, B, C, A_1 consists of two pairs of points A, B and C, C_1. The second pair differs from the first in that its elements each contain only one side of a 4-point used in the construction.

From a proposition about the complete 4-side proved earlier (Figure 75) it follows that

The pairs of elements of a harmonic four separate each other.

Thus three points A, B, C of a line determine the three harmonic fours

$$AB/CC_1 \text{ and } BC/AA_1 \text{ and } CA/BB_1$$

In the configuration that is polar in the field, the relative positions of the lines of the harmonic line set ab/cc_1 are characterized by the following property (Figure 156):

There exists a complete 4-side of which two opposite vertices lie on a, two other opposite vertices on b, a fifth vertex on c and the sixth on c_1.

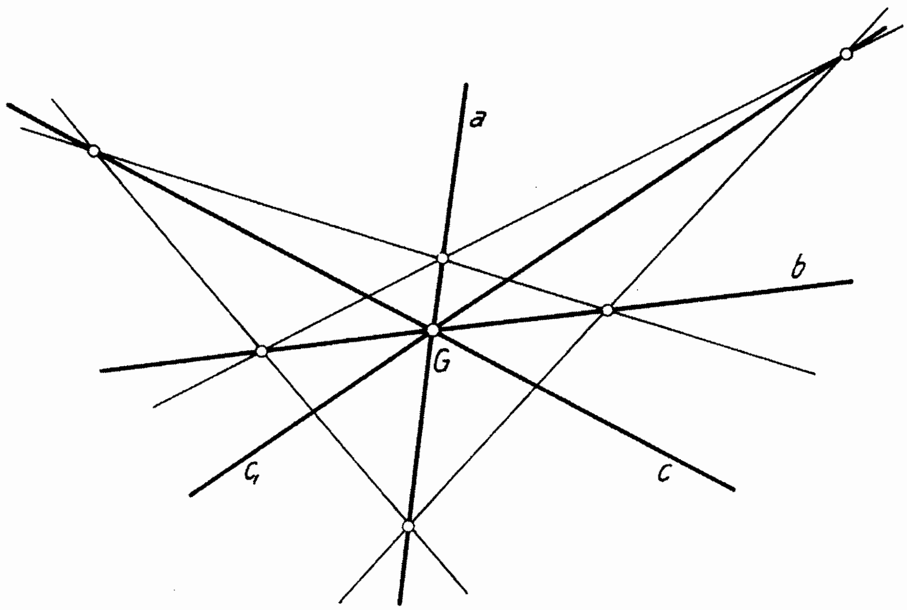


Figure 156

The two pairs A, B and C, C_1 of the harmonic four AB/CC_1 do not, according to the definition, have the same significance. Yet closer inspection shows that the harmonic relationship is symmetric with respect to the two pairs of elements. *We can thus speak of two pairs separating each other harmonically.*

Indeed, let $PQRW$ be any 4-point determining the harmonic four AB/CC_1 (Figure 157). Let the line AG meet BQ in 1 and BP in 2; let the line BG meet AP in 3 and AQ in 4. We now consider the 4-point 1234. Its sides 13 and 24 go through C, 23 and 14 go through C_1, which results from Desargues' Theorem as follows. The 3-points ABG, PQR are in perspective with respect to W; the points of intersection of corresponding sides, namely the points C, 4, 2, thus lie in a line. Similarly,

ABG and WRQ are in perspective with respect to P,
ABG and RWP are in perspective with respect to Q,
ABG and QPW are in perspective with respect to R,

from which it follows that the triples C_1, 4, 1 and C_1, 3, 2 and C, 3, 1 respectively are in line.

Of the complete 4-point 1234, two opposite sides go through C, two other opposite sides through C_1, a fifth side through A, and the sixth through B. By definition, therefore, CC_1/AB is a harmonic four, as was to be proved.

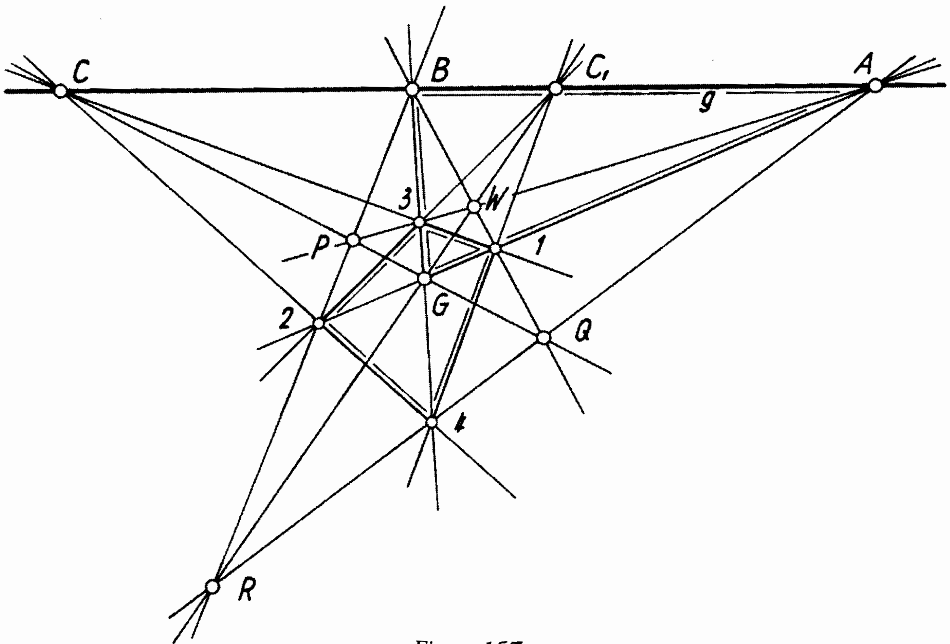


Figure 157

Figure 157 is the *Fundamental Harmonic Configuration.* Only we are now seeing it as two complete 4-points, which have an extra vertex (namely *G*) and an extra side (namely *g*) in common.

The same figure can be seen as two linked complete 4-sides, which have an extra side (namely *g*) and an extra vertex (namely *G*) in common.

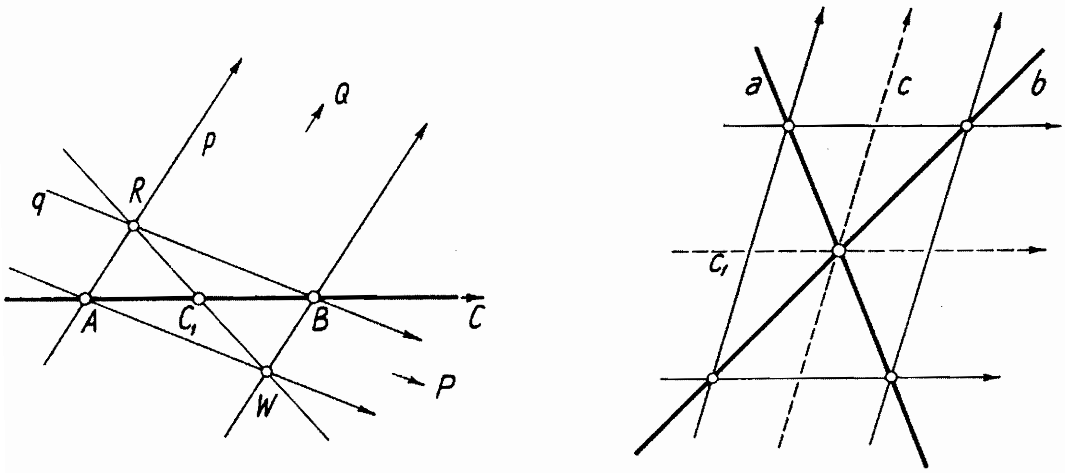
By definition and by the symmetry, just demonstrated, of the pairs involved in the harmonic four, the following is true:

If $^{BA}/_{CC_1}$ *is a harmonic four, then so are* $^{BA}/_{CC_1}$, $^{AB}/_{C_1C}$ *and* $^{BA}/_{C_1C}$, *as well as* $^{CC_1}/_{AB}$, $^{C_1C}/_{AB}$, $^{CC_1}/_{BA}$ *and* $^{C_1C}/_{BA}$.

The concept of a harmonic four can be expressed in another way: if *A*, *B* and *C*, C_1 separate each other harmonically, we call C_1 the *perspective mean* (or simply *mean*) of *A*, *B* with respect to *C*. Similarly, *C* is the perspective mean of *A*, *B* with respect to C_1.

The reason for this terminology is evident from Figure 158: of the three given points *A*, *B*, *C*, let *C* be the limit point of *AB*. If we choose the limit line of the plane of the figure as side *r* of the auxiliary 3-side *pqr*, then *A*, *R*, *B*, *W* form the vertices of a parallelogram, and C_1 becomes the mid-point of *AB* in the usual sense.

If ab/cc_1 is a harmonic set of lines (Figures 156 and 159), we call *c* the mean line of *a*, *b* with respect to c_1, and c_1 the mean line of *a*, *b* with respect to *c*.



Figures 158 and 159

REMARK. Harmonic fours meet us in sense-perceptible images everywhere. The mid-point C_1 of AB with respect to *tactile space*—measured with a fixed scale—appears in *visual space* as the perspective mean in the sense defined (Figure 160).

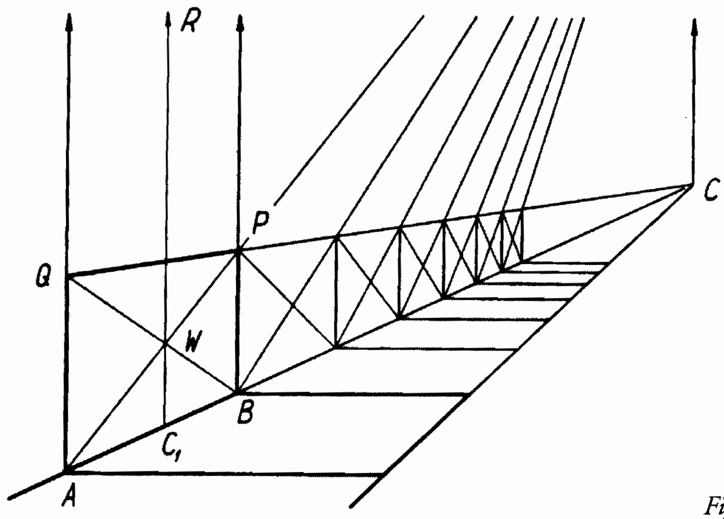


Figure 160

The following propositions about harmonic fours are fundamentally important:

If a point Z is connected with the points A, B, C, C_1 of a harmonic four AB/CC_1 on z, then the connecting lines a, b, c, c_1 constitute a harmonic four ab/cc_1 in Z.

If the lines a, b, c, c_1 of a harmonic four ab/cc_1 in Z are intersected with a line z, then the points of intersection A, B, C, C_1 constitute a harmonic four AB/CC_1 in z.

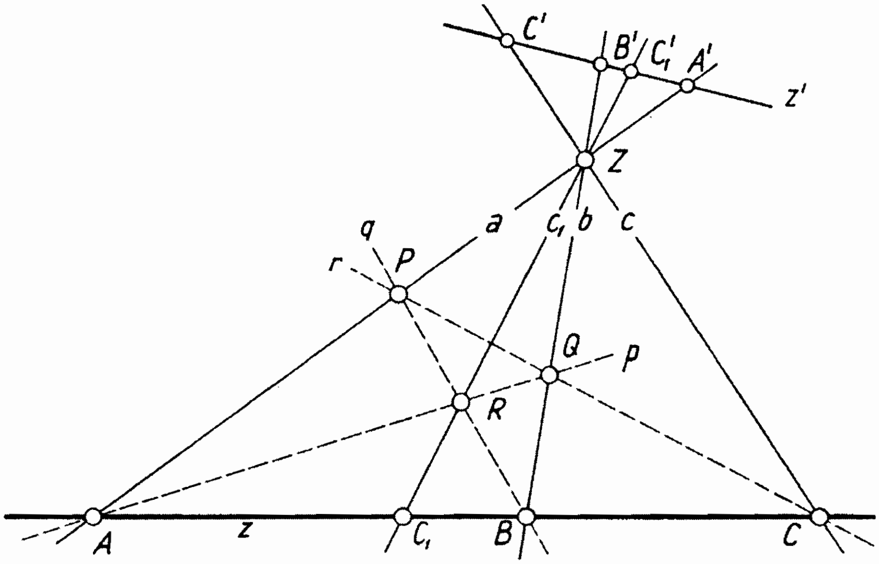


Figure 161

Proof of the left-hand proposition (Figure 161): In the plane zZ put a line p through A distinct from $ZA = a$. Let p meet $ZB = b$ in Q. Let the line $CQ = r$ meet a in P. The connecting line $BP = q$ forms, with p and r, an auxiliary 3-side pqr which determines the complete 4-point $PQRZ$ and hence the point C_1. That is, the point of intersection $pq = R$ lies on $ZC_1 = c_1$. But, in the complete 4-side $pqrz$, two opposite vertices lie on a, two other opposite vertices on b, a fifth vertex on c and the sixth on c_1. Therefore ab/cc_1 is a harmonic four. Figure 161 can also be used to prove the right-hand proposition.

If we intersect the four lines in question in the left-hand proposition with any line z', the corresponding points of intersection form a harmonic four $A'B'/C'C'_1$ once more. That is, if we project a harmonic point set from any point onto a line, then another harmonic set is produced.

The form which, in space, is polar to a harmonic point set is the harmonic plane set AB/CC_1. The four planes A, B, C, C_1 of a sheaf (g) form a harmonic plane set if there is a complete punctual 4-plane with two opposite edges lying in A, two other opposite edges in B, a fifth edge in C, and the sixth edge in C_1.

If three planes A, B, C of a sheaf (g) are given, then the plane C_1 is found by the construction polar to the one given at the beginning (Figure 162): An arbitrary point X is chosen on g. Through X there are put a line p in A, a line q in B, and a line r in C, the lines chosen so as not to belong to one plane. Let the faces of the auxiliary 3-edge pqr be called $P = qr$, $Q = rp$, $R = pq$. The plane $W = (AP, BQ)$ is also formed. The 4-plane $PQRW$ has the property that two opposite edges lie in A, two opposite edges in B and a fifth edge lies in C. When connected with g, the sixth edge RW yields the sought-for plane C_1.

To facilitate visualization, the punctual form in X has, in Figure 162, been intersected with a plane. The following can be seen from the figure (you need consider only the 4-side in which the 4-plane $PQRW$ is intersected by a plane):

If the planes of a harmonic plane set AB/CC_1 are intersected with a plane not containing the line carrying the set, then the four lines of intersection form a harmonic line set. And conversely, if the lines of a harmonic line set are connected with a point outside its plane, then a harmonic plane set is obtained.

Furthermore from this it follows, with the help of the two earlier propositions, that

If the planes of a harmonic plane set are intersected with a line skew to the line carrying it, then a harmonic point set is produced. And conversely, if the points of a harmonic point set are connected with a line skew to the line carrying it, then a harmonic plane set is obtained.

This can also be read directly from Figure 162.

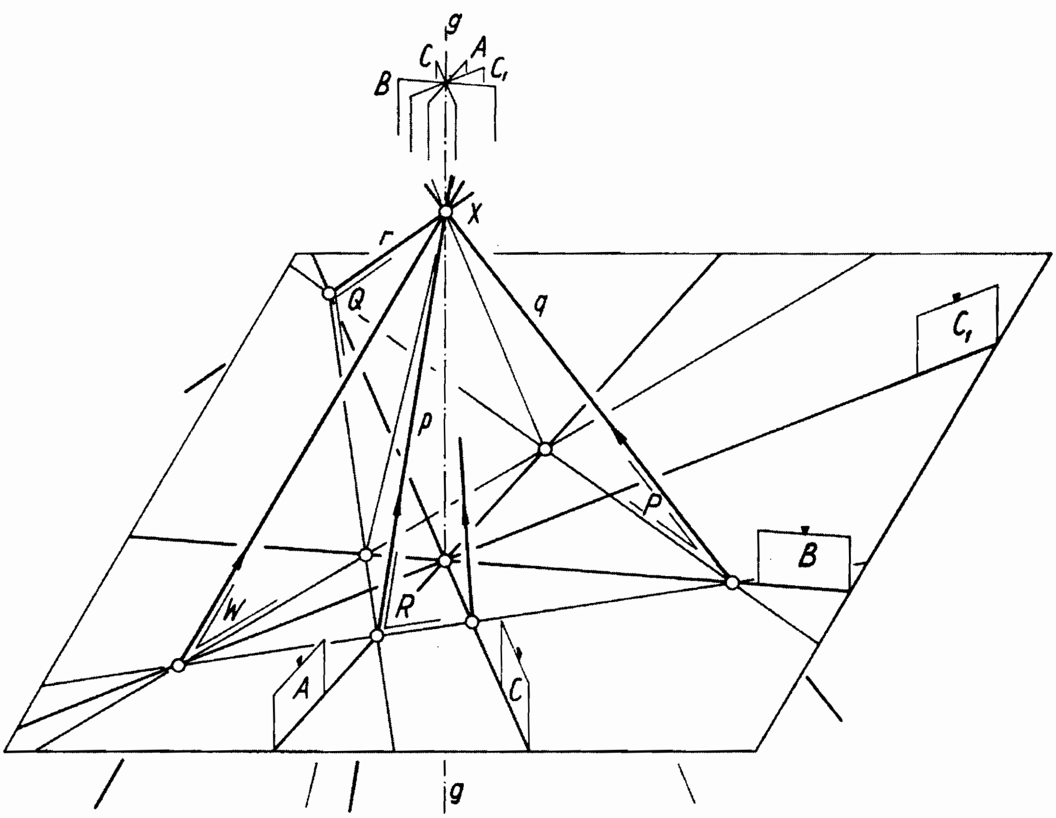


Figure 162

We have now gained an insight into the fundamental law of the preservation of the harmonic relationship by the operations of connecting and intersecting. In order to be able to express it appropriately, we introduce the concept of *elementary construction chain.*

Suppose we are given a sequence—we also call it a chain—of first-degree basic forms. Suppose, too, that any two consecutive forms in the chain arise from each other by connecting and intersecting. For two such consecutive links of the chain there are six possibilities, as follows:

1. 3. *The first form is a point range.* Hence the second is either a line pencil or a plane sheaf. Connecting the points of the range with a point not belonging to the range, we get a line pencil; connecting them with a line that is skew to the line carrying the range produces a plane sheaf. In both cases we form a projection of the point range.

2. 4. *The first form is a plane sheaf.* Hence the second is either a line pencil or a point range. Intersecting the planes of the sheaf with a plane not belonging to the sheaf, we get a line pencil; intersecting them with a line skew to the line carrying the sheaf produces a point range. In both cases we form a section of the plane sheaf.

5. 6. *The first form is a line pencil.* Hence the second is either a point range or a plane sheaf.

If we intersect the lines of the pencil with a line belonging to the plane but not to the point carrying the pencil, then we obtain a point range.

Connecting the lines of the pencil with a line belonging to the point but not to the plane carrying the pencil, then we obtain a plane sheaf.

