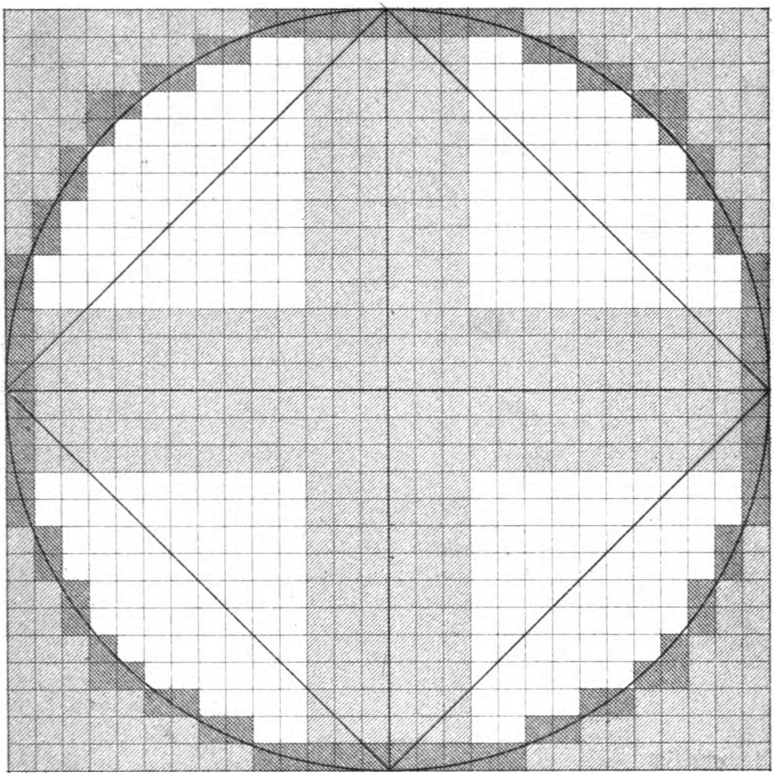

VERIFICATION OF THE QUADRATURE OF THE CIRCLE.

Let either of the above *quadrants* be divided into 196 squares, having 14 on each side; then the line which is the side of the whole inscribed square will divide the quadrant into two equal parts, each of which contains 98 squares.

And the arc of the circle belonging to the quadrant will divide the outside 98 squares into two parts, which shall be to each other as 4 is to 3.

For the arc of the quadrant cuts 21 of the squares, which form a complete *arch*; and there are 142 squares within those which are cut by the arc and 33 without; then if these 21 squares be divided in the proportion of 4 to 3, there will be 12 squares that will fall within the circle and 9 that will fall without it; then $142 + 12 = 154$, the number of squares within the arc, and there are 196 squares in the quadrant.

Then 154 is to 196 as 11 is to 14, for $\frac{196}{154} = \frac{14}{11}$.

Again, of the 154 blocks within the arc, if 98 be deducted, we shall have 56 between the side of the inscribed square and the arc of the circle; and if to the 33 blocks without the arc 9 be added, we shall have 42 blocks without the arc of the circle; then 56 is to 42 as 4 is to 3.

Again, if the sides of the squares which form the *arch* of the quadrant, viz.: $5 + 2 + 2 + 2 = 11 \times 2 = 22$. be multiplied by 4, we shall have 88 sides for the circumference of the circle, and as each of the quadrants has 14 squares on each side, we shall have 28 sides for the diameter of the circle.

Then dividing the given circumference by the given diameter, we have $88 \div 28 = 22 \div 7 = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, *which is the the true ratio of the circumference to the diameter of the given circle.*

TO

THE AMERICAN PEOPLE

WHOSE LOVE FOR LEARNING AND DEVOTION TO THE TRUTH,

ARE ONLY EQUALED

BY THE MAGNIFICENT CONTRIBUTIONS WHICH THEY HAVE MADE

TO THE CAUSE OF EDUCATION,

THIS VOLUME IS RESPECTFULLY INSCRIBED AS A CHEERFUL CONTRIBUTION TO THE CAUSE WHICH WE ALL ADVOCATE IN COMMON, AND AS A SMALL TESTIMONIAL OF THE ESTEEM IN WHICH THEY ARE HELD

BY THE AUTHOR.

L.C.

THE

QUADRATURE OF THE CIRCLE,

THE

SQUARE ROOT OF TWO,

AND THE

RIGHT-ANGLED TRIANGLE,

BY WILLIAM ALEXANDER MYERS,
President of Myers' Commercial College, Louisville, Ky.

FIRST EDITION.

"Where is the wise."—1st Cor., i, 20. "Now the serpent was more subtile than any of the beasts of the earth which the Lord God had made."—Gen. iii, 1.

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AUTHORS MADE USE OF IN THE PRESENT VOLUME.

Should the student desire more general information upon the subjects treated of in the present volume, he is referred to the following works which have been freely used by the author wherever they have been found to be of service to his cause. They will be found to be among the best of their kind

"Montuclas' History of Mathematics;" "Hutton's Recreations;" "DeMorgan on the Law of Probabilities;" "Elements of Euclid," by Todhunter; "Elements of Euclid," by Thompson; "Davies' Le Gendre;" "Robinson's Geometry;" "Chauvenet's Geometry;" "Loomis' Geometry and Trigonometry;" "Bullfinch's Beauties of Mythology;" "Minifee's Draughting and Architecture;" "Home and School Journal;" "Chambers' Encyclopædia," and the "Douay Bible."

PREFACE.

The following pages are intended to explain certain mathematical truths, which were discovered by the author while engaged in a series of investigations made during the hours of rest from the labors of the college and the counting room. They consist chiefly of new methods employed in the solution of problems which have heretofore been regarded by mathematicians as impossible; and, although the author's mind has been employed with the subject for a number of years, the result of the investigations are now published for the first time.

If the discoveries should not come up to that standard of brilliancy which *commands* attention, it is hoped that they may be found worthy of a fair and impartial consideration.

The author scarcely dares to hope, with the many examples of failure before him, that at the outset the entire mathematical world will bow in submission to his decree, or submit unconditionally to the power of his reason or the force of his logic; nor does he desire that the glorious fabric, which the mathematical genius of the world combined has reared as a monument to the memory of departed greatness, should crumble into dust by a single touch. Ah, no! Rather let the ivy of remembrance forever remain green upon their mausoleums, and the vines of gladness encircle their remains. But if *Genius*, while pursuing her walks amid these temples of departed greatness, should suddenly be inspired by *Wisdom*, and conceive *Truth*, who would be so poor as to refuse a garland with which to crown her brow, where truth sits enthroned?

The discoveries are as follows:

1. The Quadrature of the Circle.

2. A Common Measure of the Side and Diagonal of the Square.

3. An Infinite Series of Right-angled Triangles, with a Rule for their Solution.

For information concerning the History of the Quadrature of the Circle, the reader is referred to the Introduction, which begins on page 9, of this book. But before we proceed too far in our investigation of the subject, it seems proper to inquire first what is the circle. If a draughtsman or mechanic take an ordinary pair of dividers, and with one foot as a center, and the other starting at a certain point, cause it to describe a curve which is constantly receding upon itself, this point will return to the point from whence it started, when it is said to be an inclosed curve; and the curve, which is described by one point rotating around the other point within, is said to be the circumference of the circle, every point of which is equally distant from the point within; and this point within is called the center of the circle; and the plane figure which is inclosed by the circumference is said to be the circle itself. But a *mathematical circle* is more difficult to comprehend. If we say that to make a dot with a pencil that it is a point, the definition is sufficient for mechanical purposes; but a mathematical point has position only, and no magnitude, because it has no *size.* So, also, a mathematical circumference is a curved line constantly receding upon itself; but, like a mathematical straight line, it has *length only, without either breadth or thickness.* A mathematical circle, then, is a plane figure, which is inclosed by a curved line so finely defined as to be invisible, not only to the naked eye but by the means of the most powerful microscope which it is likely ever will be made, *yet its existence* can be as certainly determined, mathematically, as if it were drawn mechanically upon wood or paper, and not only its figure, but its dimensions, and consequently the ratio or proportion of its different parts one to another. But it may be objected that it has no real existence, because we can not see it. It may be answered that it has a mathematical existence, which can be so plainly demonstrated by sensible figures that the human mind finds it impossible to doubt it. Just as the Deity has an eternal existence, for neither can we see Him "*and live*," but it can be shown that He is, for "*the invisible things of Him, from the creation of the world, are clearly seen, being understood by the things that are made, His eternal power also, and divinity;*" and it is a wonderful truth that space, which is also said to be infinite, can only even be partially measured by the aid of the power which we obtain from the science of numbers, because, by their aid, we can reason mathematically and truly far beyond what we can see. It may be said with truth that the circle, the square, and even the triangle are emanations from the divine intelligence, as well as the science of numbers, by the aid of which they are measured, because they are a contrivance; and if there is contrivance there must have been design; *but there is contrivance, therefore there was design;* and if there was design, there must have been a designer; and it is very evident this designer was not man, who has expended all the talent, energy, and ability he ever had in trying to find out what the circle is, and has ignominiously failed. And after all the struggle, by simply turning our eyes to Holy Writ, there we find it, simple and beautiful as truth itself, of which it is a fit emblem, for it has no fault—it is perfect. "Verily the foolishness of God is wiser than men." The Designer, then, must have been one of infinite power and wisdom, "higher than the heavens," "who dwelleth in light inaccessible; and if so, He must have used such numbers as in His infinite wisdom would best accomplish that result, and these numbers *must* be such as is commonly found in all His works. Let us "come and see."

By PROPOSITION 2, PART 1, it is proved that, if from the circumference of the given polygon the $\frac{1}{50}$th be deducted, the remaining $\frac{49}{50}$ will be exactly equal to the circle itself; and if the square root of the $\frac{49}{50}$ be extracted, the result will be the number 7, which is the base of the system.

Again, by PROPOSITION 3, PART 1, it is proved that, if from the sums of the squares of the two sides of any square the $\frac{1}{50}$th be deducted, the square root of the remaining $\frac{49}{50}$ can be extracted exactly, which will be 7, the base of the system or the generating number; and if the square root of the $\frac{1}{50}$th be extracted, it will give the number 1, which is another generating number.

COINCIDENCE FROM HOLY WRIT.

Levit. xxiii., xxiv., and xxv., see the numbers 7, 49, and 50. Genesis, Exodus, Leviticus, Numbers, Deuteronomy, and Apocalypse for the number 7.

By *Case* 2, PART 1, it is shown that the sine No. 1 was divided into 14 parts, and each part was $\frac{1}{70}$.

COINCIDENCE FROM HOLY WRIT.

See Genesis, Exodus, Leviticus, Numbers, and St. Matthew for the number 14.

See Genesis and Daniel for the number 70.

By *Case* 2, the inscribed double triangle for double the number of sides was proved to be $\frac{2}{99}$; by *Case* 3, $\frac{2}{19601}$; by *Case* 4, $\frac{2}{768398401}$, and so on *ad infinitum*.

COINCIDENCE FROM HOLY WRIT.

See Genesis, Exodus, Leviticus, Numbers, and Apocalypse for the number 2.

See Job, chapters 1 and 2, for the circumference and diameter.

See Genesis for the circle.

See Apocalypse for the square.

"And the stone which the builders rejected was composed of three triangles."

HISTORY

OF THE

QUADRATURE OF THE CIRCLE,

TRANSLATED FROM THE FRENCH OF MONTUCLA,

BY J. BABIN, A. B., LOUISVILLE, KY.

It has seemed to us fit to treat here separately this *Great Question*, on account of its great celebrity, and we shall not hesitate to give some of the silly notions to which this problem has given rise in ill-balanced and enthusiastic minds.

To *square the circle* is to assign the geometrical dimensions of a square equal to the circle. The quadrature of the circle has been attempted in several ways, by endeavoring to find a square or any other rectilineal figure equal to the circle. As it was soon found that the rectangle of the radius, by half the circumference, is equal to the area of the circle, the problem was soon reduced to finding the length of the circumference in terms of the radius. We can not believe that Archimedes was the first who made known this truth, for it is a necessary consequence of what was already known on the measure of regular polygons of which the circle is the extreme limit, the last of all.

The circle being after rectilineal figures, the most simple in appearance, geometricians very naturally soon began to seek for its measure. Thus we find that the philosopher Anaxagoras occupied himself with this question in his prison. Then Hippocrates of Chios tried the same problem, and it led him to the discovery of what is called his lune, a surface in the shape of a crescent, bounded by two arcs and exactly equal to a given square. He also found two unequal lines which were together equal to a rectilineal figure, so that if their relation could have been found the solution of the problem would have been obtained. *But this no one has yet been able to do, nor is it likely ever to be done.* We are also indebted to Simplicius for the history of a disciple of Pythagoras, named Sextus, who claimed to have solved the problem, but his reasoning has not been transmitted to us. Finally, this inquiry at that early day became so famous that Aristophanes in ridiculing Meton makes him appear on the stage, in his "Comedy of the Clouds," as promising to square the circle about 430 years before Christ. This was all the more amusing from the fact that people generally suppose that to attempt to *square the circle* is the same thing as trying to make a circle square, which implies a manifest contradiction. Yet this was that Meton, so renowned for his discovery of the cycle of nineteen years, of whom, together with Socrates, the comedian made a public laughing stock.

Aristotle mentions two of his contemporaries, Bryson and Antiphon who worked at the quadrature of the circle. Nothing could have been more grossly inaccurate than Bryson's pretended quadrature; for he made the circumference of a circle equal to $3\frac{3}{4}$ times the diameter. But Antiphon stated that having inscribed a square in a circle, if an isoceles triangle having the chord for the base be inscribed in each of the remaining segments, and similar triangles in the remaining eight segments, and so on, the sum of all these rectilineal figures would be equal to the circle; nothing could be more true than this, and Aristotle was undoubtedly wrong in calling Antiphon a *paralogist*, for one of Archimedes' two quadratures of the parabola depends upon the same operation; but this method has not yet succeeded with the circle.

It might be supposed that Archimedes applied himself to the solution of this problem, and that he gave his approximate measure of the circumference of a circle only for want of the long-sought-for vigorously exact measure. His discoveries on the spiral, if they have preceded his book on the dimensions of the circle, might well inspire him with the hope of finding the length of the circumference. However, that may be, Archimedes showed, about the year 250 before Christ, that if the diameter of a circle is 1, its circumference is less than $3\frac{10}{70}$ or $3\frac{1}{7}$, and more than $3\frac{10}{71}$; by taking $3\frac{1}{7}$ the error is less than the $\frac{1}{497}$ of the diameter. The calculation of Archimedes is singularly skillful, and anticipates the objection made by some of those who reject his account, for the reason that he could not precisely extract the square roots of the several numbers used in his calculation. But I have known some of these individuals, and I have never found a single one who knew Archimedes otherwise than *by name.*

We still further know by the testimony of Simplicius, that Nicomedes and Appolonius had tried to square the circle; the first by means of the curve, which he calls *quadrans* or the quadratrix, the discovery of which, however, is usually ascribed to Dinostratos; and the second, by means of a certain line which he called the sister of the *tortuous* line, or the spiral, and which was nothing else but the quadratrix of Dinostratos. This quadratrix, invented in truth at first to divide the angle in any way whatever, would give the quadrature if its extreme limit on the radius could be found. Perhaps Appolonius or Nicomedes discovered this property; be this so or not, Eutocious tells us that Appolonius had carried further than Archimedes the close relation of the diameter to the circumference, and that another geometrician named Philo, of Gadares or Gades, had gone still further, so that the error did not exceed the $\frac{1}{100,000}$th. The moderns have carried this accuracy much beyond this point.

Finally, among the ancients there were many of those persons *unworthy* the name of geometrician, who pretended to have found in different ways the quadrature of the circle. Jamblicus, cited by Simplicius, says so expressly. But their false reasoning has not reached us, and no doubt did not deserve it.

The Arabs, who followed the Greeks in the Culture of the Sciences, must also have had their quadratures; but all we know about it is that some of them supposed they had discovered that the diameter being 1, the circumference is the square root of 10; a very grave error; for it exceeds 3.162, and the circumference, according to the account of Archimedes, is not quite 3.142857. For the rest we see in the catalogues of Arabian writings several works entitled *de quadratura Circuli;* like several others on the trisection of the angle, the duplication of the cube, etc.

We pass rapidly over the centuries of ignorance which produced a few treatises on the quadrature of the circle, manuscripts left remaining in the dust of libraries, until we reach the period of the revival of letters among ourselves. About this time the famous CARDINAL DE CUSA distinguished himself by his unfortunate attempts at the solution of this problem. Nevertheless he tried an ingenious method; he rolled a circle on a plane or line, and supposing that its circumference was applied to it wholly until the point which had first touched it touched it again; he therefore justly inferred that this line would be equal to the circumference. He even conceived the outline of the curve, which the point that first touched the straight line was to describe which formed the curve, since called the Cycloid. But he supposed, with Charles de Bovelle, in the following century, that this curve was itself an arc of a circle, and from this he claimed to determine it by a geometrical construction which was entirely arbitrary, resting on no real property of this movement. He also tried another method, according to which he gave the following solution of the problem: a circle being given, add to its radius the side of the inscribed square, and with this line as diameter describe a circle, in which is inscribed an equilateral triangle, the perinuter of this triangle will, says Cardinal de Cusa, be equal to the circumference of the first circle.

It was not difficult for Regiomontanus to prove that Cusa was mistaken; this relation of the circumference to the diameter fell outside the limits demonstrated by Archimedes; that is according to this relation the diameter would be to the circumference as one to a number greater than $3\frac{1}{7}$ already too large. Besides, the Cardinal learned for his age, though very much addicted to astrology, he presents in the collection of his works several geometrical tracts which are full of paralogisms.

We have just spoken of Charles de Bovelle or Carolus Bovillus, distinguished at the time by the title of noble philosopher. He signalized himself by the strangest ideas. He gave in 1507 a work entitled: *Introductionum Geometricum*, translated into French and republished in 1552 under the auspices of Oronce Finée, under the title of *Geometrie Pratique, Composée par le noble* Philosopha, *Maitre Charles de Bovelle*, etc. He claims to give there the quadrature of the circle according to the idea of the Cardinal de Cusa, which, he says, came to him by seeing a wheel moving on the pavement. But the construction by which he pretends to give the length of the line to which is applied the circumference of the rolling circle is absolutely arbitrary, and it would follow that the diameter is to the circumference as 1 is to the square root of 10, or 3.1618, which is far from the limits of Archimedes. What is also singular, is that in this same book, and in an appendix added to the first volume of the preceding works, he speaks of the quadrature of the circle made by a poor peasant, according to which the circle having 8 for diameter is equal to the square having 10 for diagonal, that is to 50, which is false; for the circle is in this case less than $50\frac{2}{7}$, and more than $50\frac{18}{71}$, and the quadrature of Bovelle does not agree with that of the peasant, which he considers as true; for the latter gives the relation of the diameter to the circumference exactly as 1.0000 to 3.1250; the noble philosopher even wanders further from the truth than the peasant does below; and he might have been told that when one is mistaken he ought not at least to contradict himself. Bovelle says, falsely, that these relations coincide. Either he had not performed the calculation himself, or he did not know enough arithmetic to extract by approximation a square root; these works of Bovelle are pitiable; his manner of cubing the sphere is preëminently absurd.

We are sorry to find in the same class a royal professor of the 16th century, named Oronce Finée, who, by his numerous works, acquired a kind of fame. He gave in his Protomathesis a quadrature of the circle, a little more ingenious, in truth, than that of Bovelle; but which is, nevertheless, a paralogism. On the point of dying, in 1555, he urgently advised his friend, Mizault de Monthuçon, to publish his discoveries, not only upon this subject, but also on the most famous problems of geometry, such as the trisection of the angle, and the duplication of the cube, and the inscription in the circle of all regular polygons. Mizault kept his word, and in 1556 published this assemblage of paralagisms under the title of *De rebus Mathematicus hactenus desideratis, libri IV.* Most of these problems are solved in various ways by him; it happens that his different solutions of the same problem do not agree with one another, nor with those of Bovelle, and of his rural geometrician which he had approved by publishing them, it was the height of false reasoning in geometry; consequently he was easily refuted by the geometrician Buteon, who had been his disciple at the College Royal, by Momus or Nunez, a Portuguese geometrician, and several others; but still he died satisfied, fully persuaded that his name would be placed on a level with those of Archimedes and Apollonius. This scandal was renewed among the royal professors in 1600, when Monantheuil, one of their number, published a quadrature of the circle.

One Simon a Quercu (doubtless Duchêne or Van Eck) appeared on the arena a few years later, in 1585, and proposed a quadrature of the circle. His pretended discovery was much less wide from the truth than those of his predecessors and fell within the limits of Archimedes. So Peter Metius, who undertook to refute him, was obliged to seek for a closer relation of the diameters to the circumference, and found that the one was to the other as 113 to 355. The pretended quadrature of Duchêne could not stand this test, and must be named only because it led to the curious and elegant discovery of Metius; for this relation of 113 to 355, reduced to decimals, is the same as 10000000 to 31415929; which is at the most but $\frac{3}{10.000.000}$ of the diameter in excess. The diameter of the earth being only 6542816 (toises) or 13936912 yards, the error made by this relation of the circumference of a circle of that size would scarcely be 2 (toises) or 4 yards. If those who connect in their minds the problem of the quadrature of the circle with that of longitudes, knew what we have just said they would soon see their mistake; for, if these problems were connected with each other, what would be an error of 4 yards on a track around the earth? The Spanyard, Sir Jaime Falcon, of the order of Our Lady (*Notre Dame*), of Montesa, published in 1587, at Antwerp, his paralogism on the quadrature of the circle. His book is rendered amusing by a dialogue in verse between himself and the circle, which thanks him very affectionately for squaring it; but the good and model knight ascribes all the honor thereof to the holy patroness of his order. The paralogism was apparently so gross that no one took the trouble to refute it.

But a man much more famous than the foregoing challenged the attention of learned Europe by his pretensions on the quadrature of the circle; it was the celebrated Joseph Scaliger. Full of self conceit, he supposed that he had only to present himself on the field of geometry and that nothing that had baffled geometricians until then could resist a man of letters with his powers. He therefore undertook to find the quadrature of the circle, and put forward, with much braggadocia, his discoveries on this subject in a book which appeared in 1592: *Nova Cyclometria;* but he had no cause to congratulate himself on having thus wished to place himself among geometricians. For he was refuted by Clavius, Viete, Adrianus Romanus, Christman, etc., who showed each in his own way that the size which he assigned to the circumference of the circle was only a little less than the inscribed polygon of 192 sides; which being absurd, demonstrated the incorrectness of Scaliger's reasoning; but he did not surrender; and never did a man who thought he had discovered the quadrature of the circle, the trisection of the angle, the duplication of the cube, or perpetual motion, give in to the plainest reasoning. He will sooner deny the most elementary propositions of geometry, like Moliensis Cana, who found no less than twenty-seven false propositions in the first book of Euclid. Scaliger replied with bitterness to the geometricians who had censured his quadrature; he treated them with contempt, especially Clavius who had already wounded him by an answer to his attack on the Gregorian Calender. Unfortunately for the honor of Scaliger abuse is not reasoning, and the established fact remains that Scaliger, an eagle in literature, was nothing in geometry.

Scarcely had Scaliger disappeared when one Thomas Gephirander came to take his place. But he had not Scaliger's pride; he acknowledged, even in the title of his book, that his discovery was simply the result of divine Grace. We shall see many others gifted with this same spirit of humility. The *paralogism* of Gephirander nevertheless was palpable; for it consisted in the pretension that if between two magnitudes there is any geometrical relation whatever, the same relation will exist if the same quantity is taken from each. Thus, according to this illuminate, the same relation exists between 2 and 5 as between 3 and 6, since only the same quantity, viz., unity is subtracted from each of these numbers. But scarcely any of the follies with which false reasoning, a false mind, and the conceit of never recanting one's errors inspire these visionaries have equaled those of Alph. Cano, of Molina, in a book entitled: *Nuevos descubrimientos Geometricos.* He remodels the whole of Euclid, and scarcely one of his propositions is spared by him. Yet who would believe it! he found another fool, named Janson or Jansen, who translated him into Latin under the title of *Nova reperta Geometrica*, etc. Moreover Cano admitted that he had not the least idea of geometry until the Deity, whose delight it is to humble the proud and enlighten the ignorant, had inspired him.

A similar wiseacre presented at the same time in France his paralogisms upon the quadrature of the circle and the duplication of the cube. It was a merchant of Rochelle, called De Laleu. This one also pretended to have received the solution of these problems by *divine revelation*, and announced that the union of the Jews, Turks, and Pagans to the Christian religion depended upon the manifestation of this truth. In fact, according to him, the quadrature of the circle was the quadrature of the heavenly Temple, and the duplication of the cube that of the elementary, terrestrial, and aquatic altar, whence was to flow the conversion of the Jews, idolaters, etc. Accordingly some religionists, overexcited by meditation, dabbled with the matter, and the superior of the Jesuits even invited some able geometricians of that time, like Mydorge, Hardy, etc., to confer with Laleu. The result can easily be foreseen—it is impossible to reason with persons who have no principles in common with us. Hardy showed the incorrectness of the solutions in a manner satisfactory to all geometricians; for, as this Laleu gave several solutions of the same problem, Hardy made it appear that these did not even agree with each other. But Laleu, backed by Perjos, his book-keeper, and by a Scotchman named J. de Dunbar, did not give up the contest until after his death. We pass lightly on some other quadratures of the circle proposed by an anonymous writer of his time, and by one Benoit Scotto, whom St. Clair, royal professor, and Hardy refuted, to come to Longomontanus, who defiled, so to speak, the last years of his life, by his pretensions on the quadrature of the circle. That astronomer, formerly a disciple of Tycho Brahé, and known by a good work on astronomy, imagined, in 1622, that he had discovered the solution of this celebrated problem, which he published under the title of *Cyclometria lunulis reciproci demonstrata*, etc. He claimed to have found that the diameter is to the circumference as 1. is to 3.14185. In vain did Snellius, Henry Briggs, Guldin, warn him with moderation of his mistake, by showing him that the diameter being 1, the circumference is more than 3.14159, and less than 3.14160. But Longomontanus was not at all inclined to give in. He heaped a thousand bad reasons against the calculations of Vieta, Adrianus, Romanus, Ludolph, Vanceulen, Snellius, who were unanimously opposed to him. *Soon he* saw the quadrature of the circle in the mysterious properties of the numbers 7, 8, 9, and of the proportion *sesqui tertiam*, or of 3 to 4; he spent the last years of his life in publishing new vagaries: his different quadratures do not even agree together. The geometrician Pell attempted, about 1644, to set him right; he made him see, by a calculation, without the extraction of any root, that his relation would make the circumference greater than the circumscribed polygon of 236: the stubborn and irritable old man died in 1647, persuaded that he alone was right against all.

About the same time a new pretender to the honor of squaring the circle appeared in France, in the person of one S. Oudart, of Ogen, author of a work entitled, *Supplementum Supplementi Continens.* He gave a geometrical construction quite ingenious, and which would, in fact, give a line equal to the circumference, if three points which he supposes in a straight line were so in fact; besides, he did not deduce from it any numerical relation. His reasoning was about the same as that of Mallemant, of Messange, who, among many insignificant works, gave, in 1685, a *quadrature of the circle*, with a pretty rational history of the *problem*. He supposed that the three points were in a circular line, for which he had no warrant. They were not deemed worthy of a refutation; both might have been undeceived by making his construction only 2 feet in diameter.

The famous Hobbes appeared shortly after, about 1650, with his claims, not only to the quadrature of the circle, but also the rectification of the parabola, etc. His pretended solution having been refuted by Wallis, he took occasion to write against geometricians and geometry itself. Almost every year he wrote something new on this subject, and went always from paralogism to paralogism. One of his writings is entitled: *Rosetum Geometricum*, or the Geometrical Boquet. He abused geometricians a great deal, and Wallis in particular, and showed in several ways that his pretended discoveries were ridiculous.

Bertrand la Costa published in 1666, and again in 1677, a work entitled: *Demonstration of the Quadrature of the Circle.* But it was of no value and was treated with contempt by the French Academy of Sciences.

Three other semi-mystical visionaries presented to the public some *vagaries on the quadrature of the circle.* One John Bachon, of Lyons, announced in 1657 his discovery by a work entitled: *Demonstratio divini Theorematis Quadraturae Circuli, theologica, philisophica, Geometrica et Mecanicacum ratione quantitatum incommensurabilium.* The geometrical quadrature would have been enough; and people may judge of the author from his mixing together all these different methods.

An anonymous writer proclaimed in 1671 that the reign of the greatest king of the universe was to be rendered illustrious by this most brilliant discovery, and undertook to prove it by a pamphlet in 4to, entitled: Demonstration of the Divine Theorem of the Quadrature of the Circle, and the relation of this theorem with the Visions of Ezechiel and the Revelation of St. John. He does not fail, after the example of his colaborers, to ascribe his discovery to a special favor of the Divinity, according to this passage of the Scriptures: *Revelasti ea parvulis.* In fact, there is found at the end of this work a large mysterious board, presenting on a common center four decreasing pyramids of circles and angles, which represent the Angelic Hierarchy.

The third fool was named Dethlef Cluver, grandson or nephew of the celebrated geographer of that name. By ransacking the science of the infinite, on which he promised a great treatise, he finitely discovered that this problem, to find the quadrature of the circle, reduces itself to this one: to ,construct a world analogous to the divine intelligence: *Construere Mundum divinae Menti Analogum;* he promised to give geometrically and vigorously the solution of the first. Meanwhile, he (unsquared) *dequarrail* the parabola, and claimed that all that the geometricians had found on the curve figures was incorrect. (*See Acta* lipsiensia Julii, 1686, October, 1687). Leibnitz propounded doubtless to amuse himself some doubts on these vagaries. He wanted to pitch this Cluver against Nieuventit, who at the same time accumulated pitiable objections against the new calculations of the infinite that would have amused the geometricians. This trick did not succeed.

But among all the discoveries of the quadrature of the circle one is a kind of phenomenon, the only one who as yet admitted his mistake. It is Richard Albinus (White), an English Jesuit, author of a work entitled: *Chrysespis sen Quadratura Circuli*, in which he gives a false solution of the problem. But some friends opened his eyes, and he also acknowledged his error on the rectification of the spiral.

A better knowledge of geometry did not keep the 18th century from similar follies; there is not even a doubt that succeeding ages will resemble in this respect the past. In 1713, a Mr. G. A. Roerberg undertook to show that the circle is equal to the square of the side of an inscribed equilateral triangle; he did not perceive that it follows that the circumference would be exactly three times the diameter, or equal to the inscribed hexagon.

The solution of the three problems which have so long puzzled narrow intellects, the quadrature of the circle, perpetual motion, and the trisection of the angle was also announced in 1714 with much emphasis. The first discovery was that of one S. Daniel Wayvel, a Dutchman, and it was a palpable paralogism. From it followed that the diameter being 1. the circumference was, 3.142 exactly, which is altogether too much.

Usually the (*quadrateurs*) come off with seeing their discoveries neglected or scoffed at by their contemporaries; but in 1728, Mathulon, of Lyons, was more unfortunate. He announced to the learned world his signal discoveries of the quadrature of the circle and perpetual motion. He was so confident of his success that he appropriated 1,000 crowns for whoever would demonstrate to him that he was mistaken on either of these points. But Nicoli, then very young, demonstrated his error, and Mathulon admitted it; but he objected to the payment of this sum which Nicoli had abandoned to the Hôtel Dieu of Lyons. The matter was brought before the courts of that city, and the 1,000 crowns adjudicated to the poor.

In spite of this ill-success a new aspirant to the honor of squaring the circle was soon after seen. It was Basselin, professor of the University; his calculations were so complicated and so long that no one would have been willing to follow and verify them. But there is in such a case a means of detecting the mistake. *Barben du Bourg*, who later devoted himself to medicine, employed it by showing that the results of Basselin fell outside the known limits; for the rest, he was such a novice in geometry that he did not know that Archimedes had squared the parabola. Yet I have seen a beautiful Latin poem which celebrated the glory of Basselin and that of the college which his discovery had made illustrious.

The abbot Falconet, brother of the celebrated academician of that name, also published, about 1740, a little work in which he claimed to have discovered the quadrature of the circle. His process was less awkward than many others, but la Land, who was his friend, vainly endeavored, a few years later, to undeceive him.