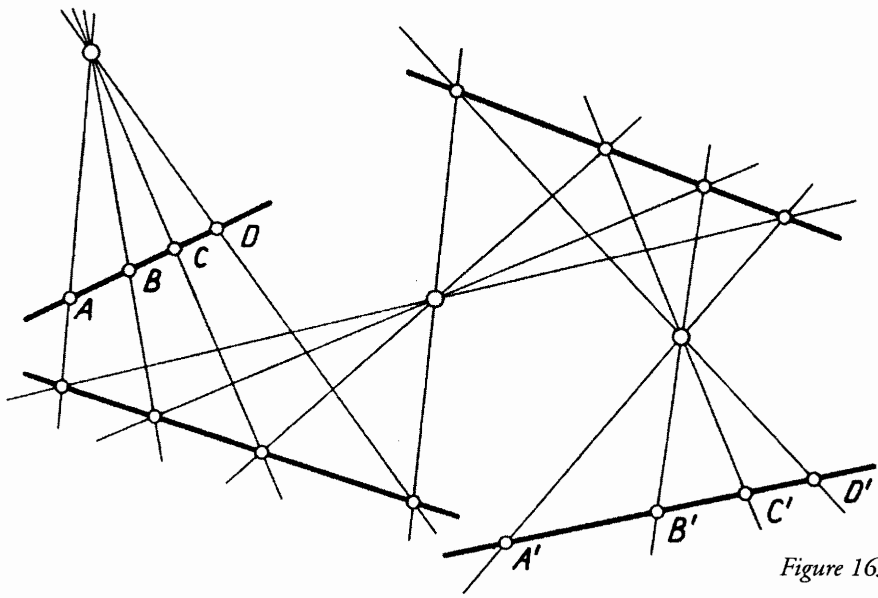


Figure 163

We could also say, taking in all six cases, that two consecutive basic forms of the chain are connected as follows:

Each element of one basic form belongs to one and only one element of the other basic form.

If a sequence of first-degree basic forms exhibits the properties mentioned, then we speak of an *elementary construction chain.* Figure 163 shows such a (seven-linked) chain in the plane; the links in this case can only be point ranges and line pencils.

Suppose A, B, C, . . . are elements of the first basic form G of an elementary construction chain, and A', B', C, . . . are the *corresponding* elements of the last basic form G' of the chain. Then we call the relationship between G and G brought about by the construction chain a projectivity; the forms G and G' themselves we call projective. We use the symbol $\overline{\wedge}$ for this relationship, and write

$$G \;\overline{\wedge}\; G \qquad \text{or} \qquad A'B'C \ldots \overline{\wedge}\, ABC \ldots$$

We can now state the *Preservation Law for harmonic relationships* as follows:

In an elementary construction chain, a harmonic four always corresponds with another harmonic four.

Or:

In projective forms, to any harmonic four of one form there corresponds a harmonic four of the other.

A consequence of this fact, assuming the continuity of basic forms, is the Fundamental Theorem of projective geometry. From this theorem the whole of the geometry of space and counterspace can be developed step by step.

REMARK. The harmonic relationship turns out to be indestructible vis-a-vis the operations of connecting and intersecting. In it we have found a key concept which, because of its properties, has a special significance from the start. In the general concept of harmonic relationship, the various concepts of symmetry are included as special cases.

In conclusion, we explain some properties of the relative positioning of the points of a harmonic four AB/CC_1 on a line g. Similar properties will hold for harmonic line and plane sets.

Suppose A and C are held fixed. It is immediately clear that if B moves towards A, then C_1 will also move towards A, since C_1 is always separated from C by A and B. Here "X moves towards Y" shall mean not only that X approaches Y

as its goal, but also that no interval, however small, prevents this process by blocking the way to Y.

What happens if, while A and B remain fixed, point C moves towards B? Suppose, in Figure 164, we keep fixed the points A and B, the auxiliary lines p and q (and hence R), and the vertex Q of the auxiliary 3-point PQR. Then the projection of point range g from Q onto q shows that P tends towards B. If we then project q from A onto $v = QW$, it follows that W also tends towards B. Finally, the projection of point range v from R onto g shows that C_1 tends towards B, too.

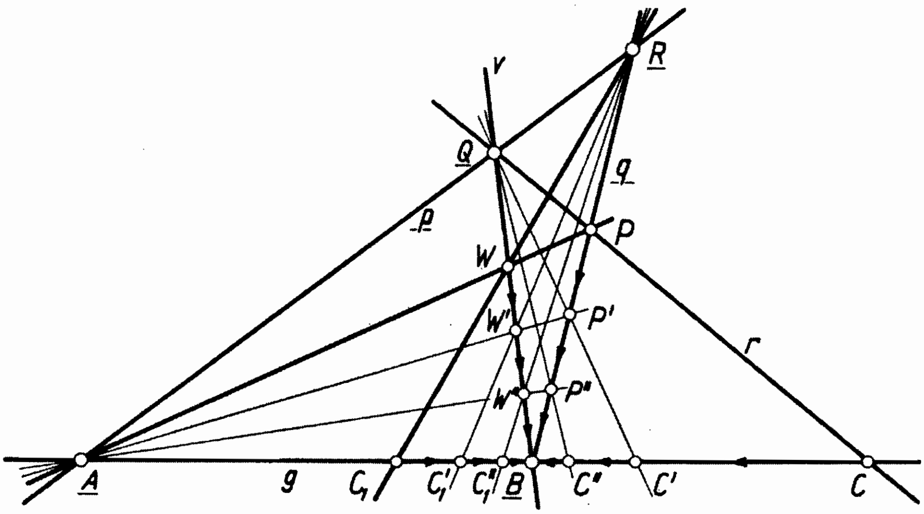


Figure 164

Expressing this more accurately, if $ACC\,C' \dots B$ indicates the natural ordering, then the same is true (Proposition III) of the sequences

$$RPP'P' \dots B \text{ and } QWW'W' \dots B \text{ and } AC_1C_1C'_1 \dots B$$

Since, in a harmonic four, the two pairs of points separating each other are equally significant, we can summarize these facts as follows:

I. *If, in a harmonic four that always remains harmonic, two elements move arbitrarily close to each other, then at least one of the other two elements moves arbitrarily close to the first two.*

The ranges

$$ACC\,C' \dots B \text{ and } AC_1C_1C'_1 \dots B$$

are the end links of the elementary construction chain

$$g \longrightarrow \text{pencil } Q \longrightarrow q \longrightarrow \text{pencil } A \longrightarrow v \longrightarrow \text{pencil } R \longrightarrow g$$

Through this chain, the point range g is related projectively to itself. Furthermore, each of the points A and B corresponds with itself: A and B are fixed points of the projective relationship. We call this relationship the harmonic reflection of the point range g in the pair of points A and B and say that C_1 arises from C by harmonic reflection in A and B. The construction chain just referred to shows that

II. *If C runs from A to B through one of the two segments determined by A and B, then the reflected point C_1 runs from A to B through the complementary segment; C and C_1 thus run in opposite senses.*

In particular:

III. *Two pairs of points that are both separated harmonically by a third pair of points cannot separate each other.*

Here the question arises, can two pairs of points E, F and K, L that do not separate each other, always be separated harmonically by the same pair of points M, N? The answer, in fact, is yes. To prove it we must invoke continuity however.

IV. *If two pairs of elements of a first-degree basic form which do not separate each other are given, then there always exists a pair of elements which separates both of the former harmonically.*

Proof. In a line g, let E, F and K, L be two pairs of elements that do not separate each other. Let EF and FE be the two segments determined by E and F. Suppose, for example, that K and L belong to the segment EF and $EKLF$ indicates the natural ordering of the points (a state of affairs which, if necessary, could be achieved by renaming the points). For brevity, we call the sense EKF of running through the elements "towards the right" (Figure 165).

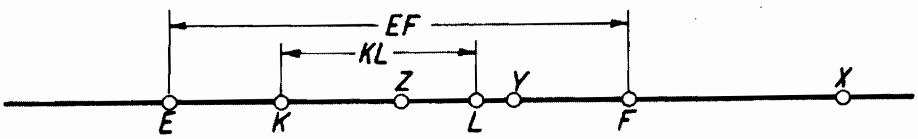


Figure 165

Now suppose X runs through the segment FE from $\underline{F}$ towards the right. At each of its positions we reflect X in E, F and call the reflected point Y. Furthermore, let Z be the reflection of X with respect to K, L. If X runs from F towards the right, then Y goes from F towards the left, and Z moves leftwards on KL. At the start of the movement, Y is to the right of Z. But at the conclusion of the movement Y is to the left of Z. The segment EF can therefore be divided into two parts in a way that preserves order: in the right-hand subset put all positions of Y for which Z still lies to the left of Y, in the left-hand subset all other positions of Y. For this division in two there must, because of continuity, exist a dividing element N. In it Y and Z coincide. If M is the corresponding position of X, then M and N constitute a pair of points in which E and F, as well as K and L, are reflections of each other. This proves Proposition IV.

EXERCISES

1. Construct, for three given points A, B, C of a line g, the point D such that the four AB/CD is harmonic. Repeat the exercise enough times to develop a true feeling for the relative positions of pairs of points that separate each other harmonically.

2. Construct, for three given lines a, b, c of a point G, the line d such that the four ab/cd is harmonic. (The same remark applies here as in Exercise 1.)

3. Construct, for three given points A, B, C of a line g, the three points A_1, B_1, C_1 associated harmoniously with them. Then construct the three points associated harmoniously with A_1, B_1, C_1.

4. Carry out the construction corresponding to Exercise 3 for three lines a, b, c of a point G.

5. Of the three points A, B, C of a line, let C be the line's limit point. Construct the points A_1,B_1,C_1 as in Figure 154.

Remark. The following three fundamental problems deserve special attention. They are about different kinds of symmetry. The following terminology, though fairly obvious, needs explaining.

"Two points and two lines separate each other harmonically" means that the two points are harmonically separated from the points of intersection of their connecting line with the two lines.

"Two points A, B are harmonically separated by a point C on AB and a line d" means that A, B are harmonically separated by C and the point of intersection of AB with d.

"Two lines a, b are harmonically separated by a line c through ab and a point D" means that a, b are harmonically separated by c and the line connecting ab with D.
And so on.

6. Symmetry in plane geometry. The points P and P' are symmetric with respect to the point M and the line m whenever P, P' and M, m separate each other harmonically. The lines p, p' are symmetric with respect to M, m whenever p, p' and M, m separate each other harmonically.
   This general symmetry becomes central symmetry when m is the limit line of the plane in question, and (in general oblique) axial symmetry when M is a limit point.
   Construct, for given M, m and for a given figure, the symmetrical figure with respect to M, m.

7. Symmetry in point geometry. The planes P and P' are symmetric with respect to the plane M and the line m whenever P, P' and M, m separate each other harmonically. The lines p, p' are symmetric with respect to M, m whenever p, p' and M, m separate each other harmonically.
   Think this out for yourself when M is a horizontal plane and m a line perpendicular to M.

8. Symmetry with respect to two skew lines. The points P, P' are symmetric with respect to the skew lines m, n whenever P, P' and m, n separate each other harmonically. That is, the line PP' meets m in M and n in N in such a way that PP'/MN is a harmonic four.
   If n is a limit line of space, an (in general oblique) axial symmetry results and, in a special case, the usual axial symmetry.
   As an example of the general case, choose two skew lines of a cube as m and n, and determine the figure that is symmetrical to the cube with respect to m and n. (Consider, in particular, the point symmetrical to the cube's middle point.)

9. Prove, with the help of Figure 161, the right-hand proposition.

10. Verify that the 13 points and 13 lines of the Fundamental Harmonic Configuration organize themselves into nine harmonic point sets and nine harmonic line sets.

11. Think out for yourself (for example, with the aid of Figure 85) the nine harmonic plane sets and the nine harmonic line sets in a punctual Fundamental Harmonic Configuration.

12. In a mental picture, form an elementary construction chain in which all three first-degree basic forms appear at least once.

13. In the text, a harmonic plane set is defined as being polar in space to a harmonic point set, and a harmonic line set is defined as being polar in the field to a harmonic point set. This still leaves us with the possibility of formulating a definition that is polar in space to the given definition of a harmonic line set, as follows. The four lines a, b, c, c_1 of a pencil constitute a harmonic four ab/cc_1, if there exists a complete punctual 4-edge with two opposite faces through a, two other opposite faces through b, a fifth face through c and the sixth face through c_1.

Careful inspection shows that this definition has the same meaning as the polar one. This can be seen from Figure 162. (Consider the four lines with arrows on them.)

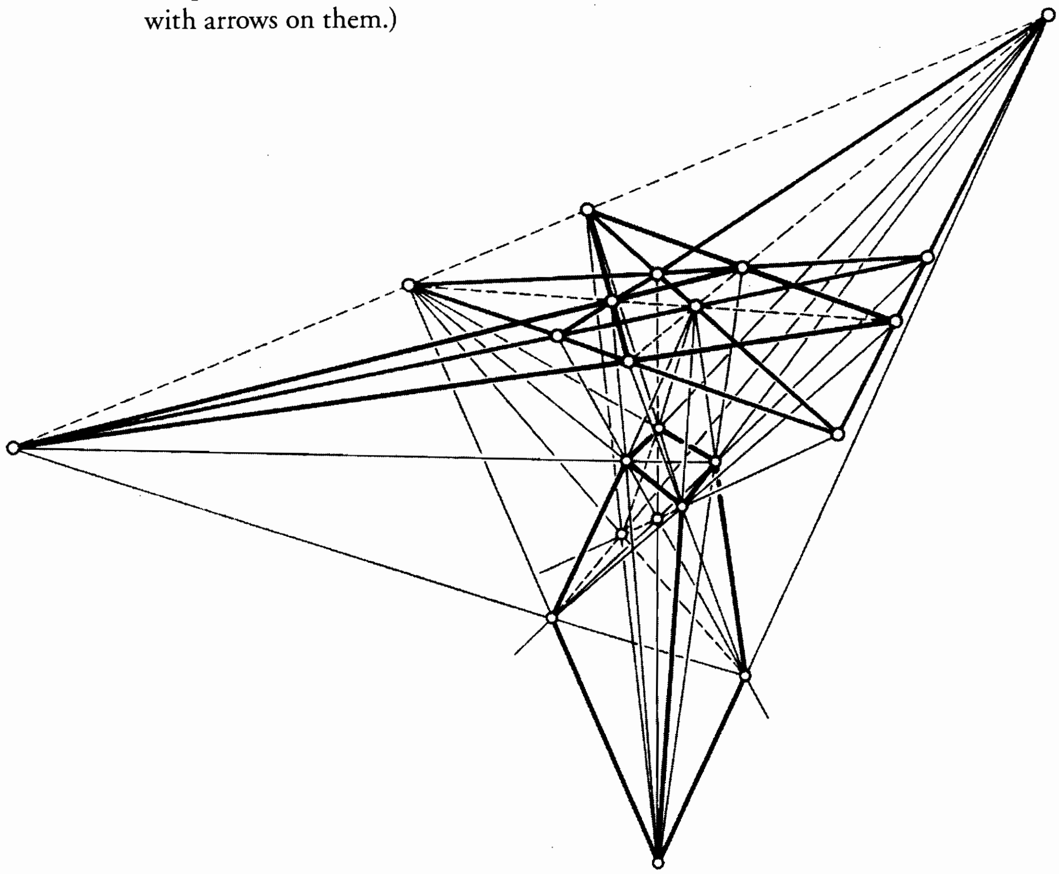


Figure 166

14. In Figure 166 a complete hexahedron is drawn. Consider first the triangle through each of whose vertices go four edge lines. Pairs of opposite faces of the hexahedron go through the sides of this triangle. Let D^- be the plane of this triangle (in Figure 16 this is the plane $A^+B^+C^+$). Each of the four cross-lines intersect D^- in a point. Consider the complete 4-point formed by these four points. Its six sides are the lines of intersection of the diagonal planes with the plane D^-. The triangle considered just now is the extra triangle of this 4-point.

Lastly, include the face diagonals of the hexahedron's six faces. These meet in pairs in points on the sides of the extra triangle. These six points are the vertices of a complete 4-side that has the extra triangle in common with the former 4-point. The four sides of the complete 4-side arise as follows: Take, for example, the lowest vertex of the hexahedron; three edges go out from this vertex. Their end-points form a plane whose line of intersection with D^{-} is a side of the 4-side. With each of the hexahedron's vertices is associated one of these planes. These produce, in pairs, the four sides of the 4-side.

The hexahedron is thus linked, in its plane D^{-}, with a Fundamental Harmonic Configuration. (In each of the twelve faces of the Fundamental Structure is to be found such a Fundamental Harmonic Configuration.)

15. Starting from the points A, B, C, determine the harmonic four AB/CC_1 with an auxiliary 3-side pqr in which p and q are parallel, and confirm (using theorems of elementary geometry) that, measured in tactile space, the harmonic four has the following property: The ratio of the lengths of the (finite) segments AC and BC is the same as the ratio of the lengths of the segments AC_1 and BC_1.

16. On a line, two pairs of points E, F and K, L, that do not separate each other, are given. Determine as accurately as possible the pair M, N that separate both of the former pairs harmonically.
(Use trial and error. The exact construction uses a method which is beyond the scope of this book.)

Chapter 19

The Fundamental Theorem

In this chapter we explain the *Fundamental Theorem of Projective Geometry*. The scope of its consequences can only, it is true, be comprehended once we have seen how other facts about space can be developed from it. You can clearly distinguish two components that combine to bring this theorem about. One, the Preservation Law of the harmonic relation, was dealt with in the last chapter. Looking at the phenomena leading to this law, we notice that the continuity of basic forms was not invoked. Continuity does play a decisive role in the second component, however.

We begin by defining what we understand by the net determined by three elements of a first-degree basic form. In other words the net determined by three points or three planes A, B, C of a line, or by three lines a, b, c of a pencil. We develop the concept in the case of three points A, B, C of a line g.

The net $\{ABC\}$ comes about as follows. We construct the three points A_1, B_1, C_1 associated harmoniously with the points A, B, C; in other words we reflect A harmonically in B and C, B in C and A, and C in A and B. Next we select three points from the six in every possible way—besides A, B, C and A_1, B_1, C_1—that is: A, B, C_1 and A, B_1, C and A_1, B, C and A, B_1, C_1, etc., and construct each time the points associated harmoniously with them. From the set of points obtained, we take, again in all possible ways, a triple of points not yet used and construct the points associated harmoniously with this triple. Imagine the process continued indefinitely. The resulting set of points is the net $\{ABC\}$.

In short, $\{ABC\}$ arises from A, B, C through successive harmonic reflections. The net $\{ABC\}$ clearly constitutes a countable set, since we could count the steps of the construction creating it.

From the whole set we now take a subset whose construction is particularly easy to see. As in Figure 154, using a 3-side pqr and the corresponding complete 4-point $PQRU$, we determine the point that is the reflection of A with respect to B, C; for soon-to-be obvious reasons we call it A_2, however (Figure 167). Then we reflect the point B harmonically in A_2, C to obtain A_3. After that A_2 is reflected in A_3, C which gives us A_4, then A_3 in A_4, C and so on.