Leistner, an officer in the service of the emperor, made more noise, and succeeded in having an imperial commission appointed to judge of the truth, real or assumed, of his discovery. He was, like many others, persuaded that in the series of numbers there are two which express the ratio of the diameter to the circumference, and that if the quadrature of the circle was not found it is because no one has been fortunate enough to put his finger on these numbers. In the first place that is a very false idea, *since it has been demonstrated that there is no two numbers which express exactly the ratio of the side of a square to the diagonal; and it is also demonstrated that there is none which express the ratio of the diameter to the circumference*, but Leistner thought he had found these highly favored numbers in the following: 1225 and 3844, for these numbers are two squares, and even prime to each other. They are derived from 35 and 31, which, according to Leistner, express the relation of the square to the circumscribed circle, and squaring them both and quadrupling the last they must express the relation of the diameter to the circumference. But Marinoni, reporter of the Commission, made it appear that this ratio of 1225 to 3844 is not even as accurate as that of 1 to $3\frac{1}{7}$, and that it gives a circumference which falls below the least limits of $3\frac{1}{7}$ and $3\frac{10}{71}$, the last of which is less than the polygon of 196 sides.

There is a quantity of other couples of numbers enjoying the properties deemed so wonderful by Leistner, and which gives a value nearer the circumference, as was shown by Lambert in his *Beytrage* or *Memoires de Mathematiques*, Vol. II, 1770, page 156. What Leistner thought he had found by a special favor of God can be found in a thousand ways by an analytical process. The Commission, on the report of Marinoni, rejected the discovery of Leistner, who, like his compeers, appealed from the judgment in a work entitled *Nodus Gordius*. It afforded Marinoni the occasion of publishing a work, where the process of determining the ratio of the diameter to the circumference is developed and expounded in a manner to convince any one but a man who thinks he has found the quadrature of the circle.

In 1751 a pastor or preacher of Kattembourg announced to the public this beautiful discovery, and soon after an inhabitant of Rostock enrolled himself as a volunteer for the same object. One of the two claimed that having entertained a Frenchman the latter had seen some of his papers and taken unfair advantage of the circumstance. Yet, about 1750, Henry Sullamar announced in England the quadrature of the circle, and found it in the number 666 of the Apocalypse ; he published periodically every two or three years some pamphlet in which he endeavored to prop his discovery. Paris soon enjoyed a similar spectacle. In 1753 an officer in the guards, Sir de Causans, who until then had never had any suspicion of geometry, suddenly found the quadrature of the circle in having a circular piece of sod cut, and then, rising from truth to truth, explain by his quadrature original sin and the Trinity. He pledged himself by a public writing to deposit at a notary to the amount of 300,000 francs to wager with those who might wish to appear against him, and actually deposited 10,000 francs for the benefit of any who might show him his mistake. That was certainly not difficult; for it followed from his discovery that the square circumscribed about the circle was equal to it and, consequently, the whole to its part. Some persons undertook to win the 10,000 francs, among others a young lady sued him; some others, in answer to his challenge, deposited different sums of money with the notaries. But the king adjudged that the fortune of a man, who was really innocent, ought not to suffer from such eccentricities of mind; for, on every other subject, Sir de Causans was a very estimable man. The suit was quashed and the bets declared void. Yet the knight found the means of obtaining the judgment of the academy which considerately refused to speak, but which was finally compelled to give its opinion.

We shall pass more rapidly over the names of other discoverers of the quadrature of the circle. Fondee, of Nangis, found it, not by measuring curves by straight lines, but straight lines by curves. Liger filled *les Mercurés* with similar follies on the quadrature of the circle.

He demonstrated it by the *Méccanisme en pleiu des figures*, which gave him, independently of the quadrature of the circle, the commensurability of the side of the square and its diagonal, by making out that 288 are equal to 289. Sir de Culant fell on the same discovery some years ago, and would also no doubt have discovered the quadrature of the circle had not death taken him away. La Frenaye, footman to the Duke of Orleans, spent twenty years in wandering from paralogism to paralogism, and in sifting the numbers 7, 8, and 9, which, according to him, contained the whole mystery of the quadrature. Clerget saw some contradiction in the relation more or less near of the diameter to the circumference given decimally, and had besides found the size of the point of contact of a sphere with a plain. Some years ago Maure used to weary all those that would listen to him by the recital of the injustice of geometricians and the Academie of Sciences. He intended to cross to England where he was sure to find more equitable judges in the Royal Society.

We must not forget one of the modern *Quadrateurs* who outdid many others in sanguineness and absurdity. It is Rohberger de Vausenville. The challenges he had made to the geometricians of all nations, even Turk and Arabian, as well as to all academies, the suit brought forward against the Academy of Sciences to secure for himself the capital of the prize established by Count de Meslay, his indecent attacks upon all geometricians who tried to enlighten or teach him, have made him famous among those who have followed this path.

His final theorem is that the square of the diameter is to that of the circumference, as 22 times the radius multiplied by the square root of 3 is to 432 times the radius. A more experienced geometrician would have said as eleven times the square root of three is to 216. That made the circumference of the circle whose radius is one equal to 3.36, which differs from the known proportion even in the 2d figure. It would follow from the pretended discovery of Vausenville that the circumference of a circle would exceed the circumscribed polygon of 12 sides.

Even at the present time, citizen Tardi, an old engineer, applies to the institute, the *Corps Legislatif* and all the world, to show his quadrature. He is having pamphlets printed, but is waiting for the proceeds of subscription. We have also just received a print with the title: Final Solution of the diameter of the Circle to its Circumference, or the discovery of the Quadrature of the Circle, by Christian Lowenstein, Architect, Cologne, 1801. His method consists in applying to a great quarter of a circle a strip of iron and he finds the circumference to be 3.1426.

These publications come to us more especially in the Spring of the year, when fits of folly are more frequent, and *cit. de la Land*, who spent a year at Berlin, says it was the season when the Academy of Berlin received most writings of that kind.

We, were, perhaps, wrong in dwelling so long upon these follies; we now pass to a more important article about this subject.

The impossibility of finding the quadrature of the circle was maintained by James Gregory, a Scotch geometrician, in a treatise entitled: *Vera Circuli et hyperbola quadratura*, Patav, 1664, in 4to, for he understood by the quadrature that which he obtained by approximation.

One is disposed to think this quadrature impossible to the human intellect when the useless efforts of geometricians of all times are considered; I do not speak of the pitiable efforts of those we have just been discussing, but of the efforts of such modern geometricians as St. Vincent, Wallis, Newton, Leibnitz, Bernoulli Euler, etc., who have found new methods of determining the area of curves, and who, by their reasoning, have found that of a quantity of other curves less complicated in appearance than the circle, whereas the latter has always eluded their efforts.

Besides, a distinction must be made in this respect: there are two quadratures of the circle, one definite, the other indefinite. The definite quadrature is the one that would give the precise measure of the entire circle or of a given sector or segment, without giving indefinitely that of any sector or segment whatever. The indefinite quadrature, which would be the most perfect by giving the quadrature of any part whatever, would evidently include the other. Scarcely any but the first is sought by *quadrateurs* in general.

The conviction is general that there is no demonstration absolutely convincing that the definite quadrature is impossible. Yet James Gregory claimed that he gave an irrefragable demonstration.

It rested upon the progressive course represented by the increase and decrease of the inscribed and circumscribed polygons whose limit is the circle itself. But this demonstration did not appear conclusive to Huygens, and it was the cause of a contest between these two geometricians which occupied the newspapers of the time. It must be admitted that though the reasoning is worthy of a head like that of Gregory, one of the forerunners of Newton, yet as the last limit of which he speaks is placed, so as to speak, in the mists of the infinite, the mind is not struck by an irresistible conviction. Still I would not put in the same category the assumed demonstration of this impossibility by Hanow. It is only a pitiable reasoning. An anonymous writer, some years ago, gave a little tract entitled: *Demonstration of the incommensurability*, etc. He claimed to have proved the impossibility of the quadrature of the circle. His calculations are exact, although more complicated than necessary; but it proves neither the incommensurability of the circumference and diameter, nor the impossibility of measuring the former; for a complication of incommensurable quantities does not prove demonstratively the incommensurability of the product or quotient. Two irrational quantities may, when multiplied together, give a rational quantity. The same is true of a larger number. A quantity may be composed of an infinite number of irrational quantities and represent only a rational quantity. But citizen Legendre, at the end of his Geometry, edition of 1800, page 320, demonstrates that the ratio of the circumference to the diameter and its square are irrational numbers, and that had been already demonstrated by Lambert, *Mein de Berlin*, 1761.

An irrational quantity is susceptible of a geometrical construction. Thus, supposing the circumference to be irrational or incommensurable with the diameter, it could, nevertheless, be determined geometrically, and this would undoubtedly be to find the quadrature of the circle.

As for the indefinite quadrature, Newton seems to have demonstrated that no enclosed (*fermée*) curve continually receding upon itself, as the circle, is capable of it. (Princ. phil. nat. math. lib. I.; Lem. XXVIII, p. 106.) This demonstration is connected with the theory of angular sections and of equations. I undertook, in 1754, to make it more plain and develop it more fully in my *Histoire des Recherches*, etc. I will have to refer to it and deem it convincing. Besides, although geometry presents numberless examples of squared curves, I know of none among the enclosed curves or curve continually receding (*retournant*) upon itself, that can be. Still D.'Alembert, in the fourth volume of his *Opuscules*, 1768, says, that he can scarcely assent to Newton's reasoning to prove the impossibility of the quadrature of the circle. I see, says he, that a similar course of reasoning, applied to the rectification of the cycloid, would lead to a false conclusion, the only difference, it seems to me, is that the circle is a receding curve and the cycloid is not. But I see nothing in Newton's reasoning which can be changed by disparity, more particularly, since the cycloid, if it is not a receding curve like the circle, is a continued curve whose sides (branches) are not separated; in a word, the reasoning of Newton rests solely on this supposition that in the circle an infinite number of areas corresponds to the same abscissa, whence he infers that the equation between the arc and the abscissa must be of an infinite degree, and consequently is not algebraically rectifiable; now, by applying his reasoning to the cycloid, I would infer that the equation between the abscissa and the corresponding arc must also be of an infinite degree, and therefore the arc is not rectifiable algebraically, which is false. D.'Alembert made the calculation and concluded by saying, it seems to me that these reflections might deserve the attention of the geometricians and induce them to look for a more vigorous demonstration of the impossibility of the quadrature and of the indefinite rectification of oval curves.

We shall now give a brief account of the principal discoveries on the quadrature of the circle, as most of them are included among the geometrical discoveries already discussed in the former volumes, I will only give them here without going into details Archimedes first discovered that the circumference is less than $3\frac{10}{70}$ or $3\frac{1}{7}$, and more than $3\frac{10}{71}$ times the diameter. Some of the ancients, as Appolonius and Philo, found nearer relations, but it is not known what they were.

About 1585 Peter Metius, in impugning the false quadrature of Duchêne, gave his near ratio of 113 to 355. It was shown above how near he was right. About the same time Viete and Adrianus Romanus also published relations expressed decimally which came much nearer to the truth. Viete carried the approximation to 10 decimal places instead of 6, and taught besides several somewhat simple constructions which gave the value of the circle, or the circumference to within a few millionths. Adrianus Romanus carried the approximation to 17 figures. But all that is far below what was done by Ludolph Van Ceulen, and which he published in his book *de Circulo et adscriptis*, of which Snellius published a Latin translation, at Leyden, in 1619. Ceulen, assisted by Petrus Cornelius, found with inconceivable labor a ratio of 32 decimals; see V. II, p. 6. Snellius found the means of shortening this calculation by some very ingenious theorems, and if he did not excel Van Ceulen he verified his result, which he put beyond attack. His discoveries on this subject are found in the book entitled *Willebrordi Snelli Cyclometricus de Circuli dimensione*, etc. Descartes also found a geometrical construction which, carried to infinity, would give the circular circumference, and from which he could easily deduce an expression in the form of a series. (See his *Opera posthuma.*)

Gregoire de Saint-Vincent is one of those who are most distinguished in this field; true, he claimed incorrectly to have found the quadrature of the circle and of the hyperbola, but the failure in this respect was preceded by so great a number of beautiful geometrical discoveries, deduced with much elegance according to the method of the ancients, that it would have been unjust to have placed him among the paralogists we have mentioned. He announced, in 1647, his discoveries in a book entitled: *Opus Geometricum quadraturae Circuli et Sectionem Coni* libris, X, *Comprehensum.* All the beautiful things contained in this book are admired; only the conclusion is impugned. Gregoire de Saint-Vincent lost himself in the maze of his proofs which he calls *proportionalities*, and which he introduces in his speculations. It was the subject of quite a lively quarrel between his disciples on the one hand, and his adversaries on the other, Huygens, Mersenne, and Leotand, from 1652 to 1664.

If that skillful geometrician had not been mistaken, it would only have followed from his investigations that the quadrature of the circle depends upon logarithms, and consequently on that of the hyperbola. That would still be a handsome discovery, but it did not even have that advantage. This furnished Huygens the occasion of divers investigations on this subject. He demonstrated several new and curious theorems on the quadrature of the circle: *Theoremata de quadratura hyper, ellipsis et Circuli*, 1651; *De Circuli Magnitude inventa*, 1654. He gave several methods of approaching his quadrature much shorter than the usual way. He demonstrated a theorem which Snellius had taken for granted. There are also many very simple geometrical constructions which give lines singularly near any given area. If, for example, the arc is 60° the error is scarcely $\frac{1}{6000}$th.

James Gregory distinguished himself in this controversy, and whatever may be our opinion of his demonstration of the impossibility of the definite quadrature of the circle, he can not be denied the authorship of many curious theorems on the relation of the circle to the inscribed and circumscribed polygons, and their relation to each other. By means of these theorems he gives with infinitely less trouble than by the usual calculations, and even those of Snellius the measure of the circle and of the hyperbola (and consequently the construction of the logarithms) to more than twenty decimal places. Following the example of Huygens, he also gave constructions of straight lines equal to arcs of the circle, and whose error is still less For example, let the chord of the arc of a circle be *a*, the sum of the two equal inscribed chords equal to *b*, then make this proportion: $A+B:B::B:C$; if you take the following quantity, $\frac{8c+8B-A}{15}$ it does not exceed the $\frac{1}{3500}$th for a semicircle, and for 120° it would be less than $\frac{1}{40000}$th; finally the error for a quarter of a circle will not be $\frac{1}{300000}$th.

The discoveries of Wallace, found in his *Arithmetica infinitum*, published in 1655, lead him to a singular expression of the relation of the circle to the square of its diameter; it is a fraction in this form, $\frac{3\times3\times5\times5\times7\times7\times9\times9\times11\times11}{2\times4\times4\times6\times6\times8\times8\times10\times10\times12}$, etc.

This fraction, carried to infinity, expresses exactly the above relation, *Arithmet. infinit.*, prop. 191; but if we confine ourselves, as is necessary, to a finite number of terms we have alternately a relation greater or less than the true one according as we take an odd or an even number of terms of the numerator and denominator. Thus $\frac{3}{2}$ gives too great a relation, and $\frac{3\times3}{2.4}$ give too small a relation. The fraction $\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}$ is too small, and $\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\cdot\frac{9}{8}$ too great. But to bring the one near to the other, Wallis directs to multiply this product by the square root of a fraction formed by the unity plus unity, divided by the last figure which ends the series. Then the product, although much nearer, will be too large if the figure is the last of the numerator, and too small if the last of the denominator. The values of $\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\sqrt{1+\frac{1}{5}}$; $\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}$; $\sqrt{1+\frac{1}{6}}$; $\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\sqrt{1+\frac{1}{7}}$; $\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}$ $\sqrt{1+\frac{1}{8}}$;

etc., alternately too small or too great, will fall within the known limits.

Here is another expression of the relation of the circle to the square of the diameter, found by Lord Brunker about the same time.

The circle being one, the square is expressed by the following fraction carried to infinity:

$$1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

It will be seen that this fraction is such that the denominator is an integer plus a fraction, whose denominator is 2 plus the square of one of the odd numbers 1, 3, 5, 7, etc.; when brought to an end the limits obtained are alternately in excess or too small.

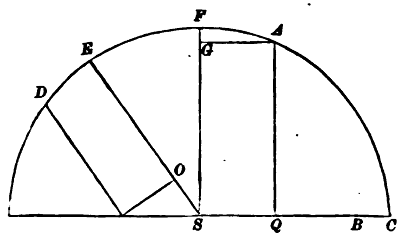
Such was the knowledge of geometricians on this famous problem when Newton and Leibnitz appeared on the arena. In 1682, Leibnitz gave out in his *Actes de Leipsig* what he had discovered as early as 1673, namely, that the square of the diameter being one, the area of the circle is expressed by the infinite series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}$, etc. It follows from his discovery about the same time that the radius of the circle being unity and the tangent of an arc t, this arc itself is $t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7$, etc. If then the arc is 45°, the tangent t is equal to the radius or one. Thus the arc of 45° is $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+$etc.; multiplying by 4 we shall have the semi-circumference, which multiplied by the radius will give the area of the circle equal $4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}$, etc.; the square of the diameter being 4. Thus the square of the diameter being made unity, the area of the circle will be $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}$, etc. to infinity.