These reflection points can be constructed as follows: ABA_2C is projected from R onto r, giving the set QPP_2C. This is projected from U back onto g, producing BA_2A_3C. By the Preservation Law, A_2C/BA_3 is a harmonic four, which also follows directly from the position of the complete 4-point PP_2RU. After that,

BA_2A_3C are projected from R onto r and the points obtained are projected from U back onto g providing $A_2A_3A_4C$. We continue in this way indefinitely: projecting from R onto r and from U back onto g. In this way a scale $ABA_2A_3A_4 \ldots C$ is produced on g with the property that *any three consecutive points together with C form a harmonic four.*

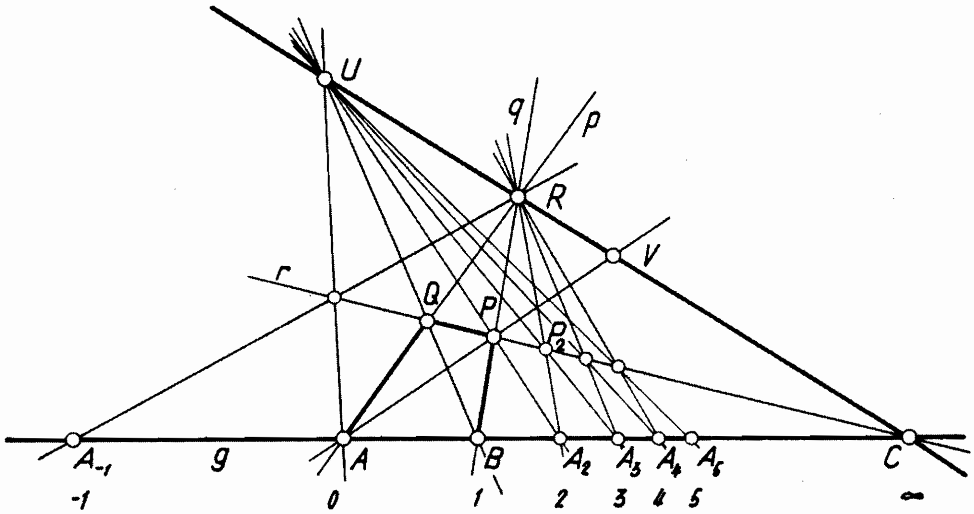


Figure 167

By a similar construction the scale can also be continued in the opposite sense CBA of running through the points of g:

$$\ldots A_{-3}A_{-2}A_{-1}ABA_2A_3 \ldots \; C$$

It is natural to assign *ordinal numbers* to the points of this scale, in particular to point A the ordinal 0, to point B, 1, and to point C, ∞ (infinity). In this way the sequence of whole numbers . . . . –3, –2, –1, 0, 1, 2, 3 . . . including ∞ is related in a one-to-one correspondence to the scale points.

Here, these numbers do not measure fixed lengths, but merely represent the order relationships of the scale. Yet one notices straight away that the scale we have constructed becomes the usual measuring scale in the tactile space of rigid bodies—with AB as unit of length—when the point C is taken as limit point.

Clearly Figure 167 is also determined by the 4-point $ABPQ$. Along g we see a sequence of 4-points that follow the basic 4-point $ABPQ$. By adding suitable lines to the drawing, a net of quadrangular meshes is produced; this is why we have called the set $\{ABC\}$ a net in g for short. Since August Ferdinand Möbius (1790 – 1868), one of the co-founders of modern geometry, was the first to study such nets, we speak of a Möbius net determined by the basic 4-point $ABPQ$ (Figure 168).

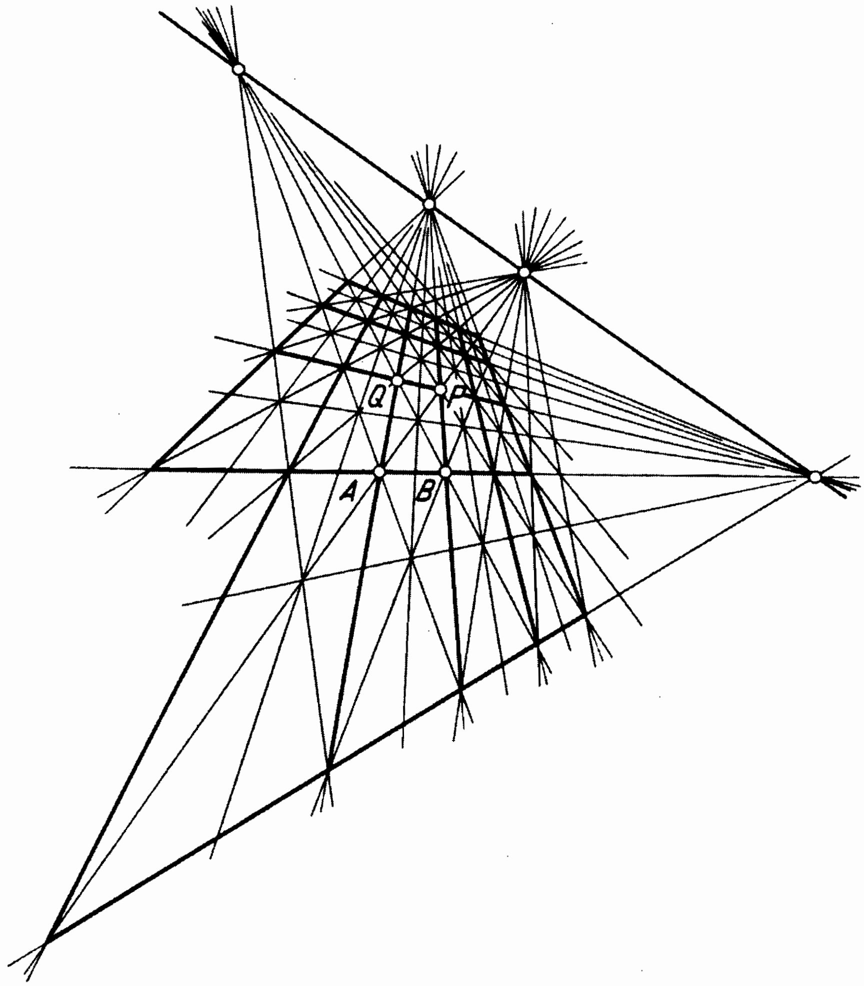


Figure 168

When the basic 4-point is a parallelogram, a rectangle or—in the most special case—a square, the construction described produces a net of parallelograms, rectangles, or squares, respectively.

From the figure it is immediately clear how we can make the net more dense. (Desargues' Theorem is used repeatedly here.) Each mesh is divided into four sub-meshes. With the second "densification" each of these new meshes again becomes four sub-meshes, so that each original mesh is divided into 16 sub-meshes, and so on.

With the first densification we assign to the appropriate new scale points the ordinal numbers

$$\ldots\ -3/2, -1/2, +1/2, +3/2, +5/2, +7/2\ \ldots.$$

With the second densification we assign the appropriate quarters, with the third densification the appropriate eighths, and so on.

Since each successive densification consists of constructions of harmonic fours, as suitable 4-points indicate, it follows that all the scale points produced in this way belong to the net $\{ABC\}$.

It is immediately clear that the net $\{ABC\}$ is dense in itself, that is, *each segment whose end-points X, Y are points of the net contains other points of the net.* Because if Z is any point of the net not belonging to this segment, all we need to do to obtain such a point is to reflect Z harmonically in X, Y.

A question arises here that everyone, from their empirical knowledge limited to tactile-space, will immediately answer in the affirmative: Is the net $\{ABC\}$ dense not only in itself but also in the line carrying it? In other words, does every segment of the line, even if its end-points are not net points, contain points of the net?

If it is true, then harmonic reflection provides us with a means, using a stepwise process starting from three points, that is, using a countable procedure, of getting into any interval of the line. This was mentioned earlier when we defined the concept of a skeleton. Expressed in terms of the latter, our question becomes: Is the net $\{ABC\}$ a skeleton? That the answer is "yes" constitutes the second component of the Fundamental Theorem:

Skeleton Theorem. The net determined by three elements of a first-degree basic form is a skeleton of this form.

If we picture, within the tactile space of rigid bodies, the process of repeated bisection, the theorem seems obvious. The problem, however, is to see whether the theorem is linked with the tactile-space concept of length, or whether it is a consequence of more general phenomena. It turns out that the theorem follows from our system **A** and **O**.

Proof. Let U and V be the end-points of an arbitrary interval of the line g, which are not separated by any two of the given points A, B, C (Figure 169). We can assume that U and V lie between A and B with respect to C, a situation that can be achieved by swapping round the names A, B, C if necessary. Further, we may assume that the sequence $AUVBC$ indicates these points' natural ordering. We have to prove that there are net points between U and V ("between" always meaning "between with respect to C").

If U and V are net points, we only need to reflect C harmonically in U, V in order to produce a net point between U and V. From now on we assume that at least one of the points U and V, say U, is *not* a net point.

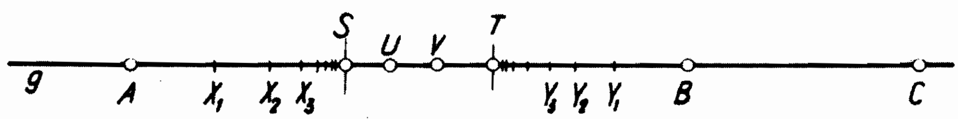


Figure 169

Starting from U, we move leftwards (in the sense VUA) towards A. In doing so we either meet for the first time a net point X (which if necessary must occur in A at the latest), or else we can give no such first net point. The second alternative is only possible if, between A and U, there are *infinitely* many net points. In this second case we can divide the interval AU into two parts in a way that preserves order: All points to the right of which no net points are to be found make up the right subset, all other points the left subset. By continuity, this division in two can only be brought about by a dividing element S. S is not a net point, because otherwise the first case would have occurred; S is nevertheless an accumulation point of net points.

Similarly we start from V and move to the right (in the sense UVB) towards B. We meet either a first net point Y (this might be V itself or, at the latest, B) or, before any net point, a point T which, though not a net point itself, is an *accumulation point* of net points.

There are now four possibilities to check, according to whether we met X or S and Y or T, namely: X and Y, S and T, X and T, S and Y.

In the first case (X and Y) we reflect C in X, Y. The reflection point Z is a net point lying between X and Y but neither between X and U nor between V and Y, and therefore between U and V.

In the second case (S and T) there exist net points in every interval whose right-hand end-point is S, as well as in every interval whose left-hand end-point is T. Hence a net point can be chosen arbitrarily close to S—let X_i be one such—and a net point can be chosen arbitrarily close to T—let Y_i be one such. We now reflect point C harmonically in X_i, Y_i. The reflection point Z_i is a net point. Suppose now that for each choice of X_i and Y_i the corresponding point Z_i does not lie between U and V. Then it must always be found either between X_i and S or between T and Y_i, that is, either arbitrarily close to X_i or arbitrarily close to Y_i. We would then have harmonic fours X_iY_i/Z_iC in which two of the elements are squeezing arbitrarily close together without a third element coming to join them arbitrarily closely. But by Proposition I this is impossible. Thus there are net points Z_i lying between U and V.

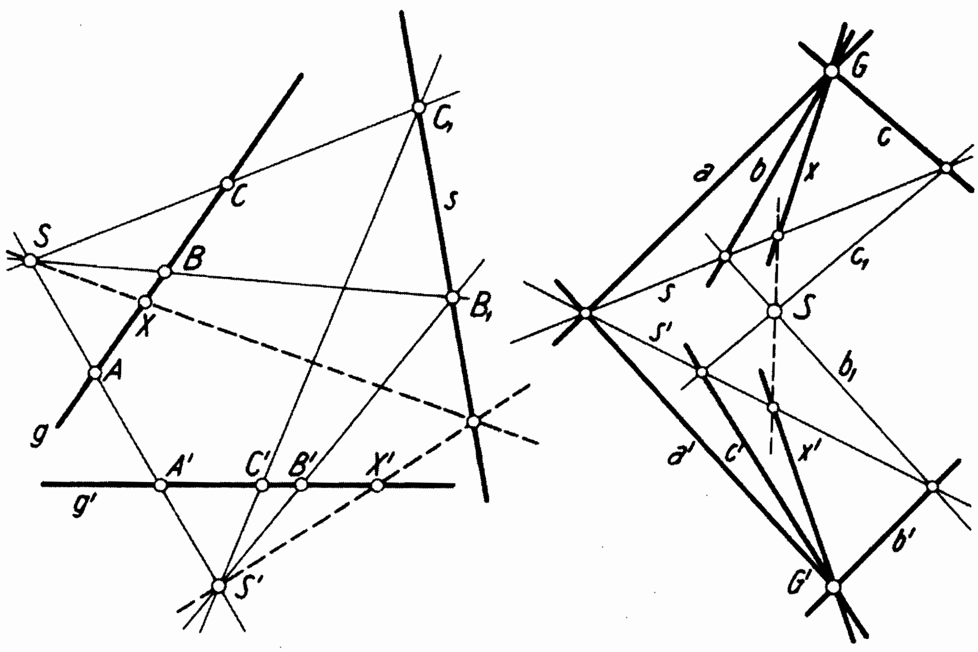
In the third case (X and T) we can assume there are net points Y_i lying arbitrarily close to T. Suppose we determine the net points Z_i harmonically separated from C by X and Y_i. By the proposition just referred to, the points Z_i so constructed cannot all lie between T and Y_i, and so there are net points Z_i between U and V. The fourth case is dealt with similarly. This proves the Skeleton Theorem.

First Remark. Notice the characteristic role continuity plays in the proof. It guarantees the existence of the points S and T. Without these points, no further conclusion can be drawn, unless we have recourse to other concepts (for example, facts true in the world of rigid bodies). Only continuity assures us that with our stepwise constructions we can penetrate into every interval of a basic form. It forms the foundation without which we would remain in ignorance of how far our constructions are able to lay hold of the elements of a basic form.

SECOND REMARK. The rigorous proof of what we call the Skeleton Theorem (and with it the proof of the Fundamental Theorem) is one of the main obstacles to modern geometry (without the use of analysis) getting a footing in teaching. Our use of Proposition I here gives the proof an expression that lends itself, particularly in the class/lecture room, to fluent and impressive shaping.

THIRD REMARK. The Möbius net determined by a basic 4-point is dense in the point field to which it belongs. That is, by continued densification, every point of the field can be seen as the limit point of a sequence of meshes enclosing the point, which draw themselves arbitrarily tightly round it. This is a simple consequence of the Skeleton Theorem.

Basic construction. Given three points A, B, C of a line g and three points A', B', C' of a line g'—in general any three elements in each of two first-degree basic forms—we can give any number of elementary construction chains leading from A, B, C to A', B', C'. If the lines g, g' lie in the same plane then five links of the chain are sufficient (Figure 170). We choose two points S and S' on the line AA', then determine the points of intersection $B_1 = (SB, S'B')$ and $C_1 = (SC, S'C')$; let their connecting line B_1C_1 be called s. The chain $g \longrightarrow S \longrightarrow s \longrightarrow S' \longrightarrow g'$ leads, by construction, from A to A', from B to B' and from C to C'.



Figures 170 and 171

In the case of three lines a, b, c of a pencil G and three lines a', b', c' of a second pencil G', where G and G' belong to the same plane (Figure 171), put two lines s and s' through the point of intersection aa' and determine the connecting lines b_1

$= (sb, s'b')$ and $c_1 = (sc, s'c')$; let their point of intersection b_1c_1 be called S. The chain $G \longrightarrow s \longrightarrow S \longrightarrow s' \longrightarrow G'$ leads from a to a', from b to b' and from c to c'.

The auxiliary points S, S' and lines s, s' can be chosen arbitrarily in AA' and in aa' respectively, except that neither S and A nor S' and A' (and neither s and a nor s' and a' respectively) should coincide. Obviously construction chains with more than five members can be given to produce the required relation. In each such chain an element X of the first form ABC . . . corresponds to a particular element X' of the last form $A'B'C'$ . . . Surprisingly enough, *if two elementary construction chains each lead from the points A, B, C of a line g to the same points A′, B′, C′ of a line g′, then every point X of g is assigned to the same point X′ on g′ by both chains.* This is the *Fundamental Theorem of Projective Geometry.* In short it says that

A projective relation between two point ranges g and g′ is uniquely determined by three pairs of corresponding elements A and A′, B and B′, C and C′.

Or, in general:

A projective relation between two first-degree basic forms is uniquely determined by three pairs of corresponding elements.

The relationship established by elementary construction chains is thus independent of the choice of chain, provided only that in each chain three elements of the first form always correspond with the same three elements of the last form.

Proof. First of all let X be a point of the net $\{ABC\}$. Because of the Preservation Law, with every construction chain leading from A, B, C to A', B', C', the point corresponding to X is the point X' of the net $\{A'B'C'\}$ arising from A', B', C' by the same sequence of constructions (repeated harmonic reflection) as does X from A, B, C. When X is a net point, our theorem is thus simply a consequence of the Preservation Law.

Now let X be a point that does not belong to the net $\{ABC\}$. Suppose, for example, that X is a point between A and B with respect to C. Then, since the constructions of the chains are order-preserving, each of the chains in question leads from X to a point lying between A' and B' with respect to C'. Suppose one chain leads to X' and another to X''.

Since the net $\{A'B'C'\}$ is a skeleton, we can assume there is a *net point* N' between X' and X'' (with respect to C'). Corresponding to this point N' there is a point N between A and B which is the same in both chains. Now either $AXNB$ or $ANXB$ represents the natural order. Since the constructions of the chains are order-preserving, in the first case both $A'X'N'B'$ and $A'X''N'B'$ indicate the natural order, which is impossible, since N' lies between X' and X'' (with respect to C'). Thus X' and X'' cannot be distinct points. In the second case, n which $ANXB$ indicates the natural order, the conclusion is the same.

REMARK. In the proof just given, the following three properties of the projective relation were used:

1. Each element of one form is related to one and only one element of the other form.
2. A harmonic four of one form always corresponds with a harmonic four of the other.
3. The relation is order-preserving.

We are about to show that a relation which has the first two properties necessarily possesses the third property.

That the harmonic linking of two pairs of elements is of fundamental importance is evident from the Preservation Law. Its significance appears still greater in the light of the following fact, which represents a *completion of the Preservation Law*:

A one-to-one relation between the elements of two first-degree basic forms, which assigns harmonic fours of one form to harmonic fours of the other, is order-preserving.

Proof. We have to show that in a relation with the properties mentioned in the proposition, each (not necessarily harmonic) set EF/KL of pairs of elements separating each other corresponds to another such set. Should it ever happen that the corresponding pair of elements E', F' and K', L' do not separate each other, then, by Proposition IV, we could produce a pair of elements M', N' that separates both E', F' and K', L' harmonically. The harmonic fours $E'F'/M'N'$ and $K'L'/M'N'$ would correspond, under the relation in question, to harmonic fours EF/MN, KL/MN in which, by assumption, E, F and K, L separate each other. But by Proposition III, this is impossible. Therefore the assumed non-separation of E', F' and K', L' can never take place.

The proposition proved allows us to define a projectivity simply as follows, instead of using elementary construction chains:

A relation between two first-degree basic forms is projective if it assigns to each element of one form one and only one element of the other, relating each harmonic four to another harmonic four.

By the last Remark, the Fundamental Theorem holds true with this definition of projectivity as well.