The area can also be expressed by $\frac{2}{3}+\frac{2}{35}+\frac{2}{99}+\frac{2}{195}$, etc., viz.: by adding together the two first terms, and the next two by two, or else in this way, $1-\frac{2}{15}-\frac{2}{63}-\frac{2}{143}$, etc., where it is easy to see that the denominators are successively in the first the squares of 2, 6, 10, etc., diminished by unity, and in the second the squares of 4, 8, 12, etc., similarly reduced. But it must be conceded that these different series do not converge rapidly enough to derive from them a value sufficiently accurate without the addition of a prodigious number of terms; but Euler found a remedy.

The discoveries made by Newton, even before Leibnitz, had also placed him in possession of various methods of expressing the circumference and the area of the circle, as also of segments by infinite series.

Nothing is better known to-day by all those who have any knowledge, even elementary, of the new calculations; but among these series, those that have been used most successively for that purpose are the following:

FIGURE 1.



Let C, A, F, be a quarter of a circle, whose radius A Q is unity and Q B an abscissa, $= x$, we find the area of the segment Q, B, A, F, represented by this series, $x - \frac{1}{2 \cdot 3} x^3 - \frac{1}{2 \cdot 3 \cdot 4} x^5 - \frac{1 \cdot 3}{1 \cdot 4 \cdot 6 \cdot 7} x^7$, etc.

Now, supposing B Q or x equal to $\frac{1}{2}$, this series is reduced to the following: $\frac{1}{2} - \frac{1}{48} - \frac{1}{1280} - \frac{1}{000000} - \frac{1}{000000000}$, etc. Calculating therefore in decimal fractions each of these terms, and taking the sum of the negatives from the first, which is positive, we obtain the value of the segment Q, B, A, F, from which it will be necessary to take the value of the triangle A, B, Q, calculated in decimals, and there will remain the sector F, A, Q, which is the third of the quarter of the circle, or the 12th part of the entire circle. Thus, multiplying this value by 12, we shall obtain that of the circle for a diameter equal to 2; and finally dividing this last one by 4, we shall have that of the circle to the diameter. The first ten terms above given, calculating in this manner, give the near ratio of the diameter to the circumference of 1 to 3.141.

The tangent of a small arc as that of 30° might also be employed for the same purpose; for this tangent is $\sqrt{\frac{1}{3}}$, or $\frac{1}{\sqrt{3}}$; besides, if the radius, unity, and the tangent t, the arc is, as already shown, $t - \frac{1}{3} t^3 + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \frac{t^{11}}{11}$. Thus t being made $= \frac{1}{\sqrt{3}}$, this series is reduced to $\frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 9\sqrt{3}} - \frac{1}{7 \cdot 27\sqrt{3}} + \frac{1}{9 \cdot 81\sqrt{3}} - \frac{1}{11 \cdot 243\sqrt{3}} + \frac{1}{13 \cdot 729\sqrt{3}} - \frac{1}{15 \cdot 2187\sqrt{3}} + \frac{1}{17 \cdot 6561\sqrt{3}} - \frac{1}{19 \cdot 19683\sqrt{3}}$, etc., whose progression can easily be seen.

This series, by transferring the radical to the numerator, becomes $\sqrt{\frac{1}{3}} - \frac{\sqrt{\frac{1}{3}}}{3.3} + \frac{\sqrt{\frac{1}{3}}}{5.9} - \frac{\sqrt{\frac{1}{3}}}{7.27}$, etc., which can be expressed thus: $\sqrt{\frac{1}{3}}\left(1 - \frac{1}{3.3} + \frac{1}{5.9} - \frac{1}{7.27}\right)$; thus $\sqrt{\frac{1}{3}}$ must be taken in as many decimals as are to be used in the approximation or a little more, to be more certain of the last figures; then this value must be divided successively by 3.3 or 9 by 5.9 or 45, by 7.27 or 189, by 9.81 or 7.29, etc.; this will give the approximate value of each term in as many decimals as there are in the value of $\sqrt{\frac{1}{3}}$. Add all the positive terms together, and from the sum subtract that of the negatives. This will give very nearly the arc of 30°, which being multiplied by 12, will be the value of the circumference with the diameter 2, consequently the half will be that of the circumference with the diameter 1.

It is by this means and others similar that Samuel Sharp extended to 75 decimals the approximate ratio of Ludolph, which had only 35. Machin, towards the beginning of the present century, carried it as far as 100. Lagny, in 1719, carried it to 128, and another to 155. Thus, the diameter of the circle being 1, followed by 128 zeros, the circumference is according to Lagny, greater than 3.14159, 26535, 89793, 23846, 26433, 83279, 50 | 288, 41971, 69399, 37510, 58209, 74944, 59230, 78164, 06286, 20899, 86280, 34825, 34211, 70679, 82-148, 08651, 32723, 06647, 09384, 46, and less than the same number increased by unity (added to the last figure). I have separated by a dash the 32 decimals of Ceulen. The error for a circle with a diameter 100 millions times greater than that of the sphere of the fixed stars. Supposing the parallax of the terrestrial orb to be only one second, would still be several billions of billions times less than the breadth of a hair. The 114th figure of the seven which is underlined, ought to be 8; Mr. Vega ascertained this as appears from his large tables of Logarithms, page 633, where he gives the values of the series. Baron de Zach saw, in a manuscript of the library of Ratcliffe at Oxford, the calculation carried still further, and as far as 155 figures; after 446 add 0955058, 22317, 25359, 40812, 848[illegible].

The expedient found by Euler for using the series which the arc by the tangent gives, will bring us nearer the truth and with less trouble. The expedient deserves to be inserted here.

It consists in the remark made by that great geometrician that every arc is rational or commensurable with the radius (the arc of 45° for example, whose tangent is 1), can be divided into 2 arcs whose tangents much less, shall also be commensurable to it. It is a consequence of the theorem which gives the tangent of the sum and difference of two arcs whose tangents are given; for there being no extraction of square roots in this formula, if the arcs have their tangents rational, the tangent of the sum will be so also; and *vice versa*, an arc with a rational tangent can be divided into two arcs, whose tangents much smaller will be rational. Thus the arc 45° can be divided into two (incommensurable really to each other), the tangent of one of which will be $\frac{1}{2}$, that of the other $\frac{1}{3}$. We shall therefore find by the series of the arc by the tangent each of these arcs, and their sum (though irrational to each other and to the radius) will, nevertheless, be to the arc of 45°, which, multiplied by 4, will give the ratio of the relation of the semi-circumference to the radius, or of the circumference to the diameter; for the first of these series will be $\frac{1}{2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\frac{1}{7.2^7}+\frac{1}{9.2^9}-\frac{1}{11.2^{11}}$, etc., or $\frac{1}{2}-\frac{1}{24}+\frac{1}{160}-\frac{1}{896}+\frac{1}{4608}-\frac{1}{22528}$, etc.; and the second will be $\frac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\frac{1}{7.3^7}$, etc., that is to say, $\frac{1}{3}-\frac{1}{81}+\frac{1}{1225}-\frac{1}{15309}+\frac{1}{177147}-\frac{1}{17537553}+\frac{1}{165526791}$, etc.

Now, in both these series, the terms diminish rapidly enough to attain very nearly their real values; for, in the second, by taking only seven terms the error would already be less than $\frac{1}{186,000,000}$th.

Euler shows that it is possible to obtain this result even more rapidly; for he remarks that the arc whose tangent is $\frac{1}{2}$ can be divided into two whose tangents will be $\frac{1}{3}$ and $\frac{1}{7}$, which gives the arc of 45° equal to twice the second of the above series, plus this one $\frac{1}{7}-\frac{1}{3.7^3}+\frac{1}{5{\cdot}7^5}$, etc., or $\frac{1}{7}-\frac{1}{1129}+\frac{1}{89035}$.

Finally, we shall observe that the arc whose tangent is $\frac{1}{3}$ can be divided into two whose tangents shall be $\frac{1}{5}$ and $\frac{1}{8}$, which affords the means of obtaining two series still more converging than the one which gives the arc answering to the tangent $\frac{1}{3}$. It would be easier to find by these 3 or 4 series the circumference of a circle to 200 decimals than it was for Viete or Romanus to calculate by their methods to 10 or 15 decimals. We pass in silence over the other artifices of calculations presented by Euler in the same treatise, and in another, Vol.

XI, for 1739, by which it would require only 80 hours of work to find 128 figures of Lagny; there are also some in Stirling, *Summatione Serierum*, in Simpson; *The Doctrine of Fluxions*, 1750. Kraft, in the 13th volume of Petersbourg for 1741, page 121, gave some mechanical constructions very simple and very near.

All these methods which, undoubtedly, mutually confirm each other, leave no doubt as to the numerical expression of the approximate ratio of the diameter to its circumference; this ratio is the true touchstone to try the pretended quadratures of the circle without entering into the maze of the pitiable and often obscure reasoning of those who claim to be the happy explainers of this enigma. Nothing better can be done than by leaving them to their pleasing allusion; for experience has shown attempts to open their eyes would be fruitless; that is what induced the Academy of Sciences to give notice that it would examine no more quadratures of the circle any more than trisections of the angle, or duplications of the cube, or perpetual motions. *Histoire de l'Academie*, 1775, page 64. This is the way in which the Secretary of the Academy, Condorset, who was himself a very great geometrician expresses himself.

This solution may be considered in two lights. In fact, we may look for the quadrature of the entire circle or the quadrature of any of its sectors whose chord is assumed as known. The last of these problems is considered as having no solution. Gregory and Newton, whose authority is so great, even in a science where authority goes for so little, have given different demonstrations of the impossibility of the indefinite equation. John Bernoulli has proved that the required sector was expressed by a real logarithmetical function, but which in form contains imaginary quantities. It follows that no real functions, either algebraical or logarithmetical and real in form, can represent the value of the sector of an indefinite circle; that the equation between the sector and the chord can not be constructed by the intersection of the lines of real or curve surfaces, and we may infer from this consideration the absolute impossibility of the indefinite quadrature.

Geometricians are not as well agreed as to the impossibility of the first problem, for it often happens that for particular values quantities are found whose expression is in general impossible; but an experience of more than 70 years taught the Academy that none of those who used to send in solutions of these problems understood either their nature or their difficulties, that none of the methods which they used would have led them to the solution. This long experience sufficed to satisfy the Academy of the little utility that would accrue to the sciences by examining all these pretended solutions. Other considerations have also decided the Academy. There is a popular report that governments have promised considerable rewards to the person who might succeed in solving the problem of the quadrature of the circle, and that this problem is the subject of the investigations of the most celebrated geometricians. On the strength of these reasons a crowd of men, much more considerable than is supposed, have given up useful occupations to devote themselves to the discovery of this problem, often without understanding it, and always without the information necessary to try its solution with success. Nothing was better calculated to undeceive them than the declaration the Academy thought proper to make. Many had the misfortune to believe they had succeeded, and would not yield to the reasoning with which geometricians attacked their solutions. Often they could not comprehend them, and ended by accusing them of envy or bad faith. Sometimes their obstinacy degenerated into madness; but it is not considered as such if the opinion which forms this folly does not clash with the received ideas of men, if it has not influence upon the conduct of life, and if it does not disturb the order of society. The madness of the *quadrateurs* would therefore have no other inconvenience for them but loss of time often useful to their families; but unfortunately madness is seldom confined to a single object, and the habit of reasoning falsely is acquired and extends like that of reasoning accurately. That happened more than once to the quadrateurs. Besides, unable not to realize how singular it would be if they should discover, without study, truths which the most celebrated men have sought in vain, they almost all become persuaded that it is by a special protection of Providence that they have succeeded, and there is only one step from this idea and believing all the strange combinations of ideas which occur to them are so many inspirations. Humanity, therefore, required that the Academy, persuaded of the uselessness of the examination it might have made of these solutions of the quadrature of the circle, should seek to dispel, by a public declaration, popular opinions which have been fatal to many families. Considerations so wise could not excite the animadversion of a writer like Linguet in his Political Annals. He had also found that it is not true that the pictures of exterior objects are taken inverted (upside down) in the retina, and the tide of the Amazon does not ascend up to Pauxis, where Condamine noticed it. Nothing surprises me more than to see persons of understanding have so little good sense as to persist in things they do not understand, with as much assurance and warmth as if they had occupied their whole lives in studying them, and had acquired a real superiority in the same; but mankind is subject to these inconsistencies.

PRACTICAL REMARKS AND EXAMPLES ILLUSTRATING THE QUADRATURE OF THE CIRCLE.

THE Author, who proposes for investigation a new theory upon any subject in the present advanced state of science, when nearly every truth concerning the same is believed to have been discovered, fully demonstrated and applied, must be happy if he succeeds in establishing the truth of what he advances; for, if he does not have to disprove all former theories he of necessity calls them in question and creates a doubt of their correctness before he can hope to succeed in establishing his own. This is especially so with regard to the science of mathematics; for, when a fact has been once established by actual mathematical demonstration which is universally admitted to be true, it is adhered to as firmly as one's own faith, and it is next to impossible to doubt its correctness. Any discovery, therefore, made after such fact has been so demonstrated and received, which goes to prove even a slight error in the result obtained is received with great caution; but if it should seriously call in question the truth of former theories or directly contradict a result already established, it must be brilliant indeed to secure the attention and merit the consideration of the learned. It would also have to be profitable, for in this age the money tree is pruned before all others; even imperfect theories are often preferred after they have been proved erroneous, because they are better understood, and there are so few who can bear the mental exertion necessary to examine newer and truer ones.

There are two classes of propositions in mathematics which deserve to be considered with attention, the first of which needs only to be referred to here, as the truths upon which they are based have been put beyond any question of doubt; such for example as

THEOREM 1. In any right-angled triangle *the square described on the side subtending the right angle is equal to the sum of the squares described on the sides which contain the right angle.*

THEOREM 2. *If the one-fiftieth of the sum of the squares of the two sides of any square be deducted therefrom, the square root of the remaining $\frac{49}{50}$ can be extracted exactly.*

THEOREM 3. *In any right-angle triangle if a perpendicular be drawn from the right angle to the base the triangles on each side are similar to the whole triangle and to one another, and the perpendicular is a mean proportional between the segments of the base, and each of the sides of the triangle is a mean proportional between the whole base and the segment adjacent to that side.*

THEOREM 4. *If three straight lines are proportionals the rectangle contained by the extremes is equal to the square on the mean, and if the rectangle contained by the extremes is equal to the square on the mean the three straight lines are proportionals.*

THEOREM 5. *Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed polygons of half the number of sides.*

THEOREM 6. *If two straight lines cut one another the opposite or vertrical angles are equal.*

THEOREM 7. *Triangles which have the same altitude are to each other as their bases, and their areas are to each other as the squares described on those bases.*

Or, by trigonometry,

The cosine is to the sine as the radius is to the tangent, consequently the secant is to the tangent as the radius is to the sine, therefore the square of the secant is to the square of the tangent as the square of the radius is to the square of the sine.

The second class embraces those problems which either have not been fully demonstrated, or have been given up as impossible. Respecting these Mr. Todhunter says:

" *There are three famous problems which are now admitted to be beyond the power of geometry, namely:* To find a straight line equal in length to the circumference of a given circle, to trisect any given angle, and to find two mean proportionals between two given straight lines. The grounds upon which the geometrical solution of these problems is admitted to be impossible can not be explained without a knowledge of the higher parts of mathematics; the student of the elements may however be content with the fact that innumerable attempts have been made to obtain solutions, and that these attempts have been made in vain.

The first of these problems is usually referred to as the quadrature of the circle. For the history of it the student should consult the article in the *English Cyclopedia* under that head, and also a series of papers in the *Atheneum* for 1863 and subsequent years, entitled *A Budget of Paradoxes*, by Professor DeMorgan. For the approximate solutions of the problem we may refer to Davies' edition of Hutton's *Course of Mathematics*, Art. I, page 400; the *Lady's* and *Gentleman's Diary* for 1855, page 86, and the *Philosophical Magazine* for April, 1862. The third of the three problems is often referred to as the *duplication of the cube.* See note on VI, 13, in *Lardner's Euclid* and a dissertation by C. H. Biering, entitled *Historia Problematis Cubi Duplicandi—Hauniæ*, 1844.