With the Fundamental Theorem we have in hand a passport giving access to the different areas of geometry.

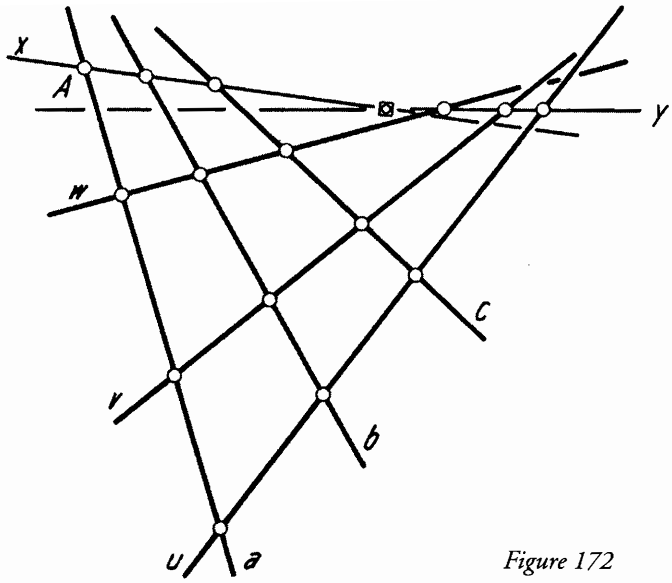


Figure 172

We turn to the proof of the Web Proposition explained earlier. Suppose a web (*abc, uvw*) is given. Let *F* be the surface generated by the meeting lines *x* of the director lines *a, b, c* and *G* be the surface generated by the meeting lines *y* of the director lines *u, v, w*.

We consider a meeting line *x* of *a, b, c* that is different from *u, v, w* but otherwise arbitrary (Figure 172). The set of planes that connect *c* with the points *au, av, aw, ax* = *A* are intersected by *b* in the points *bu, bv, bw, bx*. The point ranges *a* and *b* are thus sections of the same plane sheaf. Harmonic fours on *a* correspond to harmonic fours on *b*. In short the point ranges *a* and *b* are projectively related by the meeting lines of *a, b, c*. Obviously the same goes for *a* and *c*, as well as *b* and *c*, etc.

Now let *y* be a meeting line of *u, v, w* other than *a, b, c* but otherwise arbitrary. The question is whether the lines x and y intersect each other. If they do, then the surfaces *F* and *G* are identical.

We consider two other meeting lines: the meeting line x_1 of *a, b, y* through the point A on a and the meeting line x_2 of *a, c, y* through *A*. If *A* runs along the line *a*, then $Y_1 = yx_1$ and $Y_2 = yx_2$ run along the line *y*. Through both sets of meeting lines x_1, x_2 the range *a* is projectively related to the range *y*. Both projectivities relate the points *au, av, aw* to the points *yu, yv, yw*. Thus, by the Fundamental Theorem, in both projectivities the point *A* always corresponds to the same point *Y* on *y*. Hence the lines x_1 and x_2 coincide, with each other and with the line *x*. Therefore *x* and *y* intersect, namely in the point *Y*, and *G* is identical to *F*.

In the event that *A* belongs to the net {*au, av, aw*}, Y_1 and Y_2 are identical by the Preservation Law on its own. The significance of the Skeleton Theorem is brought home to us once more.

Perspective basic forms. Two first-degree basic forms that are related to each other through an elementary construction chain with just three members are said to be *perspective*. Such a relation, or *perspectivity*, is a particularly simple realization of a projectivity. Examples of perspectivities are

two point ranges that are sections of the same line pencil or the same plane sheaf;

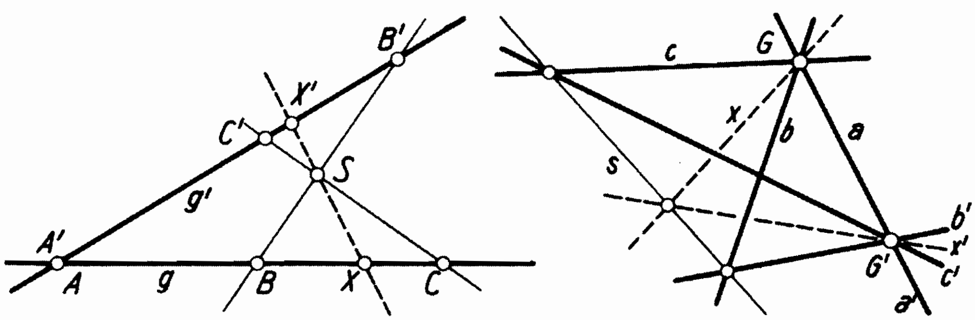
two plane sheaves that are projections of the same line pencil or the same point range;

two line pencils arising as projections of the same point range or as sections of the same plane sheaf.

An immediate consequence of the Fundamental Theorem is that

In a projectivity between two point ranges (or two line pencils of the same field or bundle, or two plane sheaves) determined by three pairs of corresponding elements, if the elements of one pair coincide, then the forms are perspective.

Proof. In the projectivity $g'(A'B'C'\ldots) \barwedge g(ABC\ldots)$, suppose for example that A and A' coincide (Figure 173). Since, as a result, g and g' belong to the same plane, the connecting lines BB' and CC' intersect in a point S. The chain $g \rightarrow S \rightarrow g'$ leads from A to A', from B to B', and from C to C'. Since the projectivity is uniquely determined by three pairs of corresponding elements, this chain—a perspectivity—provides for each X on g the corresponding point X' on g'.



Figures 173 and 174

In the case of two projective line pencils $G(abc\ldots)$ and $G'(a'b'c'\ldots)$ in the same plane (Figure 174) in which say a and a' coincide, we determine the line $s = (bb', cc')$. The chain $G \rightarrow s \rightarrow G'$ leads from a to a', from b to b' and from c to c'. The fact that the projectivity is uniquely determined by three pairs of corresponding elements implies that this chain gives, for each x, the correct x'.

Two projective point ranges or plane sheaves $g(ABC\ldots)$, $g'(A'B'C'\ldots)$ whose carrying lines are skew are always perspective.

In the case of point ranges (for example), this follows by considering the three lines AA', BB', CC' and choosing one of the lines that meet all three, say t, to be the carrier of a plane sheaf. The chain $g \longrightarrow$ sheaf $t \longrightarrow g'$ leads from A to A', from B to B' and from C to C'. By the Fundamental Theorem this chain thus leads from each point X on g to the correct point X' on g'.

Cross-line and Cross-point theorem. The following theorem is used appropriately to construct corresponding elements in projective point ranges or projective line pencils of the same field. Here X, X' and Y, Y' (x, x' and y, y' respectively) represent two arbitrary pairs of corresponding elements.

In the case of two projective point ranges g, g' of the same plane, the point of intersection of corresponding "cross-lines" XY', $X'Y$ lies in a fixed line, called the axis of the projectivity.

In the case of two projective line pencils G, G' of the same plane, the connecting line of corresponding "cross-points" xy', $x'y$ goes through a fixed point, called the center of the projectivity.

Proof (of the left-hand theorem, Figure 175). Suppose the point $K = gg'$, considered as an element of range g, corresponds in the projectivity in question to the point K' of range g'. Suppose the point $L' = g'g$, considered as an element of range g', corresponds to the point L of range g. The projectivity is uniquely determined by the three pairs K, K' and L, L' and X, X'. Let s be the connecting line LK'. Consider the chain

$$g \longrightarrow \text{pencil } X' \longrightarrow s \longrightarrow \text{pencil } X \longrightarrow g'$$

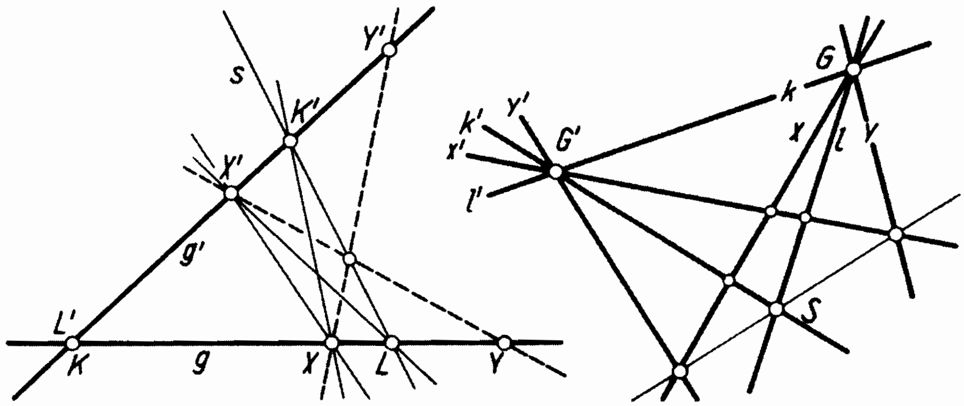
This leads, in particular, from X to X', from K to K', and from L to L', and thus provides, for each point Y on g, the correct point Y' on g'. The point of intersection of the corresponding cross-lines XY', $X'Y$ thus always lies on the fixed line s, whichever pairs X, X' and Y, Y' of corresponding elements of the projectivity we may have chosen.

Figure 176, which is polar in the field to Figure 175, one can use for the proof of the right-hand theorem.

In a projectivity $A'B'C' \ldots \barwedge ABC \ldots$ (or $a'b'c' \ldots \barwedge abc \ldots$), in order to find the point D' (line d') corresponding to D (d respectively), we construct

the axis s as connecting line of $(AB', A'B)$ and $(BC', B'C)$ or $(CA', C'A)$. The point of intersection of $A'D$ and AD' (or of $B'D$ and BD' etc.) must lie in s.

the center S as point of intersection of $(ab', a'b)$ and $(bc', b'c)$ or $(ca', c'a)$. The line connecting $a'd$ and ad' (or $b'd$ and bd', etc.) must go through S.

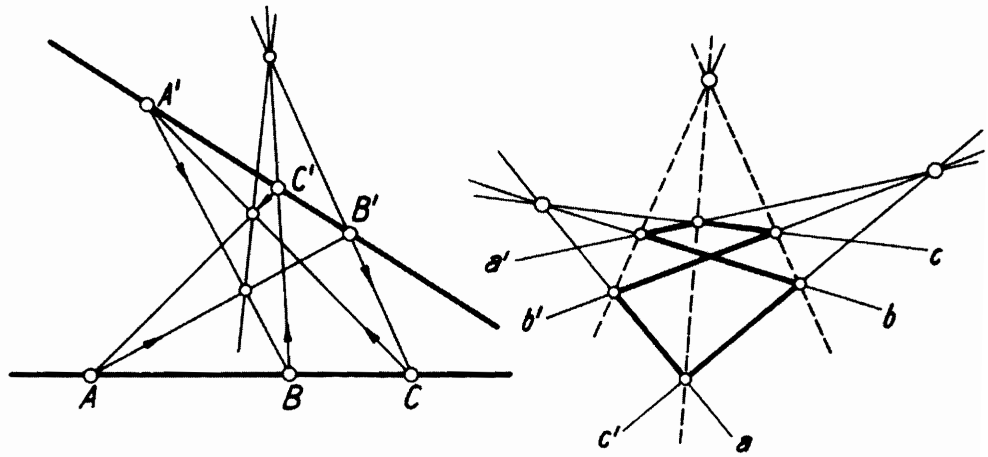


Figures 175 and 176

Arising as a special case is the famous *Theorem of Pappos.*

If the vertices of a simple planar 6-point $AB'CA'BC'A$ lie alternately on two lines, that is, A,B,C on one line and A',B',C' on another line, then the points of intersection of the opposite sides, namely $(AB', A'B)$, $(BC', B'C)$ and $(CA', C'A)$ also lie in a line (Figure 177).

This line is the axis of the projectivity $A'B'C' \barwedge ABC$.



Figures 177 and 178

For the theorem that is polar in the field we have:

If the sides of a simple 6-side $ab'ca'bc'a$ in a plane pass alternately through two fixed points, that is, a, b, c through one point and a', b', c' through another point, then the connecting lines of the opposite vertices, namely $(ab', a'b)$, $(bc', b'c)$ and $(ca', c'a)$, also go through a point (Figure 178).

REMARK. Pappos (ca. 300 AD) arrived at the theorem that today bears his name using metrical concepts. Note, however, that what this and the other propositions above describe is independent of the usual concepts of length, angle, surface area, etc. They arise as consequences of the phenomena **A** and **O**.

EXERCISES

1. Let $ABPQ$ be the basic 4-point of a Möbius net of which a number of meshes are to be constructed. Carry out the construction with basic 4-points of various shapes.

2. Divide a given 4-point $ABPQ$ into the 64 sub-meshes appearing in the Möbius net it determines.

3. A, B, C are given on g and A', B', C' on g'. g and g' lie in the same plane. Construct, as in Figure 170, for further positions of X on g the corresponding positions of X' on g' using a five-membered chain. (In particular, choose S to be A' and S' to be A.)

4. Carry out the construction corresponding to that of Exercise 3 for pencils $G(abc)$ and $G'(a'b'c')$ belonging to the same plane, as in Figure 171. (In particular choose s to be a' and s' to be a.)

5. Using two different elementary construction chains which lead from A, B, C on g to A', B', C' on g' (g and g' in the same plane), show graphically that both chains lead from a point X on g to the same point X' on g'.

6. Think Pappos' Theorem out for yourself in at least three figures.

7. Think out for yourself the theorem that is polar in the field to Pappos' Theorem in at least three figures.

8. In a plane, two lines are given with three points on each (1, 3, 5 and 2, 4, 6). With these points as vertices, form various simple 6-points (there are six of them) with vertices belonging alternately to the two lines (for example 123456, 143256, 321456, etc.) and demonstrate Pappos' Theorem graphically.

9. Carry out the exercise polar in the field to Exercise 8.

10. Let $abcd$ be any simple planar 4-side with vertices $A = cd$, $B = da$, $C = ab$, $D = bc$. The chain

$$a \longrightarrow A \longrightarrow b \longrightarrow B \longrightarrow c \longrightarrow C \longrightarrow d \longrightarrow D \longrightarrow a$$

leads from the arbitrary point 1 on a to point 5 on *a*. The same chain applied to 5 leads to 9 on *a*. The same chain applied once more leads from 9 to point 13, which turns out to be identical with 1. (Let point *ac* be called *U*. The ranges *BCU*1 and *BCU*13 are projective.)

11. A 4-side *abpq* is given in a plane. Carry out the construction polar in the field to the one in Exercise 1.

12. Starting with a 4-side *abpq*, carry out the construction polar in the field to that of Exercise 2.

13. Ponder the following singular fact. Let {*ABC*} be a net in the point range *g*. The set of points that do not belong to the net is uncountable, whereas the set of its gaps (that is, the net points) is countable!

Chapter 20

Products of Projective First-degree Basic Forms: Conic Sections

Suppose two arbitrary first-degree basic forms are given. The most primal relationship we can establish between the elements of one form and those of the other is, as the preceding explanations show, the projective relation. By the Fundamental Theorem, such a relationship is fixed as soon as three pairs of corresponding elements have been given. There are six possibilities to investigate:

1. *Two projective line pencils* that belong to a) the same field, or b) the same bundle, or c) are generally positioned.[8]

   Suppose x, x' are corresponding lines. Then we can consider in Case a) their point of intersection X, in Case b) their connecting plane X, while in Case c) x and x' are generally skew and thus determine no common elements.

   If line x runs through the first pencil then the corresponding line x' runs through the second pencil, and in Cases a) and b) the element X common to both lines x and x' takes up various positions. The totality of positions occupied by X can be regarded as the *product* of the projective pencils, or as being *generated* by them.

2. Two projective point ranges. If X, X' are corresponding points, and X runs through the first range, then the connecting line $x = XX'$ will sweep through various positions. The totality of all the positions we call the product of the two projec tive ranges, which are said to generate it.

3. Two projective plane sheaves. For each pair X, X′ of corresponding planes, we can form their connecting line x. If X runs through the first sheaf, x takes up infinitely many positions, which together constitute the product of the two projective sheaves.

4. A point range and a line pencil that do not in general belong to the same plane. For each pair X^+, x' of corresponding elements, we can form their connecting plane $X^- = X^+x'$. If X^+ runs through the range, then X^- takes up infinitely many positions, whose totality constitutes the product of the projective first-degree basic forms.

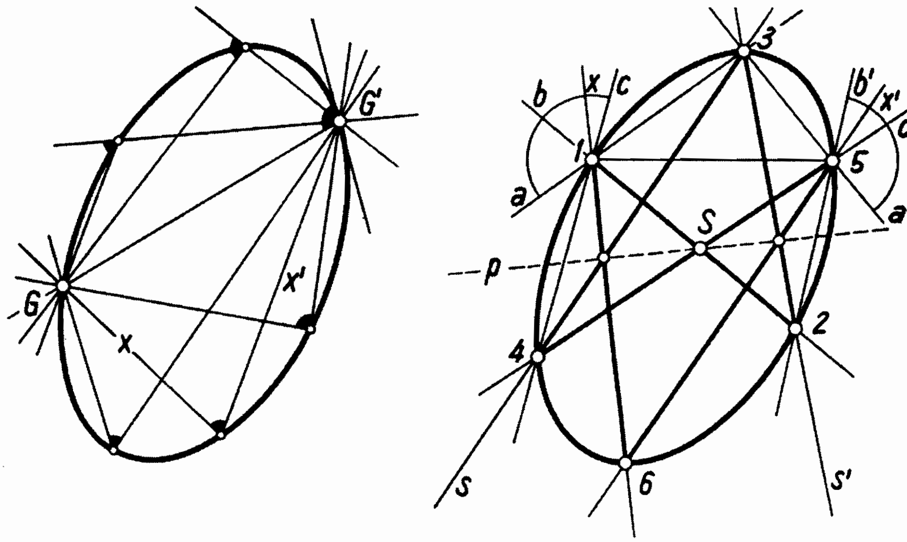
[8] Here, and in the other similar cases, the carrying forms are assumed to be distinct.

5. *A plane sheaf and a line pencil* that do not in general belonging to the same bundle. For each pair X^-, x' of corresponding elements, we can form their point of intersection $X^+ = X^-x'$. If X^- runs through the plane sheaf, then X^+ runs through various positions, whose totality constitutes the product of the projective first-degree basic forms.

6. *A point range and a plane sheaf.* Let X^+, X^- be corresponding elements. In this case, as in Case 1c, there are in general no elements common to both of these, so we cannot speak of a product in the sense of the other cases.

In the products listed above we are evidently dealing with forms that arise in the natural development of geometry as the next simplest forms after the basic forms. First we investigate the product in Case 1a.

The form generated by two projective line pencils belonging to the same field we call a conic section, the assumption being made that the generating pencils are not perspective (Figure 179). That is, the form turns out to be an ellipse, a hyperbola, or a parabola.



Figures 179 and 180

The points G, G' carrying the generating pencils, the *base points* of the conic section as we call them, themselves belong to the form generated. This is because each line x of the pencil G corresponds to exactly one line x' of pencil G'. So if x takes up the position GG', the point of intersection xx' coincides with G'; while if x', in pencil G', assumes the position $G'G$, the point of intersection xx' coincides with G. Since we have assumed that the pencils are not perspective, GG' does not correspond with the same line $G'G$.

In order to be able to implement the projective relation constructively, we imagine that we are given three arbitrary lines a, b, c of pencil $G = 1$ and the corresponding lines a', b', c' of pencil $G' = 5$. (Figure 180. The notation chosen will prove to be convenient.) The points $aa' = 3$, $bb' = 2$, $cc' = 4$ belong to the conic. To construct further points we determine the line x' of pencil 5 corresponding to a line x of pencil 1. This could be done, for example, as in Figure 171. Through $aa' = 3$ we put any two lines s, s'. Since doing so makes the construction particularly convenient, we choose 34 as s and 23 as s'. Let the point of intersection of (sb, $s'b'$) and (sc, $s'c'$) be called S. The chain $1 \rightarrow s \rightarrow S \rightarrow s' \rightarrow 5$ leads from a to a', from b to b', and from c to c'. This chain thus provides, for each x, the correct x'. The point of intersection $xx' = 6$ is another point of the conic. Notice how 6 moves when x moves in 1 causing p to move in S and x' to move in

5. We make the following assertions:

a) A conic section is uniquely determined by two points 1, 5 as base points and three further points 2, 3, 4, provided no three of the given points are in line. It is in fact the product of the projective pencils 1(2, 3, 4), 5(2, 3, 4).

b) If 1, 5 are base points and 2, 3, 4, 6 four arbitrary other points of a conic section, then the simple 6-point 123456 is a Pascal 6-point,

a Pascal 6-point being what we call a simple 6-point 123456 in which the points of intersection of opposite sides, namely (12, 45) and (23, 56) and (34, 61), are in line.

Proposition b) expresses in a simple way the connection, in a constructive sense, between point 6 and points 1, 2, 3, 4, 5, according to Figure 180.

c) If 123456 is a Pascal 6-point, then its vertices belong to a conic section for which 1 and 5 are base points.

That is, if one considers the projectivity $1(2, 3, 4) \barwedge 5(2, 3, 4)$, then, by Figure 180, point $6 = xx'$ arises through the construction of the Pascal 6-point 123456.

Among the points appearing in Propositions a), b), c) there are two that stand out from the others: the base points 1 and 5. Closer inspection shows that two arbitrary points of the conic can be looked upon as base points, that is, as carriers of generating projective pencils.

Proof. Let C be the conic section generated by the two projective pencils 1 and 5. How the projectivity is determined we leave open. In any case, let 2, 3, 4, 6 be any four points of C (distinct from 1 and 5). By b), 123456 is a Pascal 6-point.

But, like 123456, 456123 is also a Pascal 6-point. Therefore, by appropriately applying c), the six points 4, 2 and 5, 6, 1, 3 belong to a conic C' for which

4 and 2 are base points and 5, 6, 1, 3 further points. By a), C' is uniquely determined by the pencils $4(5, 6, 1) \barwedge 2(5, 6, 1)$. Since we could take, instead of 3, an arbitrary point of C other than 1, 2, 4, 5, 6, it follows that each point of C also belongs to C'.

Conversely, if X is an arbitrary point of C', then, by b), $45612X$ and hence also $12X456$ constitutes a Pascal 6-point. By c) X also belongs to the conic C generated by the pencils $1(2, 4, 6) \barwedge 5(2, 4, 6)$. Thus C and C' coincide; in other words, instead of points 1 and 5 we can choose any two other points 2 and 4 of C as base points.

We have now proved the following propositions:

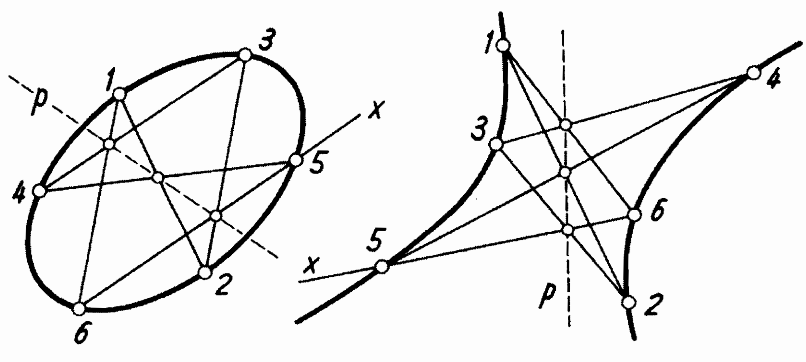
I. *If two fixed points of a conic section are connected with a point moving through it, then two projective line pencils are obtained.*

II. *Theorem of Pascal. If 1, 2, 3, 4, 5, 6 are six arbitrary points of a conic section, then the points of intersection of opposite sides of the simple 6-point 123456 lie in a line.*

The philosopher Blaise Pascal (1623 – 1662), profound as he was sagacious, discovered this fact when he was 16 years old, admittedly still using the geometry of the circle rather than in its pure form, whose elaboration became possible only in the nineteenth century.

III. *Through five arbitrary points, of which no three belong to the same line, there passes one and only one conic section.*

Proof. Two of the five points can be chosen as base points. The three lines connecting each of these to the three other points determine the projectivity between the two pencils.



Figures 181 and 182

But we could also construct further points using Pascal's Theorem directly. To do this, we label the five given points 1, 2, 3, 4, 5 *in any order* and put through 5 an arbitrary line x on which point 6 is to lie (Figure 181 and 182). (12, 45) and (23, 56 = x) determine the "Pascal line" p on which 34 and 61 must intersect.

FIRST REMARK. Note that the basic construction given allows us to construct, using only a ruler, an arbitrary number of points of a conic given by five of its points.

SECOND REMARK. In Chapter 12 it was shown that five points of a field, of which no three belong to the same line, always determine a cycle. The simplest solution to the question of how we could "interpolate" further points in a natural way into such a cycle 12345 is given by the basic construction using Pascal's Theorem.

THIRD REMARK. Six generally positioned points can be connected by a simple 6-point 123456 in 60 different ways. If the points lie on a conic, then a Pascal line goes with each of these 6-sides. With six points of a conic, there are thus associated 60 Pascal lines. These form a remarkable configuration that was first studied in detail by Jakob Steiner (1796 – 1863) who systematically developed the generation of conics by projective pencils.

The generating process by projective pencils shows that if a line x has a point G in common with a conic, then x contains at most one other point of the conic, namely the point of intersection of x with the corresponding line x'.

Through each point G of a conic C there passes exactly one line to which no other point of C belongs apart from G. Because if G' is an arbitrary other point of C, then, by (I), we can choose the points G and G' as base points. To the line $G'G$ of pencil G', there corresponds exactly one line g of pencil G. This meets $G'G$ in G. g cannot contain any other point of C apart from G, because all lines of pencil G' distinct from $G'G$ correspond with lines of pencil G distinct from g. For the same reason, g is the single line in G containing only the point G of C. g is called the *tangent* of C in G.

(I) also implies that conics are curves in the sense of Chapter 16. For example the pairs of points A, B and C, D of a conic separate each other whenever the lines drawn from a point G (and hence from any other point other than A, B, C, D) to these points separate each other. In particular, the tangent in G is the limiting position of the line GX, where X belongs to the conic, as X tends to G. To summarize, therefore:

IV. *The conic sections are second-order curves.*

Let 123456 be a simple 6-point whose vertices belong to a conic. If 1, 2, 3, 4, 5 are fixed points and we allow 6 to tend towards 5, then the limiting position of the side 56 is the tangent in 5. If only points 1, 3, 5 remain fixed and 2 tends towards 1, 4 towards 3, and 6 towards 5, then in the limit the 6-point becomes the 3-point 135 with the 3-side of tangents in 1, 3, 5.

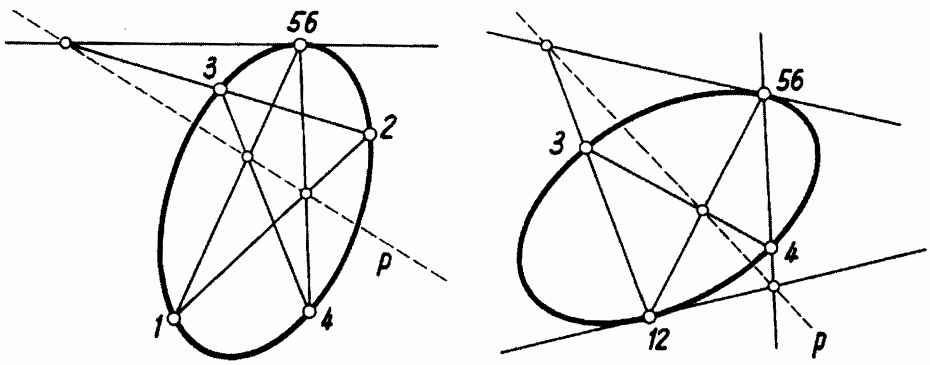
The following use of words is therefore appropriate: If in a simple 6-point 123456 two vertices coincide in one point, then the "side" determined by these two vertices is understood to be the tangent in that point.

V. *Pascal's Theorem is also true whenever, in the relevant simple 6-point, one, two or three pairs of successive vertices coincide* (Figures 183 to 186).

Proof. In Figure 180, we imagine 6 moving along the conic while 1, 2, 3, 4, 5 remain fixed. The lines 16 = x and 56 = x' are thereby describing projective pencils. The point of intersection (x, 34) moves on 34 = s, the point of intersection (x', 32) on 32 = s'. These ranges are thus perspective: corresponding points lie on a line p through the fixed point S.

By definition, the tangent in 5 is that line corresponding, in the projectivity between pencils 1 and 5, to the line 15 in 1. Giving x the position 15, we obtain Figure 183, that is, the Pascal Theorem for 1, 2, 3, 4, 56.

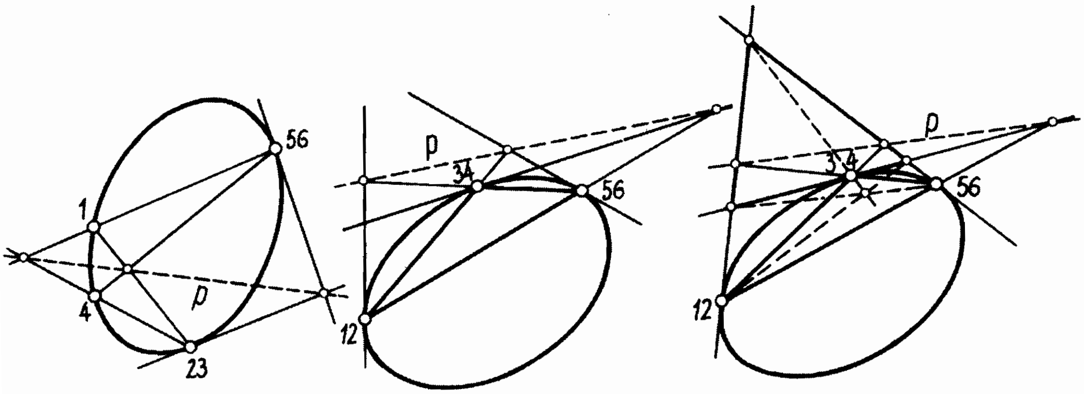
To prove the theorem for 123456 with both 1, 2 and 5, 6 coincident, we imagine in Figure 183 the elements 1, 3, 4, 56 fixed while 2 moves along the conic. By (I), pencils 12 and 32 are projective. The point of intersection (12, 45) describes the fixed range 45, the point of intersection (32, 56) describes the fixed range 56. These ranges are perspective: corresponding points lie on a line p through the fixed point (15, 34). The tangent in 1 arises as the line in 1 corresponding to the line 31 in 3. This gives us Figure 184, and thus the theorem for 12, 3, 4, 56.



Figures 183 and 184

If, in Figure 183, we imagine elements 1, 2, 4 and 56 to be fixed and let 3 move along the conic then, by (I), pencils 23 and 43 are projective. The point (23, 56) runs along the fixed range 56, while (43, 15) moves in the fixed range 15. The ranges are perspective: corresponding points lie on a line p through the fixed point (12, 45). To obtain the tangents in 2, we have to determine the line in pencil 2 corresponding to line 42 in pencil 4. This produces Figure 185, and thus the theorem for 1, 23, 4, 56.

The case of the 6-point 123456 in which 12, 34 and 56 are all tangents still remains. In Figure 184, imagine elements 12, 3, 56 to be fixed and allow 4 to move along the conic. The lines 34 and 54 thereby describe projective pencils. Point (34, 15) runs through the fixed range 15, and (54, 12) moves on the fixed range 12. These ranges are perspective:corresponding points lie on a line p through the fixed point (13, 56). If 4 reaches 3, then the line in 3 corresponding to line 53 in pencil 5 is the tangent in 3. This produces Figure 186, that is, the theorem for 12, 34, 56.



Figures 185, 186 and 186a

This completes the proof of V. The following are consequences of V:

Va. *The tangents in opposite vertices of a simple 4-point inscribed in a conic section intersect in a point that is in line with the two points of intersection of its opposite sides* (Figure 185).

Vb. *If each side of a 3-point inscribed in a conic section is intersected with the tangent in the opposite vertex, then the three points of intersection lie in a line* (Figure 186). *The 3-point and the corresponding 3-side of tangents thus form a Desargues' Configuration* (Figure 186a).

Va was demonstrated for Figure 185 for opposite vertices 2 and 5. The proposition can also be applied to the two other opposite vertices, of course. Indeed, the complete 4-point with vertices 1, 2, 4, 5 contains the three simple 4-points 1245, 1254, 1425. Applying Va to all three produces the following fundamental theorem (which contains the whole of so-called polar theory):

Vc. *The complete 4-point represented by four points of a conic section, and the complete 4-side represented by the tangents in those points, have their extra triangle in common* (Figure 187).

Using V, we have the following result, which complements III:

Vd. *A conic section is uniquely determined*

a) *by five of its points;*
b) *by four of its points and the tangent in one of them;*
c) *by three of its points and the tangents in two of these points.*

Here we assume that no three points belong to the same line.

Further points are obtained in case a) with Figure 181, in case b) with Figure 183, in case c) with Figure 184 or 185.

With the limit line of the plane, a conic can possess two distinct, or two coincident, or no common points. In the first case it is called a *hyperbola*, in the second a *parabola*, and in the third an *ellipse*. In the case of the parabola, the limit line is a tangent. (The relevant constructions are to be found in the Exercises.)

We still need to review the special case, until now excluded, in which the two generating pencils G, G' are *perspective*. In this case the line GG' corresponds to the same line $G'G$. The product consists of the line on which the points of intersection of the various corresponding lines x, x' lie, and of the points that the two coincident corresponding lines $GG' = G'G$ have in common. Two perspective pencils thus generate a pair of lines, or *line-pair*. Such a form is properly regarded as a *degenerate conic section*. Pascal's Theorem is true also for a line-pair. If the six vertices of the 6-point in question are distributed in such a way that there are three on each of the two lines, then Pappos' Theorem results as a special case.

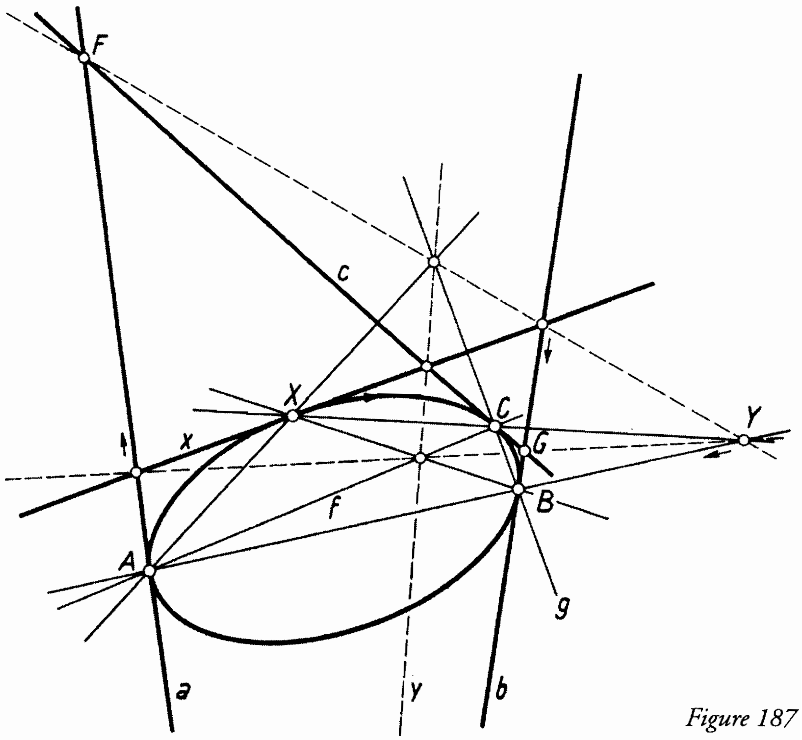


Figure 187

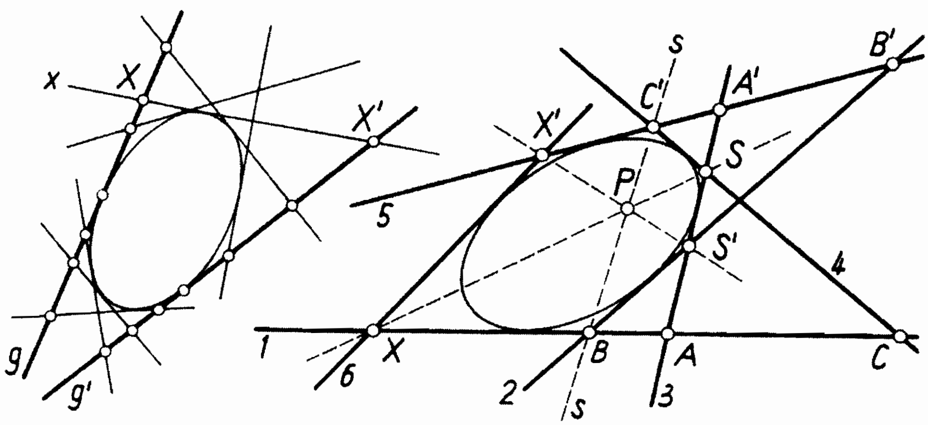
We now investigate the second case given in the list at the beginning of the chapter. First we consider two projective point ranges belonging to the same plane.

The form generated by two projective point ranges belonging to the same field we call the envelope of a conic section, or a conic envelope for short; we assume here that the ranges are not perspective.

The reason for this terminology will soon become clear.

Generating a conic envelope by means of projective ranges and generating a conic section by means of projective pencils are, in the geometry of the field, processes polar to each other. Everything we have said about conic sections could thus simply be translated word for word. We feel justified in restricting our explanations to the basic construction.

If g, g' are the projective ranges and X, X' represent corresponding points, then $XX' = x$ is a line of the envelope (Figure 188). We call the carrying lines g, g' base lines of the envelope. These are part of the envelope. This follows because for each point X of the range g, there is exactly one corresponding point X' of the range g'. If X takes up the position gg' then the connecting line XX' coincides with g'. But if X' in range g' takes up position $g'g$, then XX' coincides with g. Since, by assumption, the ranges are not perspective, point gg' does not correspond with the same point $g'g$.