Under the head of "Geometrical Analysis" Mr. Todhunter says:

"Geometrical analysis has sometimes been described in language which might lead to the expectation that directions could be given which would enable a student to proceed to the demonstration of any proposed theorem, or the solution of any proposed problem with confidence of success; *but no such directions can be given. We will state the exact extent of these directions:* Suppose that a new theorem is proposed for investigation, or a new problem for trial, *assume the truth of the theorem, or the solution of the problem,* and deduce consequences from this assumption combined with results which have already been established. If a consequence can be deduced which contradicts some result already established, this amounts to a demonstration that our assumption is inadmissible; that is the theorem is not true, or the problem can not be solved. If a consequence can be deduced which coincides with some result already established, we can not say that the assumption is not inadmissible; and it *may happen* that by starting from the consequence which we deduced, and retracing our steps, we can succeed in giving a synthetical demonstration of the theorem or a solution of the problem. These directions however are very vague, because *no certain rule can be prescribed by which we are to combine our assumption with results already established; and, moreover, no test exists by which we can ascertain whether a valid consequence which we have drawn from an assumption will enable us to establish the assumption itself.* That a proposition may be false and yet furnish consequences which are true, can be seen from a simple example. Suppose a theorem were proposed for investigation in the following words: *one angle of a triangle is to another as the side opposite to the first angle is to the side opposite to the other.* If this be assumed to be true we can immediately deduce Euclid's result in I, 19; but from Euclid's result in I, 19, we can not retrace our steps and establish the proposed theorem, and in fact the proposed theorem is false.

Thus the only definite statement in the directions respecting geometrical analysis is, that if a consequence can be deduced from an assumed proposition which contradicts a result already established, that assumed proposition must be false. We may mention, in particular, that a consequence would contradict results already established if we could show that it would lead to the solution of a problem already given up as impossible."

A brief review of the above quotations will be attempted, which, for the sake of convenience, I shall classify as follows:

1st. "No directions can be given in language which would enable a student to proceed to the demonstration of any proposed theorem or the solution of any proposed problem with confidence of success."

The above quotation will apply with more force to the demonstration of a *new* theorem or the solution of a *new* problem; for it would be necessary to know something of the nature of a theorem and the construction of a problem before directions could be given for the demonstration of the one or the solution of the other; but if a theorem is proposed for demonstration, or a problem for solution, *which have certain parts in common with similar theorems or problems before known and demonstrated*, the demonstration of such theorems and the solution of such problems would be rendered easy in the same proportion as they contained those parts in common. The mathematician, therefore, who should venture to give explicit directions for the demonstration of any theorem or the solution of any problem whatever—such for instance as the quadrature of the circle, the trisection of the angle, or the duplication of the cube—would certainly pretend to a superior knowledge concerning those things in which the ablest and wisest mathematicians have failed; for the discovery of a new method for the demonstration of the above theorems or the solution of the above problems might involve new laws which no one knows anything of except the author who makes the discovery.

2d. "Assume the truth of the problem, and deduce consequences from this assumption combined with results already established.* If a consequence can be deduced which contradicts some result already established, this amounts to a demonstration that our assumption is inadmissible; that is, the theorem is not true or the problem can not be solved."

NOTE.—*A proposition consists of various parts; we have first the general enunciation of the problem or theorem, as for example: *To describe an equilateral triangle on a given finite straight line, or any two angles of a triangle are together less than two right angles.* After the general enunciation follows the discussion of the proposition. First, the enunciation is repeated and applied to the particular figure which is to be considered, as for example: *Let* A B *be the given straight line; it is required to describe an equilateral triangle on* A B. The construction then usually follows straight lines and circles, which must be drawn in order to constitute the *solution* of the problem, or to furnish assistance in the *demonstration* of the theorem. Lastly, we have the demonstration itself, which shows that the problem has been solved or that the theorem is true.

Sometimes however, no construction is required, and sometimes the construction and demonstration are combined. The demonstration is a process of reasoning, in which we draw infer-

With reference to point 2nd, it must be confessed that so far as those problems are concerned, for which solutions have been already obtained and which are universally received as final, *it may be true* that an assumed proposition would be false which would contradict a result so well established and so universally admitted to be correct; but it does not follow as a necessary consequence that an assumed proposition would be false which would give a result different from the one already established with respect to those problems which *are admitted to be impossible*—such, for example, as the quadrature of the circle, etc.; for as long as the final solution of this problem is not obtained it is difficult to determine which solution is true and which false as long as they are confined within certain known limits. Thus it is mathematically certain that the ratio of the circumference to the diameter is either $3\frac{1}{7}$ *exactly* or very near it, and any ratio which claims to be either much above or below it *can not be true.* For example, suppose a circle is described with a radius equal to the square root of two, and cosine of the given arc is $\frac{7}{5}$, which is not quite though very near the true radius, and the sine of the same arc is $\frac{1}{5}$, which is not quite though very near the $\frac{1}{44}$ of the true circumference, now, the double of the cosine $\frac{7}{5}=\frac{14}{5}$, is not quite the true diameter though very near it, and the double of the sine $\frac{1}{5}=\frac{2}{5}$, is not quite the $\frac{1}{22}$ part of the true circumference though very near it, being the chord of double the arc, whose sine is $\frac{1}{5}$. If then we assume $\frac{14}{5}$ to be the true diameter, and $\frac{44}{5}$ to be the true circumference, and they are very near it, then dividing the circumference $\frac{44}{5}$ by the diameter $\frac{14}{5}$, we have by cancellation $\frac{44}{5} \div\frac{14}{5}=\frac{44}{5}\times\frac{5}{14}=\frac{22}{7}$, which is equal to $3.\dot{1}4285\dot{7}$, $\dot{1}4285\dot{7}$, or $3\frac{1}{7}$ to infinity, so that if $\frac{44}{5}$ *were* the true circumference and $\frac{14}{5}$ the true diameter, the true ratio of the circumference to the diameter, would be $3.\dot{1}4285\dot{7}$ or $3\frac{1}{7}$.

3rd. "These directions are very vague because no certain rule can be prescribed by which we are to combine our assumption with results already estab-

ences from results already obtained. (These results consist partly of truths established in former propositions, or are admitted as obvious in commencing the subject; *and partly of truths which follow from the construction that has been made, or which are given in the supposition of the proposition itself.* The word *hypothesis* is used in the same sense as *supposition*.)

lished, and no test exists by which we can ascertain whether a valid consequence, which we have drawn from an assumption, will enable us to establish the assumption itself."

A theorem is a truth which becomes evident by a train of reasoning called a demonstration; *the demonstration* proceeds from the premises by a regular deduction; *a deduction* is the logical consequences which follow an assumption; *an assumption* is based upon certain truths which are found in the construction of a figure, or which result from a combination of other truths. Every *combination* of truths and every construction of a figure necessarily contain within themselves the logical reasons for their demonstration; and if a supposition is made in conformity with the truths developed in the construction of a figure, or an assumption be made which is based upon truths which are the necessary consequence of a combination of other truths, this assumption will lead to a valid consequence which may result in the demonstration of the truth involved. But if, from a misconception of those truths, a false assumption is made, a valid consequence can not be drawn and the assumption will necessarily lead to an absurdity. It is also next to impossible to prove the truth of one method by another, or of combining our results with those already established; it is only when the two methods both contain certain principles in common that they can be compared; that is when they contain elements that may be reduced to the same denomination, to the same dimensions, the same weight, or the same measure.

In reference to point 4th, it is admitted that so far as those problems are concerned which have received a final solution that is universally admitted to be correct, it may be true that an assumed proposition would be false if it should contradict a result already established; but with respect to those problems which are admitted to be impossible it does not follow that an assumed proposition would be false which contradicts a result already established; for it is impossible to know which is true, or which is false, until a final solution is obtained. Consequently point 5th may very readily be acquiesced in, for the author says:

"We may mention in particular, that a consequence would contradict a result already established if we could show that it would lead to the solution of a problem already given up as impossible."

It is asserted by apologists and advocates of the present quadrature of the circle, that it is an established fact and is therefore incontrovertible; they contend that the commonly received solution is the nearest approximation that the ablest and wisest mathematicians of every age and country with their vast learning and great genius have ever been able to make towards the truth, and then we are told how many decimal places each particular author has been able to carry it to, and the greatest mathematician is the one who has carried the result to the greatest number of decimal places, and the only reason why the final solution of the problem has not been obtained is because no one has ever been able to extract the square roots to infinity. It is only necessary to return to Euclid to find that he labored very much under the same difficulty, and it looks very much like the fact that two thousand years' experience which has witnessed improvements in nearly every other branch of mathematical science has not made much progress respecting this problem, for the same ancient method is used that occupied Appolonious while in prison.

To establish a fact does not of necessity prove it to be a fact; though it may be urged in reply that a fact must be proved before it can be established. If a majority of the wisest and most intelligent of mankind were convinced of a fact which has been thoroughly tested and commonly received, it is said to be an established fact. And a fact, when we speak mathematically, may more properly be styled a truth. The establishing of a mathematical truth is usually done in the same manner as establishing an ordinary fact; two or three credible witnesses are sufficient to establish a fact in law, but a mathematical fact is first investigated by a rigid course of mathematical analysis, and after it has been thoroughly tested by the ablest mathematicians, and the result of all these various calculations agree, it is said to be an established mathematical truth. But to make such a truth binding if it depend upon the solution of a problem or the demonstration of a theorem, the truth of the theorem or the problem can not be said to be established unless there can be found a final demonstration of the theorem or a final solution of the problem which shall be mathematically demonstrated beyond a reasonable doubt; and if a problem which has been established has not been finally solved, any other solution of the same problem, which shall be a complete and final solution of the given problem, may be said to disestablish an established truth, or to successfully contradict a result already established and seriously call in question its correctness.

Numerous instances can be cited where facts (truths) have been established and believed by the great majority of the wisest and ablest of mankind which were afterwards proved to be erroneous, and in some cases they have been shown to be supremely ridiculous.

For example:

"The Greeks, and through them the Romans and other nations, believed the earth to be flat and circular, their own country occupying the middle of it, the central point being Mount Olympus, the abode of the gods or Delphi, so famous for its oracle. The circular disc of the earth was crossed from west to east, and divided into two parts by the *sea* as they called the Mediterranean, and its continuation the Euxina, the only seas with which they were acquainted. Around the earth flowed the *River Ocean*, its course being from south to north on the western side of the earth, and in a contrary direction on the eastern side. It flowed in a steady, equable current unvexed by storm or tempest. The sea and all the rivers on the earth received their waters from it.

The northern portion of the earth was supposed to be inhabited by a happy race named the Hyperboreans, dwelling in everlasting bliss and spring beyond the lofty mountains whose caverns were supposed to send forth the north wind which chilled the people of Hellas (Greece). Their country was inaccessible by land or sea. They lived exempt from disease or old age; from toils and warfare. Moore has given us the 'Song of the Hyperborean,' beginning:

> 'I come from a land in the sun-bright deep,
>   Where the golden gardens glow;
> Where the winds of the north becalmed in sleep
>   Their conch shells never blow.'

On the south side of the earth, close to the stream of ocean, dwelt a people happy and virtuous as the Hyperboreans. They were named the Æthiopians. The gods favored them so highly that they were wont to leave at times their Olympian abodes and go share their sacrifices and banquets.

On the western margin of the earth, by the stream of ocean, lay a happy place named the 'Elysian Plain,' whither mortals favored by the gods were transported, without tasting of death, to enjoy an immortality of bliss. This happy region was also called the 'Fortunate Fields' and the 'Isles of the Blessed.' We thus see that the Greeks of the early ages knew little of any real people except those to the east and south of their own country, or near the coast of the Mediterranean. Their imagination meantime peopled the western portion of this sea with giants, monsters, and enchantresses, while they placed around the disc of the earth, which they probably regarded as of no great width, nations enjoying the peculiar favor of the gods and blessed with happiness and longevity. The dawn, the sun, and the moon were supposed to rise out of the ocean on the eastern side, and to drive through the air, giving light to gods and men. The stars also, except those forming the Wain or Bear and others near them, rose out of and sank into the stream of ocean. There the Sun-god embarked in a winged-boat, which conveyed him round by the northern part of the earth back to his place of rising in the east. Milton alludes to this in his Comus:

'Now the gilded car of day
His golden axle doth allay
In the steep Atlantic stream,
And the slope sun his upward beam
Shoots against the dusky pole,
Racing toward the other goal,
Of his chamber in the east.'

' The abode of the gods was on the summit of Mount Olympus, in Thessaly. A gate of clouds kept by the goddess named the Seasons, opened to permit the passage of the celestials to earth and to receive them on their return. The gods had their separate dwellings; but all, when summoned, repaired to the palace of Jupiter, as did also those deities, whose usual abode was the earth, the waters, or the underworld.

It was also in the great hall of the palace of the Olympian King that the gods feasted each day on ambrosia and nectar, their food and drink; the latter being handed round by the lovely goddess Hebe. Here they conversed of the affairs of heaven and earth; and as they quaffed their nectar Appollo, the God of Music, delighted them with the tones of his lyre, to which the Muses sang in responsive strains. When the sun was set the gods retired to sleep in their respective dwellings. The following lines from the Odyssey will show how Homer conceived of Olympus:

'So saying Minerva, goddess azure-eyed,
Rose to Olympus, the reputed seat,
Eternal of the gods, which never storms
Disturb, rains drench, or snow invades; but calm
The expanse, and cloudless shines with purest day;
There the inhabitants, divine, rejoice
Forever.'"

The above fable though very beautiful, was one of those silly notions which the wisest of all the ancients believed as an established fact, and from it may be seen how few, even among those who profess to be wise, really think for themselves—and how the vast majority of mankind allow others to do their thinking for them; furthermore by far the greater number of mathematical truths which are now valued so highly, and referred to with so much confidence, were discovered and first demonstrated by these very Greeks and their contemporaries; and the most plausible method known up to this present time for the solution of the quadrature of the circle was discovered, it is supposed, during the highest cultivation of the arts and sciences, by these people, whose genius for invention probably exceeded that of any nation which followed them till we come to the American. Mr. Pope, in eulogizing the genius of Homer, says:

"Homer is universally allowed to have had the greatest invention of any writer whatever. The praise of judgment Virgil has justly contested with him, and others may have their pretensions as to particular excellencies, but his invention remains yet unrivalled. Nor is it a wonder if he has ever been acknowledged the greatest of poets who most excelled in that which is the very foundation of poetry.

It is the invention that in different ages distinguishes all great geniuses. The utmost stretch of human study, learning, and industry, which masters everything besides, can never attain to this. It furnishes art with all her materials, and without it judgment itself can but steal wisely; for art is only like a prudent steward that lives on managing the riches of nature. Whatever praises may be given to works of judgment, there is not even a single beauty in them to which the invention must not contribute; as in the most regular gardens art can only reduce the beauties of nature to more regularity, and such a figure which the common eye may better take in, and is therefore more entertained with. And the reason why common critics are inclined to prefer a judicious and methodical genius to a great and fruitful one, is because they find it easier for themselves to pursue their observations through an uniform and bounded walk of art, than to comprehend the vast extent of nature."

When Copernicus maintained, in opposition to the theory of the ancients, that the earth moved on its own axis, and that it moved around the sun instead of the sun moving around the earth, history teaches us with what difficulty he maintained his position, though in the end he succeeded in establishing his theory in opposition to all.

When Columbus, following in the footsteps of Copernicus, maintained that the earth being round another continent was necessary on the opposite side of the earth to maintain its equilibrium, he was treated as a visionary and madman, until through the charity of "Her Most Catholic Majesty," "Isabella, then Queen of Castile and Arragon," he was provided with means to prosecute to a successful issue his discoveries; and in return for the ingratitude of nations he gave them a "New World."

In reference to those cases where it is necessary to give instructions for the demonstration of a theorem or the solution of a problem, or of combining our assumption with results already established, it is to be observed, supposing the problem to be the solution of the *quadrature of the circle*,

1st. That the cosine must always intercept the sine at right angles, and the radius must always intercept the tangent at right angles.

2d. The tangent is always greater than the arc to which it belongs, and the sine is always less than the arc to which it belongs, for the tangent lies wholly without the circle while the sine lies wholly within the circle, and the arc to which they belong *must* lie between them; therefore, as far as the sine and the tangent agree with one another, that is as far as they can be expressed by the same quantity, either one of them may be taken for the arc of the circle to which they belong.

3d. The inscribed polygon must not at any time extend outside the given circle nor the circumscribed polygon come within it; and any method which may be adopted for the solution of the quadrature of the circle by the means of regular inscribed and circumscribed polygons, which are made to approach the circle (one from within and the other from without) by doubling the number of sides, will not give a correct solution as long as there is danger of forcing a limit between the two polygons; neither does it follow as a necessary consequence that the circle is the limit of the two polygons.

4th. If by any means the inscribed polygon could be made to approach the circle *from within*, without regard to the circumscribed polygon, by doubling the number of sides, and this result could be reduced to a commensurable quantity, it is very evident that such a method would give the true quadrature as far as it could be carried; for in that case the inscribed polygon could never be forced to extend beyond or become greater than the circle, and if it could be continued till it would reach the circle, that is to infinity, *which is impossible*, it would give the true quadrature.

5th. A common measure of two straight lines, as for example the sine and tangent, can not be regarded as a measure of the circle, as no part of it, however small, is straight.

6th. Neither is it necessary for the solution of the quadrature of the circle that the polygon should be regular, for it was demonstrated by Gauss, a mathematician of Göttingen, in his *disquisitiones*, *Arithmeticae*, published in 1801, that polygons of 17 sides, 257 sides, and in general any number of sides expressed by 2^n+1, can be inscribed in a circle when 2^n+1 is a prime number.

7th. If the square root of two be taken for the radius of the given circle, and a radius be assumed in terms of the inscribed square (the side of which will be 2 and the area 4), and this radius as a variable be made to approach as its constant the true radius, so near that the difference between it and the true radius shall be less than any assignable quantity, so that it may be taken for the true radius, and at the same time if a circumference be assumed in terms of the inscribed square and of the radius, and this circumference as a variable be made to approach as its constant the true circumference, so near that the difference between it and the true circumference shall be made less than any assignable quantity, so that it may be taken for the true circumference, provided that this radius and circumference are selected in accordance with theorem second, then will the ratio between the assumed diameter and the assumed circumference be the same as the ratio between the true diameter and the true circumference, and this ratio will be the repeating decimal $3.\dot{1}4285\dot{7}$ or $3\frac{1}{7}$ to infinity; and, consequently, when the assumed radius becomes the true radius and the assumed circumference becomes the true circumference, both of which happen at the same moment, then the circle is infinite, that is, no part, however small, is straight.

The following selection consists of the most approved methods of solving the quadrature of the circle; they are inserted here, just as they are to be found in the works of the most approved authors recently published, for the purpose of giving the student a more general knowledge of the subject.

The first method is taken from "Robinson's Elements of Geometry," which gives the approximate ratio of the circumference to the diameter, by means of inscribed and circumscribed polygons to 6144 sides, commencing with the hexagon; it is as follows:

PROPOSITION III.—THEOREM.

When the radius of a circle is unity, its area and semi-circumference are numerically equal.

Let R represent the radius of any circle, and the Greek letter π, the half circumference of a circle whose radius is unity. Since circumferences are to each other as their radii, when the radius is R, the semi-circumference will be expressed by πR.

Let m denote the area of the circle of which R is the radius; then, by Theorem 1, we shall have, for the area of this circle, $\pi R^2 = m$, which, when $R = 1$, reduces to $\pi = m$.

This equation is to be interpreted as meaning that the semi-circumference contains its unit, the radius, as many times as the area of the circle contains its unit, the square of the radius.

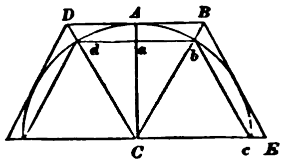
REMARK.—The celebrated problem of squaring the circle has for its object to find a line, the square on which will be equivalent to the area of a circle of a given diameter; or, in other words it proposes to find the ratio between the area of a circle and the square of its radius.

An approximate solution only of this problem has been as yet discovered, but the approximation is so close that the exact solution is no longer a question of any practical importance.

PROPOSITION IV.—PROBLEM.

Given, the radius of a circle unity, to find the areas of regular inscribed and circumscribed hexagons.

Conceive a circle described with the radius CA, and in this circle inscribe a regular polygon of six sides (Prob. 28, B. IV), and each side will be equal to the radius CA; hence, the whole *perimeter* of this polygon must be six times the radius of the circle, or three times the diameter. The chord bd is bisected by CA. Produce Cb and Cd, and through the point A, draw BD parallel to bd; BD will then be a side of a regular polygon of six sides, circumscribed about the circle, and we can compute the length of this line, BD, as follows: The two triangles, Cbd and CBD, are equiangular, by construction; therefore,



$$Ca : bd :: CA : BD.$$

Now, let us assume $CA = Cd =$ the radius of the circle, equal unity; then $bd = 1$, and the preceding proportion becomes

$$Ca : 1 :: 1 : BD \qquad (1)$$

In the right-angled triangle Cad, we have,

$$(Ca)^2+(ad)^2=(Cd)^2, \qquad \text{(Th. 39, B. I)}.$$

That is, $(Ca)^2+\frac{1}{4}=1$, because $Cd=1$, and $ad=\frac{1}{2}$.

Whence, $Ca = \frac{1}{2}\sqrt{3}$. This value of Ca, substituted in proportion (1), gives

$$\frac{1}{2}\sqrt{3} : 1 :: 1 : BD; \text{ hence, } BD = \frac{2}{\sqrt{3}}.$$

But the area of the triangle Cbd is equal to bd (= 1,) multiplied by $\frac{1}{2}Ca = \frac{1}{4}\sqrt{3}$; and the area of the triangle CBD is equal to BD multiplied by $\frac{1}{2}CA$.

Whence, area, $Cbd = \frac{1}{4}\sqrt{3}$.

and area, $CBD = \frac{1}{\sqrt{3}}$.

But the area of the inscribed polygon is six times that of the triangle Cbd, and the area of the circumscribed polygon is six times that of the triangle CBD.

Let the area of the inscribed polygon be represented by p, and that of the circumscribed polygon by P.

$$\text{Then } p = \frac{3}{2}\sqrt{3}, \text{ and } P = \frac{6}{\sqrt{3}} = \frac{2 \times 3}{\sqrt{3}} = 2\sqrt{3}.$$

$$\text{Whence } p : P :: \frac{3}{2}\sqrt{3} : 2\sqrt{3} :: \frac{3}{2} : 2 :: 3 : 4 :: 9 : 12$$

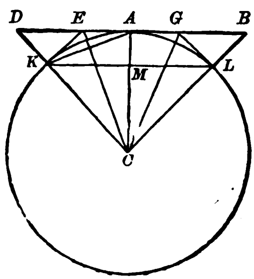
$$p = \frac{3}{2}\sqrt{3} = 2.59807621. \quad P = 2\sqrt{3} = 3.46410161.$$

Now, it is obvious that the *area* of the circle must be included between the areas of these two polygons, and not far from, but somewhat greater than, their half sum, which is 3.03 + ; and this may be regarded as the first approximate value of the area of the circle to the radius unity.

PROPOSITION V.—PROBLEM.

Given, the areas of two regular polygons of the same number of sides, the one inscribed in and the other circumscribed about, the same circle, to find the areas of regular inscribed and circumscribed polygons of double the number of sides.

Let p represent the area of the given inscribed polygon, and P that of the circumscribed polygon of the same number of sides. Also denote by p' the area of the inscribed polygon of double the number of sides, and by P' that of the corresponding circumscribed polygon. Now, if the arc KAL be some exact part, as one-fourth, one-fifth, etc., of the circumference of the circle, of which C is the center and CA the radius, then will KL be the side of a regular inscribed polygon, and the triangle KCL will be the same part of the whole polygon that the arc KAL is of the whole circumference, and the triangle CDB will be a like part of the circumscribed polygon. Draw CA to the point of tangency, and bisect the angles ACB and ACD, by the lines CG and CE, and draw KA.



It is plain that the triangle ACK is an exact part of the inscribed polygon of double the number of sides, and that the $\triangledown$ ECG is a like part of the circumscribed polygon of double the number of sides. Represent the area of the $\triangle$ LCK by a, and the area of the $\triangle$ BCD by b, that of the $\triangle$ ACK by x, and that of the $\triangle$ ECG by y, and suppose the $\triangle$'s, KCL and DBC, to be each the nth part of their respective polygons.

Then, $\quad na = p, \quad nb = P, \quad 2nx = p',$

and, $\quad 2ny = P';$

But, by (Th. 33, B. I), we have

$$CM \, . \, MK = a \qquad (1)$$
$$CA \, . \, AD = b \qquad (2)$$
$$CA \, . \, MK = 2x \qquad (3)$$

Multiplying equations (1) and (2), member by member, we have

$$(CM \, . \, AD) \quad (CA \, . \, MK) = ab \qquad (4)$$

From the similar $\triangle$'s CMK and CAD, we have

$$CM : MK :: CA : AD$$

whence $\quad CM \, . \, AD = CA \, . \, MK$

But from equation (3) we see that each member of this last equation is equal to $2x$; hence equation 4 becomes

$$2x \, . \, 2x = ab$$

If we multiply both members of this by $n^2 = n \,.\, n$, we shall have

$$4n^2x^2 = na.nb = p.P$$

or, taking the square root of both members,

$$2nx = \sqrt{p.P}$$

That is, *the area of the inscribed polygon of double the number of sides is a mean proportional between the areas of the given inscribed and circumscribed polygons* p *and* P.

Again, since CE bisects the angle ACD, we have, by, (Th. 24, B. II),

$$AE : ED :: CA : CD$$
$$:: CM : CK$$
$$:: CM : CA$$

hence, $$AE : AE + ED :: CM : CM + CA.$$

Multiplying the first couplet of this proportion by CA, and the second by MK, observing that $AE + ED = AD$, we shall have

$$AE.CA : AD.CA :: CM.MK : (CM + CA)\ MK.$$

But $AE.CA$ measures the area of the $\triangle\ CEG$, which we have called y, $AD.CA = \triangle\ CBD = b$, $CM.MK = \triangle\ CKL = a$, and $(CM + CA)\ MK = \triangle\ CKL + 2\ \triangle\ CAK = a + 2x$, as is seen from equations (1) and (3). Therefore, the above proportion becomes

$$y : b :: a : a + 2x.$$

Multiplying the first couplet by $2n$, and the second by n, we shall have

$$2ny : 2nb :: na : na + 2nx$$

That is, $$P' : 2P :: p : p + p'$$

whence, $$P' = \frac{2Pp}{p + p'}$$

and as the value of p' has been previously found equal to $\sqrt{Pp}$, the value of P' is known from this last equation, and the problem is completely solved.

PROPOSITION VI.—PROBLEM.

To determine the approximate numerical value of the area of a circle, when the radius is unity.

We have now found (Prob. 4) the areas of regular inscribed and circumscribed hexagons, when the radius of the circle is taken as the unit; and Prob. 5 gives us formulæ for computing from these the areas of regular inscribed and circumscribed polygons of twelve sides, and from these last we may pass to polygons of twenty-four sides, and so on, without limit. Now, it is evident that, as the number of sides of the inscribed polygon is increased, the polygon itself will increase, gradually approaching the circle, which it can never surpass. And it is equally evident that, as the number of sides of the circumscribed polygon is increased, the polygon itself will decrease, gradually approaching the circle, less than which it can never become.

The circle being included between any two corresponding inscribed and circumscribed polygons, it will differ from either less than they differ from each other; and the area of either polygon may then be taken as the area of the circle, from which it will differ by an amount less than the difference between the polygons.

It is also plain that, as the areas of the polygons approach equality, their perimeters will approach coincidence with each other, and with the circumference of the circle.

Assuming the areas already found for the inscribed and circumscribed hexagons, and applying the formulæ of Prob. 5 to them and to the successive results obtained, we may construct the following table:

NUMBER OF SIDES.	INSCRIBED POLYGONS.	CIRCUMSCRIBED POLYGONS
6	$\frac{3}{2}\sqrt{3} = 2.59807621$	$2\sqrt{3} = 3.46410161$
12	$3 = 3.0000000$	$\frac{12}{2+\sqrt{3}} = 3.2153904$
24	$\frac{6}{\sqrt{2+\sqrt{3}}} = 3.1058286$	3.1596602
48	3.1326287	3.1460863
96	3.1393554	3.1427106
192	3.1410328	3.1418712
384	3.1414519	3.1416616
768	3.1415568	3.1416092
1536	3.1415829	3.1415963
3072	3.1415895	3.1415929
6144	3.1415912	3.1415927

Thus we have found, that when the radius of a circle is 1, the semi-circumference must be more than 3.1415912, and less than 3.1415927; and this is as accurate as can be determined with the small number of decimals here used. To be more accurate we must have more decimal places, and go through a very tedious mechanical operation; but this is not necessary, for the result is well known, and is 3.1415926535897, *plus* other decimal places to the 100th, without termination. This result was discovered through the aid of an infinite series in the Differential and Integral Calculus.

The number, 3.1416, is the one generally used in practice, as it is much more convenient than a greater number of decimals, and it is sufficiently accurate for all ordinary purposes.

In analytical expressions it has become a general custom with mathematicians to represent this number by the Greek letter π, and, therefore, when any diameter of a circle is represented by D, the circumference of the same circle must be πD. If the radius of a circle is represented by R, the circumference must be represented by $2\pi R$.

SCHOLIUM.—The side of a regular inscribed hexagon subtends an arc of 60°, and the side of a regular polygon of twelve sides subtends an arc of 30°; and so on, the length of the arc subtended by the sides of the polygons, varying inversely with the number of sides.

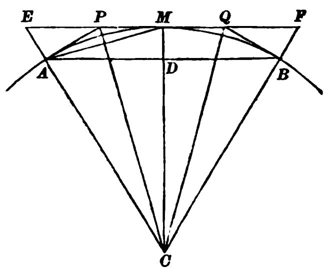
Angles are measured by the arcs of circles included between their sides; they may also be measured by the chords of these arcs, or rather by the half chords called *sines* in Trigonometry. For this purpose it becomes necessary to know the length of the chord of every possible arc of a circle.

The second method is taken from "Davies' Legendre," and gives the approximate ratio of the circumference to the diameter by means of inscribed and circumscribed polygons to 16384 sides, commencing with the square.

PROPOSITION XI.—PROBLEM.

The area of a regular inscribed polygon, and that of a similar circumscribed polygon being given, to find the areas of the regular inscribed and circumscribed polygons having double the number of sides.

Let AB be the side of the given inscribed, and EF that of the given circumscribed polygon. Let C be their common center, AMB a portion of the circumference of the circle, and M the middle point of the arc AMB.



Draw the chord AM, and at A and B draw the tangents AP and BQ; then will AM be the side of the inscribed polygon, and PQ the side of the circumscribed polygon of double the number of sides (P. VII.) Draw CE, CP, CM, and CF.

Denote the area of the given inscribed polygon by p, the area of the given circumscribed polygon by P, and the areas of the inscribed and circumscribed polygons having double the number of sides, respectfully by p' and P'.

1°. The triangles CAD, CAM, and the CEM, are like parts of the polygons to which they belong: hence, they are proportional to the polygons themselves. But CAM is a mean proportional between CAD and CEM (B. IV., P. XXIV., C. 2); consequently p' is a mean proportional between p and P: hence,

$$p' = \sqrt{p \times P}. \quad . \quad . \quad . \quad (1.)$$

2°. Because the triangles CPM and CPE have the common altitude CM, they are to each other as their bases: hence,

$$CPM : CPE :: PM : PE;$$

and because CP bisects the angle ACM, we have (B. IV., P. XVII.),

$$PM : PE :: CM : CE :: CD : CA;$$

hence (B. II., P. II.),

$$CPM : CPE :: CD : CA \text{ or } CM.$$

But, the triangles CAD and CAM have the common altitude AD; they are therefore, to each other as their bases: hence,

$$CAD : CAM :: CD : CM;$$

or, because CAD and CAM are to each other as the polygons to which they belong,

$$p : p' :: CD : CM;$$

hence (B. II., P. IV.), we have,

$$CPM : CPE :: p : p',$$

and, by composition,

$$CPM : CPM+CPE \text{ or } CME :: p : p+p';$$

hence (B. II., P. VII.),

$$2CPM \text{ or } CMPA : CME :: 2p : p+p'.$$

But, $CMPA$ and CME are like parts of P' and P, hence,

$$P' : P :: 2p : p+p';$$

or,

$$P' = \frac{2p \times P}{p + p'} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2.)$$

Scholium. By means of Equation (1), we can find p', and then, by means of Equation (2), we can find P'.

PROPOSITION XII.—PROBLEM.

To find the approximate area of a circle whose radius is 1.

The area of an inscribed square is equal to twice the square of the radius, or 2 (P. III., S.), and the area of a circumscribed square is 4. Making p equal to 2, and P equal to 4, we have, from Equations (1) and (2) of Proposition XI.,

$$p' = \sqrt{8} = 2.8284271 \quad . \; . \; . \quad \text{inscribed octagon};$$

$$P' = \frac{16}{2 + \sqrt{8}} = 3.3137085 \quad . \; . \; . \quad \text{circumscribed octagon.}$$

Making p equal to 2.8284271, and P equal to 3.3137085, we have from the same equations,

$$p' = 3.0614674 \quad . \; . \; . \quad \text{inscribed polygon of 16 sides.}$$

$$P' = 3.1825979 \quad . \; . \; . \quad \text{circumscribed polygon of 16 sides.}$$

By a continued application of these equations, we find the areas indicated in the following

TABLE.