Figures 188 and 189

In order to implement the projective relation in practice, we imagine three points A, B, C of range $g = 1$ and the corresponding points A', B', C' of range $g' = 5$ (Figure 189). These three pairs fix the projectivity. The lines $AA' = 3$, $BB' = 2$, $CC' = 4$ are lines of the envelope. To construct another line we must determine, for a given point X of range 1, the corresponding point X' of range 5. This could be done, for example, as in Figure 170. On $AA' = 3$ we choose any two points S, S'.

With 34 as S and 32 as S' the construction is particularly convenient. Let the line connecting $(SB, S'B')$ and $(SC, S'C')$ be s. The chain $1 \longrightarrow S \longrightarrow s \longrightarrow S' \longrightarrow 5$ leads from A to A', from B to B', and from C to C'. This chain thus supplies the point X' corresponding to X. The connecting line $XX' = 6$ is another line of the envelope. Notice how 6 moves when X moves in 1 and at the same time P moves in s and X' moves in 5. We make the following assertions:

a) The envelope of a conic is uniquely determined by two lines 1 and 5 as base lines and three further lines 2, 3, 4, provided no three lines go through the same point.

b) If 1 and 5 are base lines and 2, 3, 4, 6 four arbitrary other lines of the envelope of a conic, then the simple 6-side 123456 is a Brianchon 6-side,

a Brianchon 6-side being what we call a simple 6-side 123456 if the lines connecting its opposite vertices, namely the lines (12, 45) and (23, 56) and (34, 61) go through a point.

Proposition b) is a simple expression of the connection that line 6 has, through the construction in Figure 189, with lines 1, 2, 3, 4, 5 .

c) If 123456 is a Brianchon 6-side, then its sides belong to the envelope of a conic with base lines 1 and 5.

On the basis of these propositions, everything else follows—as noted above—by "polarizing" the corresponding proofs and propositions for conic sections, which to the interested reader will present no special difficulty. In particular:

I′. *If two fixed lines of the envelope of a conic are intersected with a line sweeping through the envelope, then two projective ranges are obtained.*

II′. *Brianchon's Theorem. If 1, 2, 3, 4, 5, 6 are six arbitrary lines of the envelope of a conic, then the lines connecting opposite vertices of the simple 6-side 123456 go through a point.*

The theorem named after Charles Julien Brianchon (1783 – 1864) was discovered by him in 1810. If one is cognizant of the principle of polarity, then with one of the two theorems of Pascal and Brianchon, one also has the other.

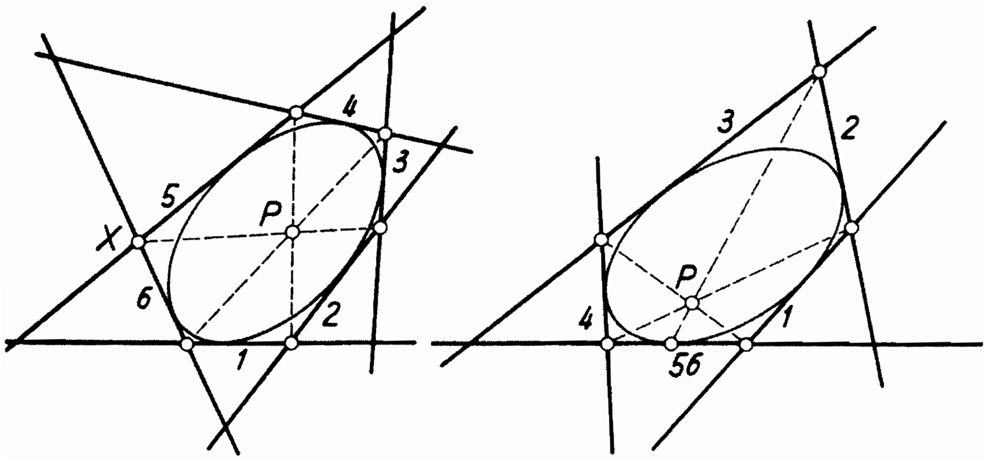
III′. *Five lines, no three of which go through the same point, belong to a unique envelope of a conic.*

Two of the lines can be chosen as base lines. The three points in which each of these is intersected by the other lines determine the projectivity between the two ranges.

Alternatively, we could work directly with Brianchon's Theorem. To do this, label the lines 1, 2, 3, 4, 5 in any order (Figure 190) and choose, on 5, an arbitrary point X through which line 6 is to go. (12, 45) and (23, 56 = X) determine the "Brianchon point" P, which also contains (34, 61).

REMARK. The three remarks about Pascal's Theorem also apply here in a corresponding sense.

The basic construction in Figure 190 has already been developed in Chapter 15. There it was the picture of a skew 6-side *aubvcw* whose sides form a web (*abc*, *uvw*). If, therefore, the lines contained in the surface determined by such a web are projected onto a plane, the envelope of a conic is obtained.



Figures 190 and 191

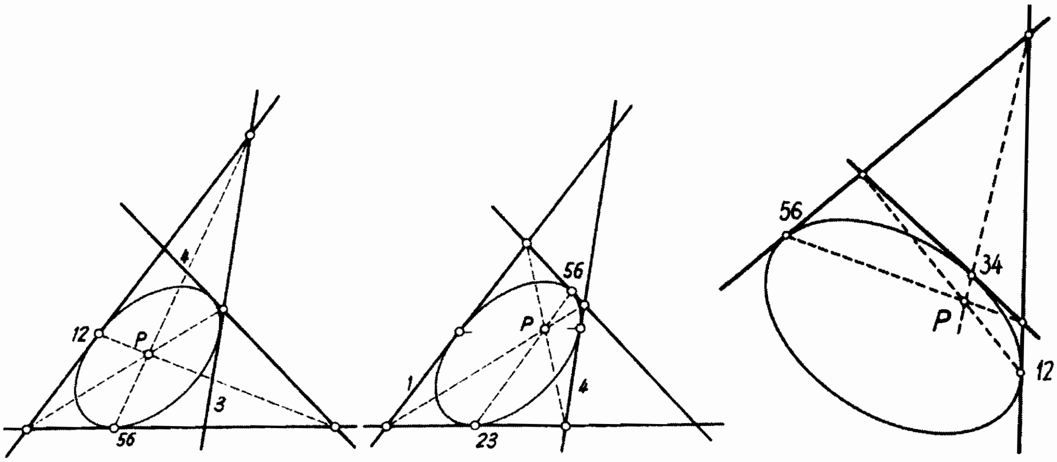
IV′. *The envelope of a conic section is a second-class curve envelope.*

An arbitrary point in the plane of the envelope contains at most two lines of the envelope.

In each line g of the envelope there is exactly one point G that contains no other line of the envelope other than g. This point is the point of contact or support point of the line g.

Let 123456 be a simple 6-side whose sides belong to the envelope of a conic. If 1, 2, 3, 4, 5 are fixed lines and we let 6 tend towards 5, then the limiting position of vertex 56 is the point of contact in 5. If only lines 1, 3, 5 remain fixed and 2 tends towards 1, 4 towards 3 and 6 towards 5, then, in the limit, the result is the 3-side 135 with the 3-point of support points in 1, 3, 5. If, in a simple 6-side, two sides coincide, it is therefore natural to take as vertex determined by these sides, the support point (point of contact) of the coincident sides.

V′. *Brianchon's Theorem is also true when, in the simple 6-side in question, one, two, or three pairs of successive sides coincide* (Figures 191, 192, 193, 194).



Figures 192, 193 and 194

The following fact justifies the name "envelope of a conic section":

VI. VI′. *The tangents of a conic section form the envelope of a conic section, and the support points of the envelope of a conic section form a conic section.*

Proof: In Figure 187, keep $A \cdot a$, $B \cdot b$, $C \cdot c$ fixed and let $X \cdot x$ run through the conic section. As a consequence, the points *ax* and *bx* move along the ranges *a* and *b*. Consider the fixed points $ac = F$ and $bc = G$. The lines from *F* to *bx* and from *G* to *ax* always meet in *Y* on *AB*. The point *Y* thus moves along the range *AB* and the pencils *FY*, *GY* are perspective. Their points of intersection with *a* and *b* therefore yield projective ranges. *That is, a moving tangent x intersects two fixed tangents a and b in projective ranges.* This proves the first part of the proposition.

By the polar train of thought (using the same Figure 187 with *AX*, *BX*, $AC = f$, $BC = g$, etc.), it follows in the case of the envelope of a conic that if a moving support point *X* is *connected with two fixed support points A and B,* then projective pencils are created. This proves the second part.

A conic section with its tangents is thus, in the field, a self-polar form. Hence the proposition polar to Va can be expressed as follows:

Va′. *The line connecting the points of contact of two opposite sides of a 4-side circumscribed about a conic section and the lines connecting the opposite vertices go through a point* (Figure 193).

Propositions Vb, Vc and VI are self-polar. And in place of Vd, we can now formulate the following self-polar proposition:

VII. VII′. *A conic section is uniquely determined*

a) *by five of its points;*

b) *by four of its points and the tangent in one of them;*

c) *by three of its points and the tangents in two of those points;*
d) *by three of its tangents and the points of contact in two of those tangents;*
e) *by four of its tangents and the point of contact in one of them;*
f) *by five of its tangents.*

We assume that no three of the points belong to the same line, and no three of the lines to the same point.

In the case, which up to now was excluded, when the two generating point ranges g, g' are perspective, the point gg' corresponds with the same point $g'g$. The product generated consists of the pencil whose lines connect various corresponding points X, X' and of the lines common to the two corresponding but coincident points gg', $g'g$. Two perspective ranges thus generate the lines of a *pair of points*, which is to be regarded as a degenerate envelope of a conic. Brianchon's Theorem is true even for this degenerate envelope.

The products enumerated at the beginning of this chapter not yet reviewed can now be surveyed more easily.

Two skew projective point ranges. Such ranges are always perspective. They generate a ring-shaped *hyperboloid* (hyperboloid of one sheet) or, in a special case in which the limit points of the ranges are corresponding points, a *hyperbolic paraboloid* (see Chapter 15).

Two projective plane sheaves.
First case: The carriers g, g' intersect, say in L. If the two sheaves are intersected with a plane that belongs neither to g nor to g', then projective line pencils are produced in this plane, which generate a conic section. Connecting the points of the conic with L, we obtain the required product: a conical surface.
Second case: The carriers g, g' are skew. By what was said in Chapter 15 (in particular by the second construction mentioned in connection with Figure 103, in which B^* is chosen on b, etc.) the product is a ring-shaped hyperboloid or a hyperbolic paraboloid.

Two projective line pencils belonging to the same bundle L. If the pencils are intersected with a plane not containing L, then two projective point ranges are produced that generate the envelope of a conic. Connecting the lines of the envelope with L, we obtain the required product; it is the envelope of a cone.

A point range g and a projectively related line pencil with a generally positioned carrying point L. If the pencil is intersected with a plane containing g but not going through L, we obtain two projective ranges which generate the envelope of a conic. If its lines are connected with L we have the required product; it is the envelope of a cone.

A plane sheaf g and a projectively related line pencil with a generally positioned carrying plane L. L intersects the sheaf in a pencil, which together with the given pencil generates a conic.

REMARK. If we are given, for example, three mutually projective point ranges (plane sheaves) with skew carrying lines, then we can form the connecting plane (point of intersection) of each triple of corresponding elements; the resulting product is a so-called cubic developable (cubic space curve). Other spatial forms can be obtained in similar ways.

EXERCISES

In the following A, B, C, . . . represent points of a conic section and a, b, c, . . . the tangents in these points. Construct further points and tangents of the conic using Pascal's Theorem and Brianchon's Theorem. These give an abundance of constructions.

1. Given A, B, C, D, E where
   a) none of the five points is a limit point,
   b) E is a limit point,
   c) D and E are limit points.

2. Given A, B, C, D, a where
   a) none of the five elements is a limit element,
   b) D is a limit point,
   c) A is a limit point,
   d) a is the limit line.

3. Given A, B, C, a, b where
   a) none of the five elements is a limit element,
   b) C is a limit point,
   c) A and C are limit points,
   d) A and B are limit points,
   e) a is the limit line.

4. Given A, B, a, b, c where
   a) none of the five elements is a limit element,
   b) c is the limit line,
   c) B is a limit point,
   d) A and B are limit points.

5. Given A, a, b, c, d where
   a) none of the five elements is a limit element,
   b) d is the limit line,
   c) a is the limit line,
   d) A is a limit point.

6. Given a, b, c, d, e where
   a) none of the five lines is the limit line,
   b) e is the limit line.

7. Arrange the forms generated by two projective first-degree basic forms in the general arrangement on page 31.

Chapter 21

THE THREE ARCHETYPAL SCALES

The three forms segment, angle field, and angle space represent *quantities.* Using appropriate units, we can measure them. For this, scales are necessary in which to fit the quantities to be measured. In the following pages we show that for each of the three types of quantity mentioned there are three kinds of scale that arise from the nature of spatial relationships. At the same time, no concept that is foreign to the nature of space is brought in. For this reason, we can speak, of archetypal scales. We elaborate the train of thought leading to these scales in the case of the point range. Scales in the line pencil and plane sheaf are then easily produced by forming views.

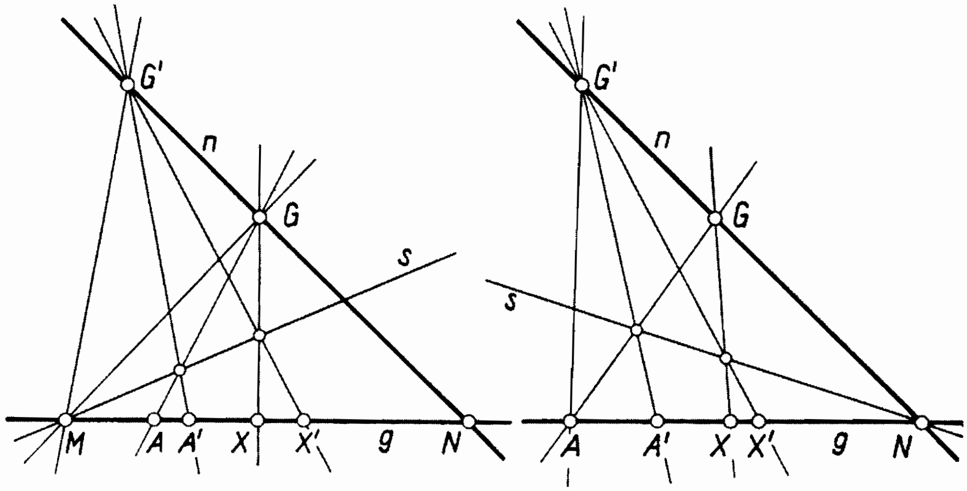
We picture an elementary construction chain whose first and last members are point ranges on the same carrying line g. Such a chain maps the point range g projectively onto itself. It can happen as a result that a point M coincides with its corresponding point M'. In that case, M is called a *fixed point* of the projective relation of g with itself. Harmonic reflection in two points M, N represents a projective mapping with two fixed points M and N.

From the Fundamental Theorem it follows immediately that a projective self-mapping with *three* fixed points L, M, N is the identity mapping; that is, each point of g corresponds to itself. This is because the projective relation is *uniquely* determined by the three pairs (L, L, and M, M, and N, N) of corresponding points. The identity mapping is a projective mapping possessing in particular the required fixed points; there cannot be a different one.

Thus there are only three possibilities for a projective mapping of a range g to itself—in which not *every* point corresponds with itself—that need to be considered: two distinct fixed points M and N, one fixed point N (it is more appropriate to say: two coincident fixed points N, N), and no fixed points.

We give the simplest possible constructions for such self-mappings:

Two fixed points (Figure 195). To determine the projectivity uniquely we assume three pairs of corresponding elements: M, M, and N, N, and A, A'. To find, in $MNA \barwedge MNA'$, the point corresponding to a point X, choose a line n through N and on it two arbitrary points G, G' distinct from N. Draw the line s connecting M and (GA, $G'A'$). The chain $g \rightarrow G \rightarrow s \rightarrow G' \rightarrow g$ leads from M to M, from N to N, and from A to A', and thus from each point X on g to the correct point X'.



Figures 195 and 196

Two coincident fixed points (Figure 196). If, in Figure 195, the fixed point M moves to the fixed point N, the result is Figure 196. In this special case, the projectivity is determined by N, N and A, A' alone, for which reason we write $NNA \barwedge NNA'$. We put a line n through N, choose on it two arbitrary points G, G' distinct from N, and draw the line s connecting N and $(GA, G'A')$. The chain $g \to G \to s \to G' \to g$ realizes the required self-mapping.

General construction (Figure 197). If it is not known whether fixed points exist, then we have to fix the projectivity by means of three general pairs A, A' and B, B' and C, C' of corresponding points. With two points G, G' that do not belong to g but are otherwise arbitrary, we form the pencils $G(ABC\ldots)$ and $G'(A'B'C'\ldots)$. The projectivity between these pencils can be established using the construction of Figure 171 or, more conveniently for what follows, with the help of the Cross-point Theorem. To do this you determine the center S of the projectivity as the point of intersection of $(ab', a'b)$ and $(bc', b'c)$. For arbitrary pairs x, x' and y, y' of corresponding lines, the line connecting xy' with $x'y$ goes through S. In Figure 197 the pair c, c' was chosen to determine x' corresponding to an arbitrary given x.

We now have simple constructions at our disposal for projectively relating a range g with itself.

How could one form, in a natural way, in a line g, a scale

$$AA_1A_2A_3A_4\ldots \text{ or } 01234\ldots$$

for short? The answer to which the facts naturally lead is as follows:

The sequences 0123456.... and 1234567.... should be projective. That is, the projective self-mapping should take 0 into 1, 1 into 2, 2 into 3, and so on.

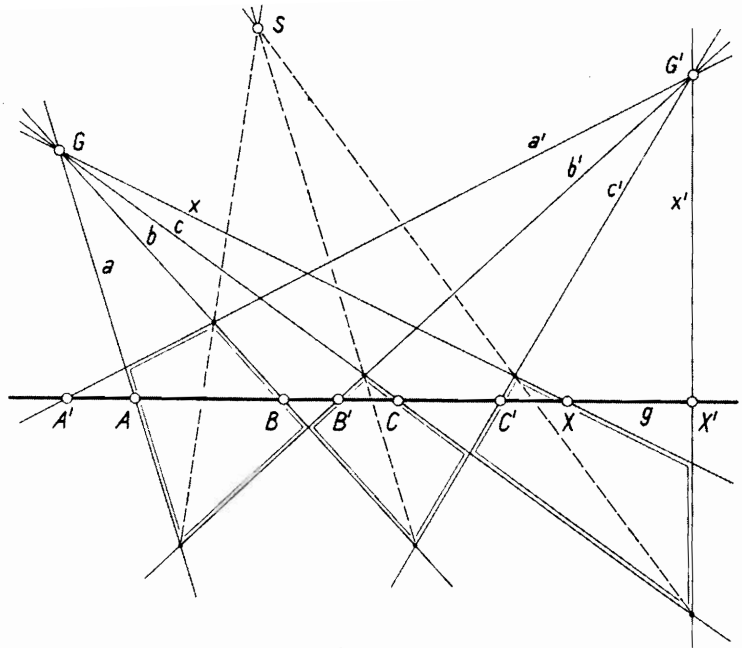
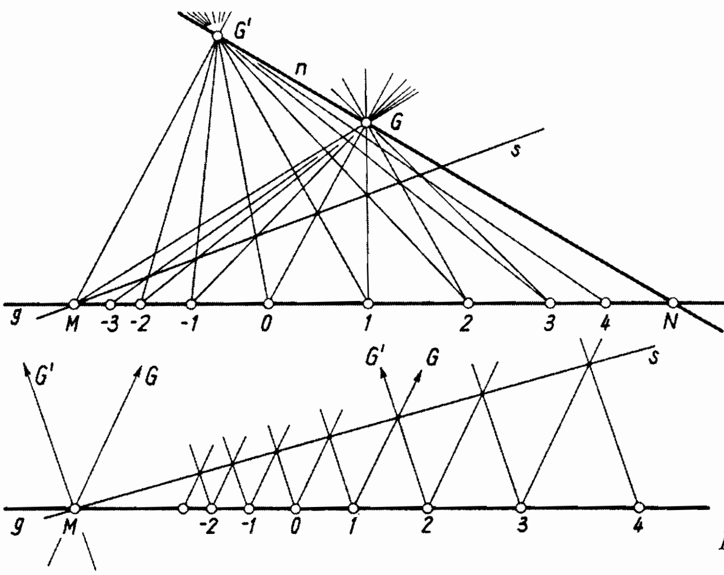


Figure 197

The resulting scale we call *multiplicative*, *additive*, or *periodic* according to whether there are two distinct, two coincident, or no fixed points.

Multiplicative scale. The projectivity is determined by the pairs M, M and N, N and 0, 1. At the same time we assume that M, N and 0, 1 do not separate each other. Figure 198 shows the construction according to Figure 195. We connect 0 with G and 1 with G'. The line connecting M with $(G', G1)$ is s. s is now intersected with $G1$ and the point of intersection projected from G' onto g; this gives 2. We repeat the process with 2, and so on.



Figures 198 and 199

The scales can be continued without limit on both sides. The fixed points M and N are accumulation points of the scale, as can easily be proved by invoking the continuity of g.

Figure 199 shows the case in which n is the limit line of the plane in question. If we measure the segments $M0$, $M1$, $M2$, . . . in the usual sense with M0 as unit, and let a be the number measuring the length of $M1$, then the numbers measuring the segments $M0$, $M1$, $M2$, $M3$, . . . are

$$1 = a^0, \quad a = a^1, \quad a^2, \quad a^3, \quad a^4, \ldots.$$

Because of this we speak of a *multiplicative scale*. In Figures 200 and 201 the multiplicative scale construction is fitted into a total picture.

Additive scale. This is a limiting case of the multiplicative scale. The projectivity $NN0 \barwedge NN1$, constructed according to Figure 196, leads to the scale of Figure 202, which we already know from the Möbius net (Figures 167 and 168):

Three consecutive scale points, together with N, always form a harmonic four.

If N is chosen to be the limit point of g the name additive becomes clear.

General construction. If it is not known whether fixed points exist, then, to determine the projectivity $012 \barwedge 123$, we have to give four scale points 0, 1, 2, 3. Figure 203 shows the construction according to Figure 197, starting from the four points 0, 1, 2, 3. The lines $G0$, $G1$, $G2$ we call a, b, c and $G'0$, $G'1$, $G'2$ we call a', b', c'. We now construct 4, 5, 6, etc., successively. Figure 204 shows the construction as part of a totality.

If, on a line, four points 0, 1, 2, 3 in their natural order are given, then only after some practice will you know how to judge what kind of scale they determine. (A criterion is given in Exercise 3.) Figures 203 and 204 produce scales *without fixed points. In contrast to the multiplicative and additive cases, these scales cover the carrying line infinitely often, in one running-through sense as well as the other. For this reason we call it periodic.* The scale is strictly periodic if, after a number of scale points, we reach point *0* once again.

Scales in the line pencil and in the plane sheaf are produced by means of the polar constructions. But you can also obtain them simply by projecting the scales in a range from a point or from a line skew to the range.

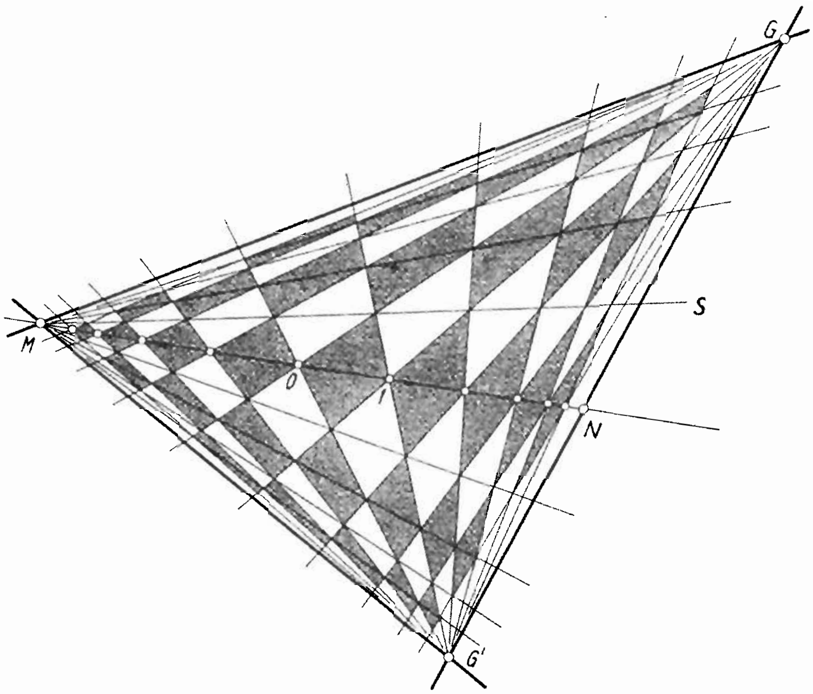


Figure 200

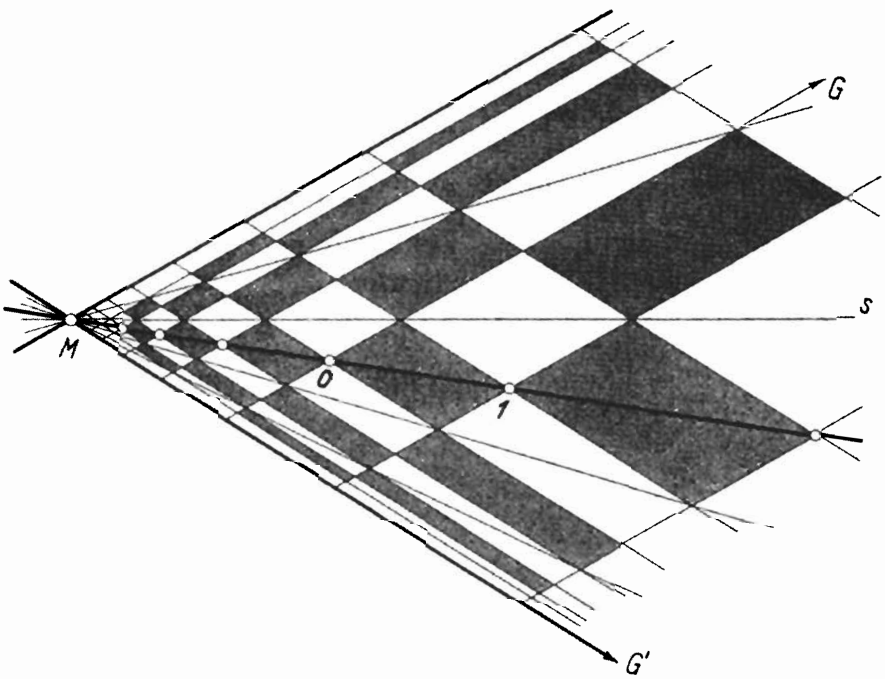


Figure 201

With this we have gained the insight that the archetypal phenomena of spatial relationships lead to three types of scale, which serve for measuring in space and counterspace. It is important to note the position of the additive scale intermediate between the multiplicative and periodic scales.

The additive scale is well-known to us from visual space (Figure 160); in the special case when the fixed point is a limit point, it is used in tactile space. Using a periodic scale in a line pencil is the accepted way of measuring angles. The multiplicative and periodic scales in a range are employed in the various non-Euclidean geometries.

In the special case in which one of the two fixed points is a limit point, the multiplicative scale appears in Euclidean similarity theory.

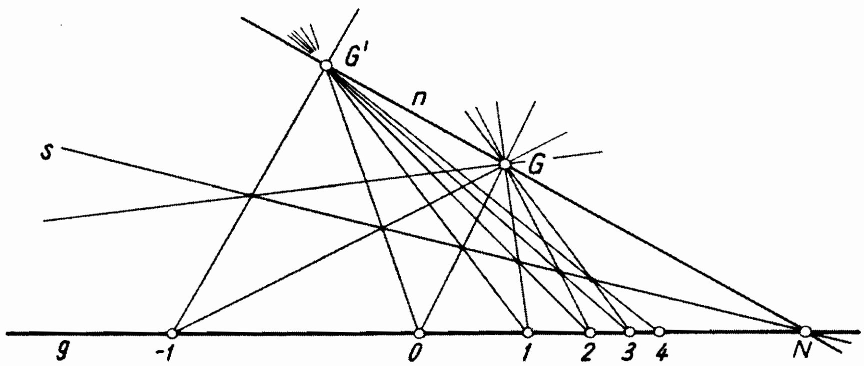


Figure 203

Figures 200, 201, and 204 contain very much more than one first thinks. They represent the construction for pencils of conic sections. If the vertices of the meshes are connected in the right way, the conics (as products of projective pencils) become visible.

For a complete insight into scales, you also have to bring into play the *companion scale* which exists for each scale. This arises from 0123456 . . . as follows. Let 1′ be the reflection point[9] of 1 in 0, 2; let 2′ be the reflection point of 2 in 1, 3; let 3′ be the reflection point of 3 in 2, 4; and so on. The points 1′, 2′, 3′, 4′, . . . form the companion scale of 1234. . . . It stands in a characteristic relationship with the original scale; together they constitute a so-called involution. Only in the case of the additive scale is the companion scale reduced to a single point, namely the fixed point.

This takes us to the very subject that leads from first-order to second-order theory, to polar theory, the involutions that are part of it, and the corresponding movement-forms in which imaginary numbers manifest themselves spatially. All the same, it may be said in conclusion that the fundamentals and exercises elucidated, as well as the first-order theory dealt with in the present volume stand one in good stead to pursue the path leading further without difficulty.

9. That is, the harmonically reflected point.

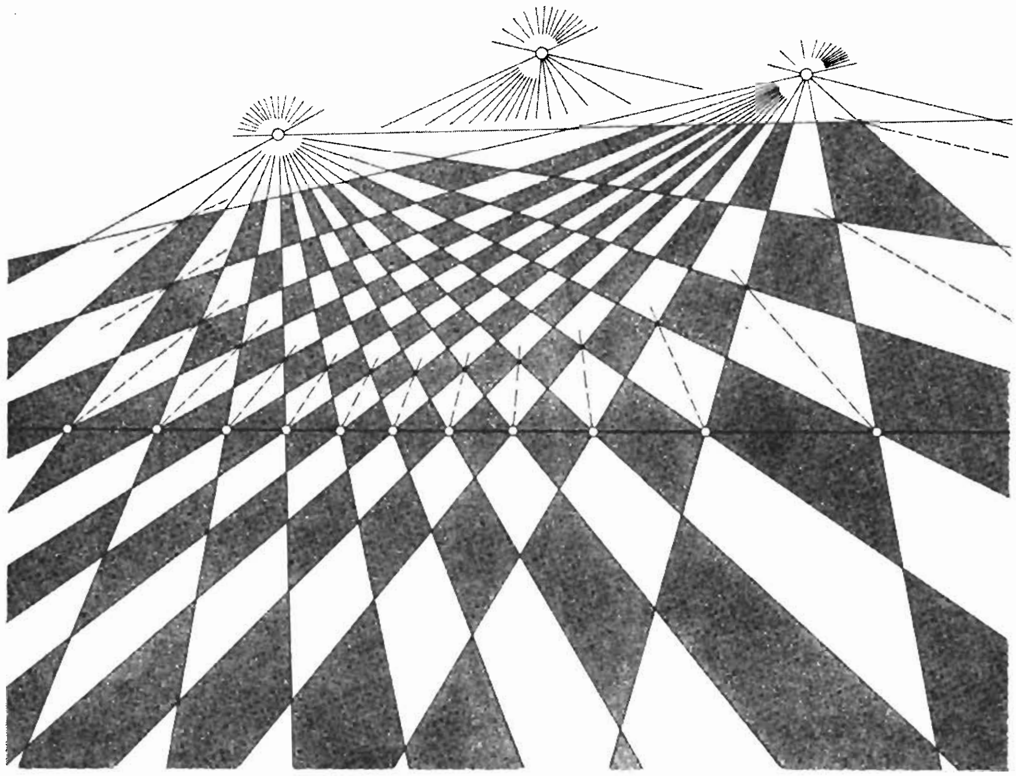


Figure 204

Surveying what as been developed in these chapters, the thought that again lights up in the mind is impressive: The archetypal phenomena may be simple and the consequences of these phenomena transparent, yet the configurations appearing to consciousness retain a certain intangibility, and each time we think about them appear as "morning-fresh" as on the First Day.

EXERCISES

1. On g, two pairs of points M, N and 0, 1 that do not separate each other are given. Construct the corresponding multiplicative scale.

2. Carry out the construction of Figure 198 for the case in which the pairs M, N and 0, 1 *do* separate each other. Why did we not include the form obtained among the archetypal scales?

3. On a line take four points 0, 1, 2, 3 (in the natural order) and construct the scale they determine. Establish the following fact in some drawings: Let U be the reflection point of 1 in 0, 2 and V the reflection point of 1 in 2, U. If 3, 1 and 2, V separate each other, then 0, 1, 2, 3 determine a multiplicative scale. If 3, 2 and 1, V separate each other, then 0, 1, 2, 3 determine a periodic scale. In the special case in which 3 and V coincide, the scale will be additive, whence the additive scale's position of balance between the other two scales becomes beautifully visible.

4. Construct Figures 200, 201 and 204 and draw some conics connecting "corresponding" mesh vertices. With the construction used in Figure 204, we obtain conics either intersected by g or not, according to whether the scale is multiplicative or periodic. Try in each case to survey the situation in the whole plane.

PART FOUR: REFERENCES

Chapter 22

References and Notes

General References. In the first third of the nineteenth century, the duality of certain spatial relationships was discovered. In the course of time it was realized ever more clearly that this duality is already given in the polar structure of the axioms of projective geometry.

Already in the first half of the nineteenth century individual researchers had clearly seen that there are other geometries besides classical Euclidean geometry. There then followed roughly between 1870 and 1910 the full clarification on a broader basis and the systematic development of various non-Euclidean geometries. What was achieved was elaborated in numerous textbooks.

The polar-Euclidean geometry of measure was also considered in some few works. But it was studied purely formally, pursued, as it were, only for the sake of completeness. The great majority of researchers never thought of recognizing in the remarkable property of duality of space a law meaningful for the outer world. It was established purely within the confines of formal mathematics. Chasles alone expressed, if only as a feeling, that here is something fundamental for science. The reason lies chiefly in the fact that the polar-Euclidean measures have, in the domain of rigid bodies to which attention is primarily directed, no immediate applicability.

In 1921 Rudolf Steiner sketched the idea of counterspace. As far as the author could check, he originally coined the word counterspace in a lecture on January 15, 1921, within a course for the teachers of the Waldorf school in Stuttgart on *Das Verhältnis der verschiedenen naturwissenschaftlichen Gebiete zur Astronomie* [GA 323]. Here are two sentences from the vivid description:

> You will see, when you proceed conscientiously to the concept of the phenomena, that you don't answer the purpose merely with the three-dimensional picture of space. You must have in mind the working together of one space that has the three usual dimensions and which you can represent ideally as spreading out radially from a central point, with another space that perpetually annihilates this three-dimensional space, and which may not now be thought of as spreading out from a point, but must be thought of as proceeding from the infinitely distant sphere.

The usual concept of space is attuned to those forces that can be characterized by the so-called central forces. To describe other kinds of forces mathematically, forces that are active for example in organic forms, and whose effect is in many respects opposite to that of central forces, one must use a concept of space that is suited to them: the concept of a counterspace whose separate dimensions—provided one looks at the working of the corresponding forces—cancel those of ordinary space.

Already in the Wärme-Lehre [GA 321] course of March 1920—again held for the Stuttgart Waldorf school teachers—Rudolf Steiner, in characterizing the states of matter, developed the concept of the negative form proper to each form.

To anyone acquainted with the natural continuous transformation that connects point space with plane space,[10] it seems as though Rudolf Steiner, in the above-mentioned context, had vividly sketched this transformation.

In a wonderful lecture of April 1922 given in the Hague in the cycle *Die Stellung der Anthroposophie in den Wissenschaften* [GA 82], Rudolf Steiner characterized counterspace in art: that is, how the sculptor in forming a human head is in reality working out of a space that, in mathematical terms, must be described as being counter to ordinary space. In doing so he uses the geometrically significant expression "forces in *surfaces*," which work sculpturally from without from all sides of the cosmos on forms here on the earth.

Only a handful of people attempted, in the years that followed, to follow up these suggestions mathematically. In the version edited by W. Kaiser of the above-mentioned January 1921 course (Verlag der Kommende Tag, Stuttgart, 1925) the idea of counterspace is again reviewed briefly. A more extensive attempt to understand counterspace mathematically is found in E. A. K. Stockmeyer: *Ein Versuch über die Universalkräfte der Kristallgestaltung* in the Mathesis collection published by the Mathematisch-Astronomischen Sektion am Goetheanum (Stuttgart, 1931, pages 241–260). With this study of the figures given by the simplest crystal forms in the limit plane, a step is taken towards understanding spatial forming from the "periphery." At the same time, and quite independently, G. Adams-Kaufmann pursued these ideas in the comprehensive paper *Synthetische Geometrie, Goethesche Metamorphosenlehre und mathematische Physik* (in the above mentioned Mathesis collection, pages 119–174), which was elaborated by him into the beautiful and important work Strahlende Weltgestaltung. *Synthetische Geometrie in geisteswissenschaftlicher Beleuchtung* (Mathematisch-Astronomische Sektion am Goetheanum, Dornach, 1934). In it there is much of an essential nature about the polarity of space that is important for an understanding of counterspace. E. Blümel, in a short work *Mathematische Transformationen und die vier Aggregatzustände* (in the above mentioned Mathesis collection, pages 87–94), attempts to approach the idea of counterspace from another direction.

10. L. Locher-Ernst: *Polarsysteme und damit zusammenhängende Berührungstransformationen. Das Prinzip von Huygens in der nichteuclidischen Geometrie.* Publications de l'Institute Mathématique de l'Académie Serbe des Sciences, Belgrade, 3 (1950), 101–118. Also: *Stetige Vermittlung der Korrelationen.* Monatshefte für Mathematic 54 (1950) 235–240. (Both are to be found in L. Locher-Ernst, *Geometrische Metamorphosen*, Philosophisch-Anthroposophischer Verlag, Dornach, 1970.)