NUMBER OF SIDES.	INSCRIBED POLYGONS.	CIRCUMSCRIBED POLYGONS.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441184
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928
16384	3.1415925	3.1415927

Now, the areas of the last two polygons differ from each other by less than the millionth part of a unit, but the area of the circle differs from either by less than they differ from each other; hence, the value of the area of either will differ from that of the circle by less than a millionth part of a unit. Taking the figures as far as they agree, and denoting the number of units in the required area by π, we have, approximately,

$$\pi = 3.141592;$$

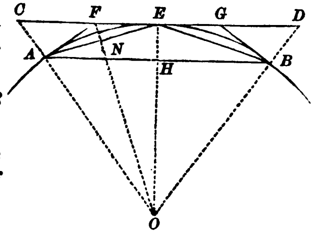
that is, *the area of a circle whose radius is* 1, *is* 3.141592.

Scholium. For practical computation, the value of π is taken equal to 3.1416.

The third method is taken from "Chauvenet's Elementary Geometry," and gives the approximate ratio of the circumference to the diameter, by two different methods, to 8192 sides each. The first is called the Method of Perimeters, and the second the Method of Isoperimeters. The above approximate ratio is followed by a chapter on the Doctrine of Limits taken from the same author.

PROPOSITION X.—PROBLEM.

25. *Given the perimeters of a regular inscribed and a similar circumscribed polygon, to compute the perimeters of the regular inscribed and circumscribed polygons of double the number of sides.*



Let AB be a side of the given inscribed polygon, CD a side of the similar circumscribed polygon, tangent to the arc AB at its middle point E. Join AE, and at A and B draw the tangents AF and BG; then AE is a side of the regular inscribed polygon of double the number of sides, and FG is a side of the circumscribed polygon of double the number of sides (4).

Denote the perimeters of the given inscribed and circumscribed polygons by p and P respectively; and the perimeters of the required inscribed and circumscribed polygons of double the number of sides by p' and P' respectively.

Since OC is the radius of the circle circumscribed about the polygon whose perimeter is P, we have (10),

$$\frac{P}{p} = \frac{OC}{OA} \text{ or } \frac{OC}{OE};$$

and since OF bisects the angle COE, we have (III. 21),

$$\frac{OC}{OE} = \frac{CF}{FE};$$

therefore,

$$\frac{P}{p} = \frac{CF}{FE}.$$

whence, by composition,

$$\frac{P+p}{2p} = \frac{CF + FE}{2FE} = \frac{CE}{FG}.$$

Now FG is a side of the polygon whose perimeter is P', and is contained as many times in P' as CE is contained in P, hence (III. 9),

$$\frac{CE}{FG} = \frac{P}{P'},$$

and therefore,

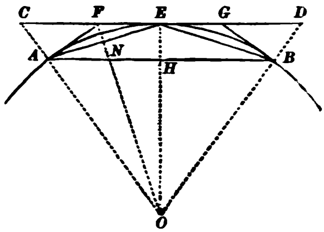
$$\frac{P+p}{2p} = \frac{P}{P'},$$

whence

$$P' = \frac{2pP}{P+p}. \qquad [1]$$

Again, the right triangles AEH and EFN are similar, since their acute angles EAH and FEN are equal, and give

$$\frac{AH}{AE} = \frac{EN}{EF}.$$



Since AH and AE are contained the same number of times in p and p', respectively, we have

$$\frac{AH}{AE} = \frac{p}{p'},$$

and since EN and EF are contained the same number of times in p' and P', respectively, we have

$$\frac{EN}{EF} = \frac{p'}{P'};$$

therefore, we have

$$\frac{p}{p'} = \frac{p'}{P'},$$

whence

$$p' = \sqrt{p \times P'}. \qquad [2]$$

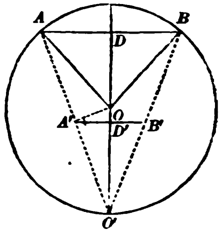
Therefore, from the given perimeters p and P, we compute P' by the equation [1], and then with p and P' we compute p' by the equation [2].

26. *Definition.* Two polygons are *isoperimetric* when their perimeters are equal.

PROPOSITION XI.—PROBLEM.

27. *Given the radius and apothem of a regular polygon, to compute the radius and apothem of the isoperimetric polygon of double the number of sides.*

Let AB be a side of the given regular polygon, O the center of this polygon, OA its radius, OD its apothem. Produce DO to meet the circumference of the circumscribed circle in O'; join $O'A$, $O'B$; let fall OA' perpendicular to $O'A$, and through A' draw $A'B'$ parallel to AB.



Since the new polygon is to have twice as many sides as the given polygon, the angle at its center must be one-half the angle AOB; therefore the angle $AO'B$, which is equal to one-half of AOB (II. 57), is equal to the angle at the center of the new polygon.

Since the perimeter of the new polygon is to be equal to that of the given polygon, but is to be divided into twice as many sides, each of its sides must be equal to one-half of AB; therefore $A'B'$, which is equal to one-half of AB (I. 121), is a side of the new polygon; $O'A'$ is its radius, and $O'D'$ its apothem.

If, then, we denote the given radius OA by R, and the given apothem OD by

r, the required radius $O'A'$ by R', and the apothem $O'D'$ by r', we have

$$O'D' = \frac{O'D}{2} = \frac{OO' + OD}{2},$$

or

$$r' = \frac{R + r}{2}. \qquad [1]$$

In the right triangle $OA'O'$, we have (III. 44),

$$\overline{O'A'}^2 = OO' \times O'D',$$

or

$$R' = \sqrt{R \times r'}; \qquad [2]$$

therefore, r' is an *arithmetic mean* between R and r, and R' is a *geometric mean* between R and r'.

MEASUREMENT OF THE CIRCLE.

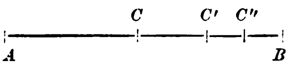
The principle which we employed in the comparison of incommensurable ratios (II. 49) is fundamentally the same as that which we are about to apply to the measurement of the circle, but we shall now state it in a much more general form, better adapted for subsequent application.

28. *Definitions.* I. A *variable quantity,* or simply, a *variable,* is a quantity which has different successive values.

II. When the successive values of a variable, under the conditions imposed upon it, approach more and more nearly to the value of some fixed or constant quantity, so that the difference between the variable and the constant may become less than any assigned quantity, without becoming zero, the variable is said to *approach indefinitely* to the constant; and the constant is called the *limit* of the variable.

Or, more briefly, the *limit* of a variable is a constant quantity to which the variable, under the conditions imposed upon it, approaches indefinitely.

As an example, illustrating these definitions, let a point be required to move from A to B under the following conditions: it shall first move over one-half of AB, that is to C; then over one-half of CB, to C'; then over one-half of $C'B$, to C''; and so on indefinitely; then the distance of the point from A is a *variable,* and this variable approaches indefinitely to the *constant* AB, as its *limit,* without ever reaching it.



As a second example, let A denote the angle of any regular polygon, and n the number of sides of the polygon; then, a right angle being taken as the unit, we have (8),

$$A = 2 - \frac{4}{n}.$$

The value of A is a variable depending upon n; and since n may be taken so great that $\frac{4}{n}$ shall be less than any assigned quantity however small, the value of A approaches to two right angles as its limit, but evidently never reaches that limit.

29. PRINCIPLE OF LIMITS. *Theorem. If two variable quantities are always equal to each other and each approaches to a limit, the two limits are necessarily equal.*

For, two variables always equal to each other present in fact but one value, and it is evidently impossible that one variable value shall at the same time approach indefinitely to two unequal limits.

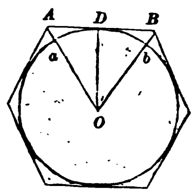
30. *Theorem. The limit of the product of two variables is the product of their limits.* Thus, if x approaches indefinitely to the limit a, and y approaches indefinitely to the limit b, the product xy must approach indefinitely to the product ab; that is, the limit of the product xy is the product ab of the limits of x and y.

31. *Theorem. If two variables are in a constant ratio and each approaches to a limit, these limits are in the same constant ratio.*

Let x and y be two variables in the constant ratio m, that is, let $x = my$; and let their limits be a and b respectively. Since y approaches indefinitely to b, my approaches indefinitely to mb; therefore we have x and my, two variables, always equal to each other, whose limits are a and mb, respectively, whence, by (29), $a = mb$; that is, a and b are in the constant ratio m.

PROPOSITION XVI.—THEOREM.

42. *The area of a circle is equal to half the product of its circumference by its radius.*



Let the area of any regular polygon circumscribed about the circle be denoted by A, its perimeter by P, and its apothem which is equal to the radius of the circle by R; then (22),

$$A = \tfrac{1}{2} P \times R, \text{ or } \frac{A}{P} = \tfrac{1}{2} R.$$

Let the number of the sides of the polygon be continually doubled, then A approaches the area S of the circle as its limit, and P approaches the circumference C as its limit; but A and P are in the constant ratio $\tfrac{1}{2} R$; therefore their limits are in the same ratio (31), and we have

$$\frac{S}{C} = \tfrac{1}{2} R, \text{ or } S = \tfrac{1}{2} C \times R. \qquad [I]$$

43. *Corallary* I. *The area of a circle is equal to the square of its radius multiplied by the constant number* π. For, substituting for C its value $2\pi R$ in [1], we have

$$S = \pi R^2.$$

44. *Corollary* II. *The area of a sector is equal to half the product of its arc by the radius.* For, denote the arc ab of the sector $a\ O\ b$ by c, and the area of the sector by s; then, since c and s are like parts of C and S, we have (III. 9),

$$\frac{s}{S} = \frac{c}{C} = \frac{\tfrac{1}{2} c \times R}{\tfrac{1}{2} C \times R}.$$

But $S = \tfrac{1}{2} C \times R$; therefore $s = \tfrac{1}{2} c \times R$.

45. *Scholium. A circle may be regarded as a regular polygon of an infinite number of sides.* In proving that the circle is the *limit* towards which the inscribed regular polygon approaches when the number of its sides is increased indefinitely, it was tacitly assumed that the number of sides is always *finite.* It was shown that the difference between the polygon and the circle may be made less than any assigned quantity by making the number of sides sufficiently great; but an *assigned* difference being necessarily a finite quantity, there is also some finite number of sides sufficiently great to satisfy the imposed condition. Conversely, so long as the number of sides is finite, there is some finite difference between the polygon and the circle. But if we make the hypothesis that the number of sides of the inscribed regular polygon is *greater than any finite number,* that is, *infinite,* then it must follow that the difference between the polygon and the circle is *less than any finite quantity,* that is *zero;* and consequently, the circle is identical with the inscribed polygon of an infinite number of sides.

This conclusion, it will be observed, is little else than an abridged statement of the theory of limits as applied to the circle; the abridgment being effected by the hypothetical introduction of *the infinite* into the statement.

PROPOSITION XVII.—PROBLEM.

46. *To compute the ratio of the circumference of a circle to its diameter approximately.*

First Method, called the Method of Perimeters. In this method, we take the diameter of the circle as given and compute the perimeters of some inscribed and a similar circumscribed regular polygon. We then compute the perimeters of inscribed and circumscribed regular polygons of double the number of sides, by Proposition X. Taking the last-found perimeters as given, we compute the perimeters of polygons of double the number of sides by the same method; and so on. As the number of sides increases, the lengths of the perimeters approach to that of the circumference (36); hence, their successively computed values will be successive nearer and nearer approximations to the value of the circumference.

Taking, then, the diameter of the circle as given = 1, let us begin by inscribing and circumscribing a square. The perimeter of the inscribed square $= 4 \times \frac{1}{2} \times \sqrt{2} = 2\sqrt{2}$ (13); that of the circumscribed square = 4; therefore, putting

$$P = 4.$$
$$p = 2\sqrt{2} = 2.8284271,$$

we find, by Proposition X., for the perimeters of the circumscribed and inscribed regular octagons,

$$P' = \frac{2p \times P}{P + p} = 3.3137085,$$
$$p' = \sqrt{p \times P'} = 3.0614675.$$

Then taking these as given quantities, we put

$$P = 3.3137085, p = 3.0614675,$$

and find by the same formulæ for the polygons of 16 sides

$$P' = 3.1825979, p' = 3.1214452.$$

Continuing this process, the results will be found as in the following

TABLE.*

NUMBER OF SIDES.	PERIMETER OF CIRCUMSCRIBED POLYGON.	PERIMETER OF INSCRIBED POLYGON.
4	4.0000000	2.8284271
8	3.3137085	3.0614675
16	3.1825979	3.1214452
32	3.1517249	3.1365485
64	3.1441184	3.1403312
128	3.1422236	3.1412773
256	3.1417504	3.1415138
512	3.1416321	3.1415729
1024	3.1416025	3.1415877
2048	3.1415951	3.1415914
4096	3.1415933	3.1415923
8192	3.1415928	3.1415926

From the last two numbers of this table, we learn that the circumference of the circle whose diameter is unity is less than 3.1415928 and greater than 3.1415926; and since, when the diameter = 1, we have $C = \pi$ (40), it follows that

$$\pi = 3.1415927$$

within a unit of the seventh decimal place.

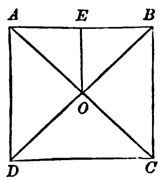
SECOND METHOD, called the METHOD OF ISOPERIMETERS. This method is based upon Proposition XI. Instead of taking the diameter as given and computing its circumference, we take the circumference as given and compute the diameter; or we take the semi-circumference as given and compute the radius.

Suppose we assume the semi-circumference $\frac{1}{2} C = 1$; then since $C = 2\pi R$, we have

$$\pi = \frac{\frac{1}{2} C}{R} = \frac{1}{R};$$

that is, the value of π is the reciprocal of the value of the radius of the circle whose semi-circumference is unity.

Let $ABCD$ be a square whose semi-perimeter = 1; then each of its sides = $\frac{1}{2}$. Denote its radius OA by R, and its apothem OE by r; then we have



$$r = \tfrac{1}{4} \qquad = 0.2500000,$$
$$R = \tfrac{1}{4}\sqrt{2} = 0.3535534.$$

* The computations have been carried out with ten decimal places in order to ensure the accuracy of the seventh place as given in the table.

Now, by Proposition XI., we compute the apothem r' and the radius R' of the regular polygon of 8 sides having the same perimeter as this square; we find

$$r' = \frac{R + r}{2} = 0.3017767,$$

$$R' = \sqrt{R \times r'} = 0.3266407.$$

Again, taking these as given, we put

$$r = 0.3017767,\ R = 0.3266407,$$

and find by the same formulæ, for the apothem and radius of the isoperimetric regular polygon of 16 sides, the values

$$r' = 0.3142087,\ R' = 0.3203644.$$

Continuing this process, the results are found as in the following

TABLE.

NUMBER OF SIDES.	APOTHEM.	RADIUS.
4	0.2500000	0.3535534
8	0.3017767	0.3266407
16	0.3142087	0.3203644
32	0.3172866	0.3188218
64	0.3180541	0.3184376
128	0.3182460	0.3183418
256	0.3182939	0.3183179
512	0.3183059	0.3183119
1024	0.3183089	0.3183104
2048	0.3183096	0.3183100
4096	0.3183098	0.3183099
8192	0.3183099	0.3183099

Now, a circumference described with the radius r is inscribed in the polygon, and a circumference described with a radius R is circumscribed about the polygon; and the first circumference is less, while the second is greater, than the perimeter of the polygon. Therefore the circumference which is equal to the perimeter of the polygon has a radius greater than r and less than R; and this is true for each of the successive isoperimetric polygons. But the r and R of the polygon of 8192 sides do not differ by so much as .0000001; therefore the radius of the circumference which is equal to the perimeter of the polygons, that is, to 2, is 0.3183099 within less than .0000001; and we have

$$\pi = \frac{1}{0.3183099} = 3.141593$$

within a unit of the sixth decimal place.