George Adams-Kaufmann in his paper *Von dem ëtherischen Raume* (in the journal *Natura* 6, 1933, Nos. 5/6, Dornach) was probably the first to realize clearly how counterspace, even including its metrical nature, can be understood mathematically.

Unfortunately, this paper became known to the author only in 1946; otherwise his own attempts might perhaps have progressed more rapidly. In lectures, papers and books (*Urphänomene der Geometrie*, Zürich 1937—*Geometrisieren im Bereiche wichtigster Kurvenformen*, Zürich, 1938—*Projektive Geometrie und die Grundlagen der euklidischen und polareuklidischen Geometrie*, Zürich, 1940) examples were given for understanding spatial forms resulting from the working together of flows of a centripetal and of a centrifugal nature, for example, the forms that (in the little book *Geometrisieren* mentioned) were called logarithmoid.

George Adams, together with Olive Whicher, has worked on the lofty task of connecting space and counterspace with plant nature. As a result we have the following publications: *The Living Plant and the Science of Physical and Ethereal Spaces* (Goethean Science Foundation, Clent, 1949) and *The Plant between Sun and Earth* (Goethean Science Foundation, Clent, 1952). In them, for example, out of the ability to implement the idea of counterspace, the unfolding of leaves in an opening bud is studied.

Of essential significance for our theme, even if not for the mathematical formulation, then all the more so for the content, are the great works of Wachsmuth, above all his comprehensive statements *Erde und Mensch* (Archimedes Verlag, Second edition, Kreuzlingen, 1952) and *Die Entwicklung der Erde* (Philosophisch-Anthroposophischer Verlag, Dornach, 1950), where what the rigidified intellect perceives as the emptiness of the universe surrounding the earth is shown to be an abundance of welling forces, a fountain of youth for the evolving of the organism of the earth. Also important in this connection are several works of H. Poppelbaum; attention may be drawn in particular to the following two papers: *Begriff und Wirkungsweise des Ätherleibes* (*Anthroposophisch-medizinisches Jahrbuch*, Volume III, Hybernia-Verlag, Stuttgart, 1952, pages 7–22) and *Tierseele und Lichtraum* (Sternkalender 1955, Dornach, pages 57–62).

In the course of time, a small circle of interested people has come into being, which is now in a position to implement the idea of counterspace even mathematically. As this happened, it became ever clearer where the obstacle lies: An abstract understanding leads nowhere. If it is purely a matter of whether the intellect applies this or that formal mathematical structure, the temptation is there to work with old habits. That is why in the present work it is frequently emphasized that what matters is a radical transformation of thought habits.

We stand at the beginning of a new development: to use *concretely*, with the corresponding consciousness, the instrument of mathematics. In judging such a beginning, it would be unfair to compare a shoot with a fully grown tree. First attempts are tentative, but this should not make one blind to which of the application's possibilities are already becoming apparent today. Those who really make, for example, the metamorphoses dealt with in Chapter 11 their own, will have no doubt that fields of forces until now unconsidered can be captured in a geometrical image.

C. F. Powell, in the paper *Freiballonflüge in großen Höhen: Neue Elementarformen der Materie* (Naturwissensch. Rundschau 6 , Stuttgart, 1953, page 404), said that the enormous variety of types of so-called particles of matter in cosmic radiation appears as an accumulation of empirical data without any coherence whatsoever. "This shows us clearly the inadequacy of our current theoretical view. . . . It is probable that the immediate theoretical difficulty can be resolved only by a totally new perspective, that we must wait for one of the big discoveries in theoretical physics like the theory of relativity or of quantum mechanics, and that as a result a complete break with accustomed ways of thinking will be brought about."

Why do people not experiment with the idea of counterspace and with the forces that find expression in it?

Fourth remark on page 41. In the last decades we have come to the view that the function of mathematics in its whole compass consists in providing structural schemes thought out by the human intellect for reasons of expediency in conformity with the phenomenal world. This signifies a step forward from the long prevalent view, held since before Kant, that space is an ideal form, coined once and for all, which one must accept ready-made. A further step forward will be signified in realizing how the intellect comes to create one structural scheme rather than another. To gain this insight we need to consider the human being's development. In the first years of its life it works its way—without conceptual consciousness—into the vertical, experiences in the co-operation of functions of the right and left organisms, the width dimension, and realizes in the act of seeing with two eyes—also in clasping the hands together—the depth dimension. As a foundation underlying these inner experiences—not as an abstraction from the phenomenal world—it is able, according as the form-producing creative power has been partially released from its activity in the physical body, to form abstract space in thought. The familiar dimensions of space thus appear as later, abstract reflected images of earlier organic activity.

If it succeeds, maybe without being directly conscious of it, in creating in thought, reflected images of organic activity still further back in the past—perhaps in the years before birth, when the individuality descending towards the earth envelops itself with the creative powers surrounding the earth—then another concept of space comes into being, namely that of counterspace.

Page 64. It was mentioned that characterizing the various divisions of the plane by n lines for $n > 5$ is a difficult problem. It is not immediately obvious where the difficulty lies. An initial insight can be gained from the paper of G. Ringel: *Über Geraden in allgemeiner Lage in the journal Elemente der Mathematik* (Birkhäuser Verlag, Basel) 12 (1957), no. 4.[11]

[11] See Ernst Schuberth, *Konfigurationen von Gerade in der eukidischen und der projectiven Ebene.* Staatsexamensarbeit, Bonn, 1964.

Remark on page 89. Of the consequences mentioned there, note the following fact proved by R. M. Robinson in 1947:

A spherical ball can be divided into five pair-wise, point-wise disjoint parts A, B, C, D, E in such a way that, by suitable rotations of the different parts, a spherical ball is obtained both from the two parts A, B and from the other three C, D, E, each of them congruent to the original one.

The division is certainly highly complicated. The proof is presented in detail in the paper L. Locher-Ernst: *Wie man aus einer Kugel zwei zu ihr kongruente Kugeln herstellen kann.* El. Math. 11, pages 25–35, (1956).

In this connection it may be mentioned that G. Cantor, for whom a preoccupation with the continuum became the life's destiny, pursued the idea of relating the first two orders of infinity (the first power is the power of countable sets, the second power that of the continuum) to the different manifestations and modes of operation of the states of matter. At the end of a paper written in 1885 (G. Cantor: *Gesammelte Abhandlung,* page 276, Verlag Springer, Berlin, 1932) can be found the sentence:

> . . . in this respect, years ago already I formed the hypothesis that the power of material substance is what in my investigations I call the first power, and that, on the other hand, the power of ethereal substance is the second.

This touches on something that we can anticipate will soon be given a real meaning.

Just how intensively individual thinkers in ancient times experienced the relationship of the rationals to the irrationals, in modern geometrical terms the relationship of a skeleton to the whole continuum is shown by the following sentence handed down in the *Euclidean Scholia*:

> The Pythagoreans told of how the first person to make this theory (of the irrationals) public would be the victim of a shipwreck, and perhaps they wished therewith to indicate that all that is irrational in the universe, even as an "inexpressible and formless thing" should be kept hidden, and that anyone meeting, in his soul, with such a form of life and making it accessible and public is drawn into the Sea of Becoming and engulfed by its unresting currents.

Page 98. At the end of Appendix One it was mentioned that it is impossible to bring the points of a segment and the points of a square surface into a one-to-one correspondence that *is also continuous.* The proof goes as follows: Let *UV* be the segment and *AB* a side of the square in question. We start from the assumption that a correspondence with the required properties does exist. Then, by continuity, the points of segment AB would have to correspond to the points of a subsegment $A'B'$ of UV. The middle point M' of $A'B'$ would be the image of a point M of AB. Consider the points of the segment MN of the square surface that is perpendicular to side AB. Because of continuity, the image of MN must reach into each arbitrarily small neighborhood of M' in the segment $A'B'$. But this is impossible if to each point of $A'B'$ there corresponds one and only one point of the square surface. Our initial assumption thus leads to a contradiction.

Chapter 13. Knowledge of how the ancient Greek sculptors worked cannot be overestimated as a stimulus for the exercises in comprehending counterspace. An insight is offered by the beautiful book by C. Blümel: *Griechische Bildhauer an der Arbeit*, Verlag Walter De Gruyter, Fourth edition, Berlin, 1953, page 16 in particular. A concise, essential account is given in the article by F. Durach: *Griechische Bildhauer an der Arbeit* in the weekly Das Goetheanum 34 (1955), No. 33.

Chapter 14. G. Adams-Kaufmann points out the essential significance of the division of point space by six planes in the above-mentioned work *Strahlende Weltgestaltung*. The characteristic features of the configuration produced by six planes was investigated—probably for the first time—by V. Eberhard in the paper *Eine Classification der allgemeinen Ebenensysteme* [Crelle's *Journal für die reine und angewandte Mathematik*, 106 (1890), 89–120].

The formulas given on page 142 are not hard to derive. We start from the division by three planes:

$$V_3 = 1\,, \qquad S_3 = 3\,, \qquad F_3 = 6\,, \qquad C_3 = 4\,.$$

We also use the fact that m lines in a plane have $\binom{m}{2}$ points of intersection and that they divide the point field into $1 + \binom{m}{2}$ domains with $2\binom{m}{2}$ boundary segments.

We now consider how the numbers V_m, S_m, F_m, C_m grow when to the m planes an $(m + 1)$th plane is added.

Since the m planes determine $\binom{m}{2}$ lines of intersection, adding the $(m+1)$th plane causes the number V_m to be increased by $\binom{m}{2}$.

Since the m planes intersect the $(m+1)$th plane in m lines, $1+\binom{m}{2}$ domains are formed in the latter plane. Each such domain divides a spatial core of the division brought about by the m planes into two parts. Thus the addition of the $(m + 1)$th plane causes the number C_m to be increased by $1 + \binom{m}{2}$.

The division that the m planes bring about in the $(m+1)$th plane produces $2\binom{m}{2}$ segments. Furthermore the $(m + 1)$th plane divides one segment on each of the $\binom{m}{2}$ lines of intersection of the m planes into two segments. Thus S_m increases by $2\binom{m}{2} + \binom{m}{2}$.

The division that the m planes bring about in the $(m+1)$th plane gives rise in the latter to $1 + \binom{m}{2}$ domains. Furthermore each of the $2\binom{m}{2}$ segments of the latter division divides a face portion (planar domain) of the division of space by m planes into two domains. Thus F_m increases by $1 + \binom{m}{2} + 2\binom{m}{2}$.

As a result the following recurrence relations are obtained:

$$V_{m+1} = V_m + \binom{m}{2}, \qquad S_{m+1} = S_m + 3\binom{m}{2},$$
$$F_{m+1} = F_m + 1 + 3\binom{m}{2}, \qquad Cm_{+1} = C_m + 1 + \binom{m}{2}.$$

These give rise to

$$V_{m+1} - S_{m+1} + F_{m+1} - C_{m+1} = V_m - S_m + F_m - C_m.$$

The value of this expression is thus independent of m. Since $V_3 - S_3 + F_3 - C_3 = 0$, we thus have that for all n greater than 2:

$$V_n - S_n + F_n - C_n = 0\,.$$

Clearly $V_n = \binom{n}{3}$. Thus the recurrence relations imply, for example, that

$$C_{m+1} - V_{m+1} = C_m - V_{m+1} = C_{m-1} - V_{m-1} + 2 = \ldots$$
$$C_3 - V_3 + m-2 = m + 1\,.$$

Thus, in general we have

$$C_n = V_n + n = \binom{n}{3} + n = \tfrac{1}{6}n(n-1)(n-2) + n\,.$$

The other formulas are obtained by similar reasoning.

Page 169. There is a short description, partly verbatim extracts, of O. Simony's works in the booklet by F. Dingeldey: *Topologische Studien* (Leipzig, 1890), from which the results mentioned in the Exercises were taken. A biographical sketch and other remarks about his investigations are provided in the paper of E. Müller: *Oskar Simony und seiner topologischen Untersuchungen*, appearing in the Mathesis collection of the Mathematisch-Astronomischen Sektion am Goetheanum, 1931, pages 175–226.

Part Two: Schooling. Further material, in particular also material for fundamental drawing exercises, is contained in the following articles by the author in the journal *Elemente der Mathematik* (Birkhäuser Verlag, Basel):

Polarentheorie der Eilinien. 6 (1951), 1–7.
Natürliche Umformung einer Kurve in ihre Evolute. 8 (1953), 73–75.
Bilder zur Geometrie der regelmäßigen Figuren. 8 (1953), 97–102.
Konstruktionen des Dodekaeders und Ikosaeders. 10 (1955), 73–81.

Second Remark on page 195. The usual proof of the Skeleton Theorem, the basic idea of which goes back to Lüroth and Zeuthen, is reproduced in detail in the author's textbook *Projektive Geometrie.*

The Fundamental Theorem. When projective geometry is based on metrical arguments—which is how it is mostly still taught—importance attaches mainly to proving the invariance of the cross-ratio with respect to the operations of connecting and intersecting. This is done essentially as follows. Let A, B, C, D be four points of a line g and A', B', C', D' their projections onto a line g'. To form the cross-ratios with A, B and A', B' as base points respectively, we draw, through each of A and B, a parallel to g' (Figure 205). We thus have that

$$(AC : BC) : (AD : BD) = (a : b) : (u : v)\,, \qquad (1)$$
$$(A'C' : A'D') : (B'C' : B'D') = (a : u) : (b : v)\,. \qquad (2)$$

That $(AC : BC) : (AD : BD)$ and $(A'C' : B'C') : (A'D' : B'D')$ have the same value now follows from (1) and (2) together with the fact that the ratio of two ratios remains unchanged when the two inner terms are interchanged, that is

$$(a : u) : (b : v) = (a : b) : (u : v)\,.$$

From this it is apparent that the Fundamental Theorem is equivalent to the commutativity of multiplication in the relevant area of arithmetic.

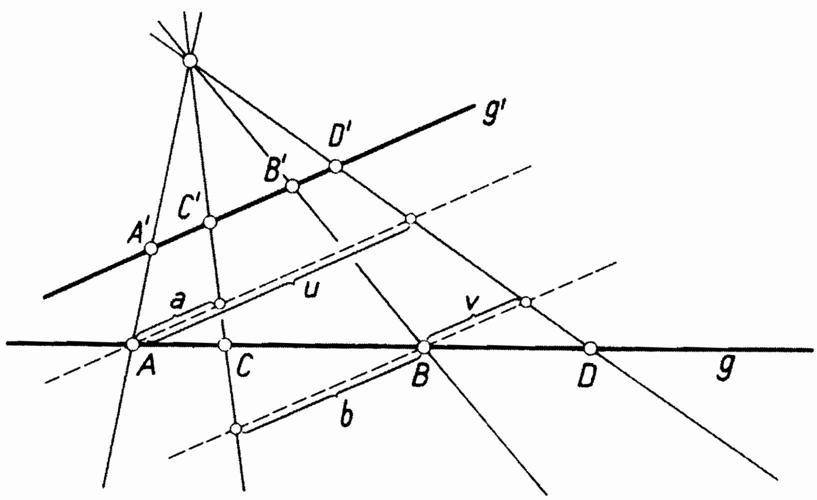


Figure 205

Textbooks. Of the many synthetic presentations of projective geometry, we mention in addition to the books already spoken of only the following:

Th. Reye: *Die Geometrie der Lage*
Part 1, 5th edition, Leipzig 1909. Part 2, 4th edition, Leipzig, 1907.
This masterpiece of fluent description is today still the first recommendation.

L. Godeaux: *Géométrie Projective*
Sciences at Lettres. Second edition, Liège, 1952.
A clear, concise account after the classical pattern of the methods developed by von Staudt and Enriques.

H. M. S. Coxeter: *The Real Projective Plane*
Cambridge University Press. Second edition, 1955.
As the title says, this only deals with planar geometry. For an axiomatic development of the latter, this book is among the best available.

H. Prüfer: *Projektive Geometrie*
Akademische Verlagsanstalt Geest & Portig. Second edition, Leipzig, 1953.
A work that is independent in its axiomatic structure and in a good many details, with a particularly pronounced emphasis on logical structure.

H. F. Baker: *Principles of Geometry*
Cambridge University Press. Volume I: Foundations. Second edition, reprinted 1954.
—Volume II: Plane Geometry. Second edition, reprinted 1954.
—Volume III: Solid Geometry. Second edition, 1934.
A comprehensive work, as original as it is profound, providing the mathematician with a wealth of stimulating material for further study of the subject.

Concluding Remark. Having indicated the larger aims in some general remarks, it is fitting to return to the modest limits set at the beginning for the present work. It seemed to the author an urgent task to provide a usable, basic, mathematically unobjectionable introduction to the elements, which above all offers the teacher and lecturer suffciently extensive material, to develop in their classes an awareness for the interplay of space and counterspace. Second-order theory and a corresponding schooling deals, using the archetypal scales presented in Chapter 21, with measures in detail and develops movement-forms that lead to a realm embracing space and counterspace as well as form-transformations that appear only partially in real space.[12]

[12] The papers collected in the volume *Geometrische Mwetamorphosen* (Dornach 1970) are first steps toward the second order indicated here.

Chapter 23

BIBLIOGRAPHY

Adams, George. *Strahlende Weltgestaltung. Synthetische Geometrie in geisteswissenschaftlicher Beleuchtung, zugleich als Vorarbeit gedacht für eine geistgemäße mathematische Physik*, Second unrevised edition. Dornach, 1965.

—*Von dem ätherischen Raume. Studien und Versuche,* No. 6. Stuttgart, 1964.

Adams, George and Olive Whicher. *The Plant Between Space and Counterspace*, Goethean Science Foundation, Clent, Stourbridge, Worcester, 1960.

—*The Plant Between Sun and Earth. 24 color illustrations,* by O. Whicher, Text by G. Adams and O. Whicher. Goethean Science Foundation, Clent, Stourbridge, Worcester, 1954.

Locher-Ernst, Louis. *Geometrische Metamorphosen, Beiträge zu einer geisteswissenschaftlichen Metamorphosenlehre*, Dornach, 1970.

— *Mathematische Meditationen*, Vollständige regelmäßige Vielecke. 7 zweifarbige Tafeln, Second edition, Winterthur, 1962.

— *Einführung in die freie Geometrie ebener Kurven*, Basel, 1952.

— *Differential und Integralrechnung im Hinblick auf ihre Anwendungen. Ein Lehr- und Übungsbuch zur Infinitesimalrechnung und zur analytischen Geometrie*, Basel, 1948.

— *Urphänomene der Geometrie*, Zürich, 1937. (Out of print.)

— *Geometrisieren im Bereiche wichtigster Kurvenformen*, Zürich, 1938.

— *Projektive Geometrie und die Grundlagen der euklidischen und polareuklidischen Geometrie*, Zürich, 1940.

Wachsmuth, Guenther, *Erde und Mensch–ihre Bildekräfte, Rhythmen und Lebensprozesse*, Third, revised edition. Dornach, 1965.

— *Die Entwicklung der Erde. Kosmologie und Erdgeschichte–ein organisches Werden*, Second, enlarged edition, Dornach, 1960.