47 *Scholium* I. Observing that in this second method the value of $r = \frac{1}{4}$, for

the square, is the arithmetic mean of 0 and $\frac{1}{2}$, and that $R=\frac{1}{4}\sqrt{2}$ is the geometric mean between $\frac{1}{2}$ and $\frac{1}{4}$, we arrive at the following proposition:

The value of $\frac{1}{\pi}$ *is the limit approached by the successive numbers obtained by starting from the numbers* 0 *and* $\frac{1}{2}$ *and taking alternately the arithmetic mean and the geometric mean between the two which precede.*

48. *Scholium.* II. ARCHIMEDES (born 287 B. C.) was the first to assign an approximate value of π. By a method similar to the above "first method," he proved that its value is between $3\frac{1}{7}$ and $3\frac{10}{71}$, or, in decimals, between 3.1428 and 3.1408; he therefore assigned its value correctly within a unit of the third decimal place. The number $3\frac{1}{7}$, or $\frac{22}{7}$, usually cited as Archimedes' value of π (although it is but one of the two limits assigned by him), is often used as a sufficient approximation in rough computations.

METIUS (A. D. 1640) found the much more accurate value $\frac{355}{113}$, which correctly represents even the sixth decimal place. It is easily remembered by observing that the denominator and numerator written consecutively, thus 113 355, present the first three odd numbers each written twice.

More recently the value has been found to a very great number of decimals by the aid of series demonstrated by the Differential Calculus. CLAUSEN and DASE, of Germany), about A. D. 1846), computing independently of each other, carried out the value to 200 decimal places, and their results agreed to the last figure. The mutual verification thus obtained stamps their results as thus far the best established value to the 200th place. (See SCHUMACHER'S *Astronomische Nachrichten*, No. 589.) Other computers have carried the value to over 500 places, but it does not appear that their results have been verified.

The value to fifteen decimal places is

$$\pi = 3.141592653589793.$$

For the greater number of practical applications, the value $\pi = 3.1416$ is sufficiently accurate.

I shall close the introduction to the present volume by the following account of the quadrature of the circle, which I have been permitted to copy from "Chambers Encyclopedia," through the courtesy of the Superintendent of the Young Men's Mercantile Library of Cincinnati, Ohio, from the copy of the same in their possession:

QUADRATURE OF THE CIRCLE.

This is one of the grand problems of antiquity, which unsolved and probably unsolveable, continue to occupy even in the present day, the minds of many curious speculators. The trisection of the angle, the duplication of the cube, and the perpetual motion have found, in every age of the world since geometry and physics were thought of, their hosts of patient devotees. The *physical* question involved in the perpetual motion (q v) is treated of under that head; and we shall now take the opportunity of noticing the *mathematical* questions involved in the other problems above mentioned; but more especially that of the quadrature of the circle, in which the difficulty is of a different nature from that involved in the other two geometrical ones. A few words about them, however, will help as an introduction to the subject. According to the postulates of ordinary geometry, all constructions must be made by the help of the circle and straight line. Straight lines intersect each other in but *one* point; and a straight line and circle, or two circles, intersect in *two* points only. From the analytical point of view we may express these facts by saying that the determination of the intersection of two straight lines involves an equation of the *first* degree only; while that of the intersection of a straight line and a circle, or of two circles, is reducible to an equation of the second degree.

But the trisection of an angle, or the duplication of the cube, requires for its accomplishment, the solution of an equation of the *third* degree; or, geometrically, requires the intersections of a straight line and a curve of the third degree, or of two conics, etc., *all of which are excluded by the postulates of the science.* If it were allowed that a parabola or ellipse could be described with a given fucus and directrix, as it is allowed that a circle can be described with a given radius about a given center, the trisection of an angle and the duplication of the cube would be at once brought under the category of questions resolvable by pure geometry; so that the difficulty in these cases is one of mere restriction of the postulates of what is to be called geometry.

It is very different in the case of the quadrature of the circle, which (the reader of the preceding article will see at once) means the determination of the area of a circle of given radius, literally, the assigning of the side of a square whose area shall be equal to that of the given circle.

The common herd of "squarers of the circle," which grows more numerous every day, and which includes many men of undoubted sanity, and even of the highest business talents, rarely have any idea of the nature of the problem they attempt to solve. It will, therefore, be our best course to show, first of all, *what has been done* towards the solution of the problem; we shall then venture a few remarks as to *what may yet be done,* and in what direction philosophic "squarers of the circle" must look for real advance In the first place, then, we observe that *mechanical processes are utterly inadmissible.* A fair approximation may, no doubt, be got by measuring the diameter of a circular disc of uniform material, and comparing the weight of the disc with that of a square portion of the same material of a given side. But it is almost impossible to execute any measurement to more than six places of significant figures; hence, as will soon be shown, this process is at best but a rude approximation. The same is to be said of such obvious processes as wrapping a string round a cylinder post of known diameter and comparing its length with the diameter of the cylinder; only a rude approximation to the ratio of the circumference of a circle to its diameter can thus be obtained. Before entering on the history of the problem, it must be remarked that the Greek geometers knew that the area of a circle is half the rectangle under its radius and circumference (see CIRCLE), so that the determination of the length of the circumference of a circle of given radius is precisely the same problem as that of the quadrature of the circle.

Confining ourselves strictly to the best ascertained steps in the history of the question, we remark that Archimedes proved that the ratio of the diameter to the circumference is greater than 1 to $3\frac{10}{71}$ or $3\frac{1}{7}$, and less than 1 to $3\frac{10}{70}$; the difference between these two extreme limits is less than the 1000th of the whole ratio. Archimedes' process depends upon the obvious truth, that the circumference of an inscribed polygon is less, while that of a circumscribed polygon is greater, than that of the circle. His calculations were extended to regular polygons of 96 sides.

Little more seems to have been done by mathematicians till the end of the 16th century, when P. Métius gave expression for the ratio of the circumference to the diameter as the fraction $\frac{355}{113}$, which, in decimals, is true to the seventh significant figure inclusive; curiously enough it happens that this is one of the fractions which express in the lowest possible terms the best approximation to the required number. Métius seems to have employed, with the aid of far superior arithmetical notation, a process similar to that of Archimedes. Viete shortly afterwards gave the ratio in a form true to the tenth decimal place, and was the first to give, though of course in infinite terms, an exact formula. Designating, as is usual in mathematical works, the ratio of the circumference to the diameter π, Viete's formula is

$$\frac{1}{\pi} = \tfrac{1}{2}\sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}} \times \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}}} \times \text{etc.}$$

Shortly afterwards, Adrianus Romanus, by calculating the length of the side of an equilateral inscribed polygon of 1073741824 sides, determined the value of π to 16 significant figures; and Ludolph Van Ceulen, his contemporary, by calculating that of the polygon of 36.893488147419103232 sides, arrived (correctly) at 36 significant figures. It is scarcely possible to give, in the present day, an idea of the enormous labor which this mode of procedure entails even when only 8 or 10 figures are sought; and when we consider that Ludolph was ignorant of logarithms, we wonder that a life time sufficed for the attainment of such a result by the method he employed.

The value of π was thus determined to $\frac{1}{3 \times 10^{35}}$ of its amount, a fraction of which, after Montucla, we shall attempt to give an idea, thus: suppose a circle whose radius is the distance of the nearest fixed star (250,000 times the earth's distance from the sun), the error in calculating its circumference by Ludolph's result would be so excessively small a fraction of the diameter of a human hair as to be utterly invisible, not merely under the most powerful microscope yet made, but under any which future generations may be able to construct.

These results were, as we have pointed out, all derived by common arithmetical operations, based on the obvious truth that the circumference of a circle is greater than that of any inscribed, and less than any circumscribed polygon.

They involve none of those more subtle ideas connected with limits, infinitesimals, or differentials, which seem to render more recent results suspected by modern "squarers." If one of that unhappy body would only consider this simple *fact*, he could hardly have the presumption to publish his 3.1250 or whatever it may be, as the accurate value of a quantity which, by common arithmetical processes, founded on an obvious geometrical truth, was several centuries ago shown to be the greater than 314159265358979323846264338327950288 and less than 314159265358979323846264338327950289.

We now know by far simpler processes its exact value to more than 600 places of decimals; but the above result of Van Ceulen is much more than sufficient for any possible practical application even in the most delicate calculations in astronomy.

Snellius, Huyghens, Gregory de St. Vincent, and others, suggested simplifications of the polygon process, which are in reality some of the approximate expressions derived from modern trigonometry. In 1668 the celebrated James Gregory gave a demonstration of the impossibility of effecting exactly the quadrature of the circle, which, although objected to by Huyghens, is now received as quite satisfactory. We may merely revert to the speculations of Fermat, Roberval, Cavalleri, Wallis, Newton, and others, as to quadrature in general; their most valuable result was the invention of the Differential and Integral Calculus, by Newton, under the name of Fluxions or Fluents. Wallis, however, by an ingenious process of interpolation, showed that $\frac{\pi}{4}=\frac{2.4.4.6.6.8.8.10.10, \text{ etc.},}{3.3.5.5.7.7.9.9.11, \text{ etc.},}$ which is interesting as being the first recorded example of the determination, in a finite form, of value of the ratio of two infinite products.

Lord Bruncker, being consulted by Wallis as to the value of the expression, put it in the form of an infinite continued fraction, thus:

$$\frac{\pi}{4}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+, \text{ etc.}}}}}}$$

in which 2 and the squares of the odd numbers appear. This formula has been employed to show that not only π, but its square, is incommensurable. Perhaps the neatest of all the formulas which have been given for the quadrature of the circle, is that of James Gregory, for the arc in terms of its tangent—namely, $\theta = \tan.\,\theta - \frac{1}{3}\tan.^3\theta + \frac{1}{5}\tan.^5\theta$—etc.

This was appropriated by Liebnitz, and formed, perhaps, the first of those audacious series of peculations from English mathematicians which have forever dishonored the name of a man of real genius.

If we notice that, by ordinary trigonometry, the arc, whose tangent is unity, (the arc of 45° or) $\frac{\pi}{4}$ falls short of four times the arc whose tangent is $\frac{1}{5}$ by an angle whose tangent is $\frac{1}{239}$, we may easily calculate $\frac{\pi}{4}$ to any required number of decimal places by calculating from Gregory's formula the values of the arcs corresponding to $\frac{1}{5}$ and $\frac{1}{239}$ as tangents. And it is, in fact, by a slight modification of this process (which was originally devised by Machin) that π has been obtained by independent calculators to 600 decimal places. It is not yet proved, and it may not be true, that the area or circumference of a circle can not be expressed in finite terms; if it can be, these must, of course, contain irrational quantities. The Integral Calculus gives, among hosts of others, the following very simple expression in terms of a definite integral:

$$\frac{\pi}{2}=\int_0^\infty \frac{dx}{1+x^2}$$

Now it very often happens that the value of a definite integral can be assigned when that of the general integral can not; and it is not impossible, so far as is yet known, that the above integral may be expressed in such form as

$$\sqrt{x}+\sqrt{y}$$

where $\sqrt{x}$ and $\sqrt{y}$ are irrational numbers. Such an expression, if discovered, would undoubtedly be hailed as a solution of the grand problem. But this, we need hardly say, is *not* the species of solution attempted by "squarers." We could easily, from our own experience alone, give numerous instances of their helpless absurdities, but we spare the reader and refer him, for further information on this painful yet ridiculous subject, to a recent series of papers, by Professor De Morgan, in the *Athenæum*, and to the very interesting work of Montucla, *Histoire des Recherches sur la Quadrature du Cercle.*

PART FIRST,

CONTAINING THE

GEOMETRICAL AND FINAL SOLUTION

OF THE

QUADRATURE OF THE CIRCLE,

BY AN ENTIRELY NEW METHOD, TOGETHER WITH AMPLE PROOFS OF THE SAME.

QUADRATURE OF THE CIRCLE.

Definitions.

Of Lines, Angles, etc., see Plate 1.

1. Science is knowledge systematized.

2. Art is the skill with which the principles of a science are practically applied.

3. Quantity is anything which can be increased, diminished, or measured.

4. Mathematics is the science of quantity.

5. Geometry is that branch of mathematics which treats of the properties of *extension* and *figure.*

Lines.

6. A line has only one *dimension*, namely, *length;* without either breadth or thickness; lines are either straight or curved.

7. The extremities of a line are points.

8. A point has *no dimensions*, and therefore it has no *size;* and is usually represented to the eye by a dot, as at *AA*.

9. A straight line is the shortest distance between any two given points, as at *B*.

10. A curved line does not lie evenly between its extreme points; but constantly changes its course as at *C*.

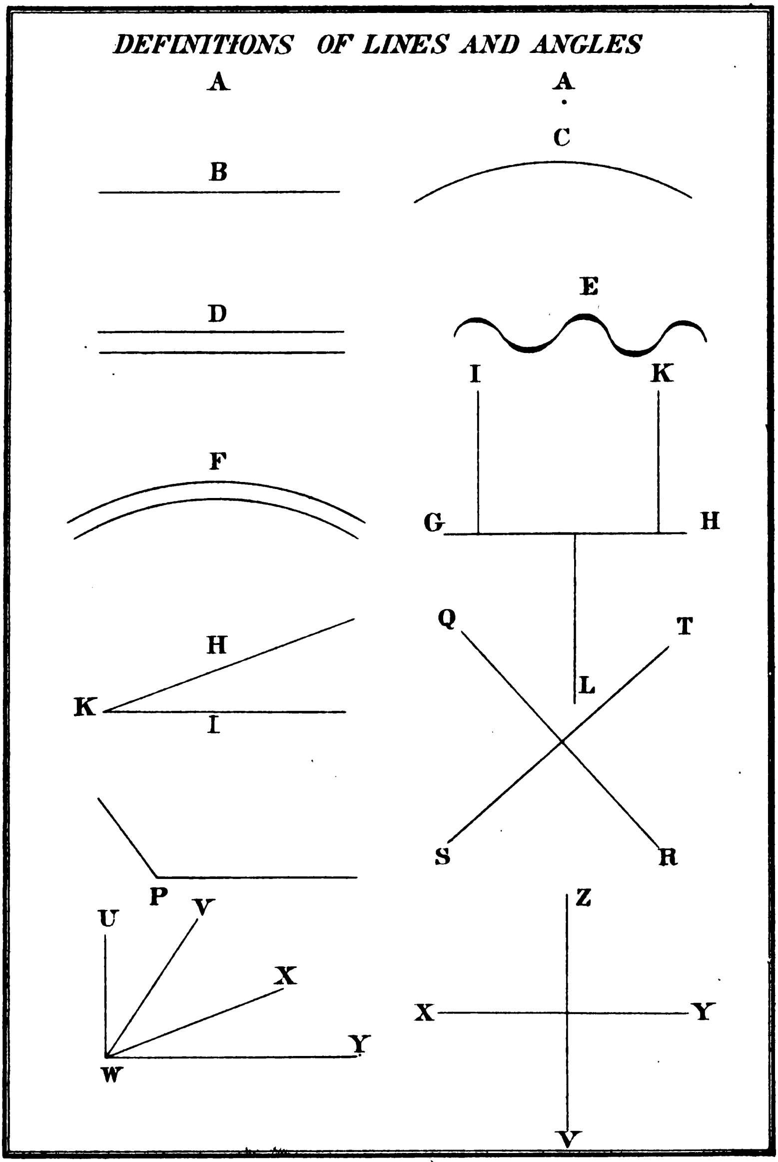
11. A waved line is composed of curved lines as at *E*.

12. Parallel straight lines are in the same plane, but have no inclination towards one another; and being produced ever so far both ways can not meet, as at *D*.

13. Parallel curved lines if produced would form two concentric circles; that is, having a common center, as at *F*.

14. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is a right angle, and the straight line which stands on the other is called a perpendicular to it; *IKL* are perpendiculars to the line *GH*.

PLATE 1.


DEFINITIONS OF LINES AND ANGLES
A
A
B
C
D
E
F
I
K
G
H
H
K
I
L
Q
T
S
R
P
U
V
X
Y
W
Z
X
Y
V

15. Inclined lines, as *H* and *I* if produced, would meet in a point as *K* forming an angle of which the point *K* is the vertex, and the lines *H* and *I* the legs of the angle.

16. When two straight lines cut each other so as to make the adjacent angles equal to one another, each of the angles is a right angle; and the straight lines which so cut each other are perpendicular to one another; thus *QR* is perpendicular to *ST* and *XY* to *ZV*; but perpendicular lines are not always vertical and horizontal; *XY* is a horizontal line, and *ZV* is a vertical line.

Angles.

17. If there is only one angle at a point it may be denoted by a letter placed at the vertex, as at *K*. But if several angles are at one point, any one of them is expressed by three letters, of which the middle one is at the vertex; thus the angle which is contained by the straight lines *UWY* is called the angle *UWY* or *YWU*.

Remark.—Angles like other quantities may be added, subtracted, multiplied, and divided; thus the angle *UWY* is the sum of the angles *UWV*, *VWX*, *XWY*, and the angle *UWV* is the difference between the two angles, *YWU* and *YWV*

18. An acute angle is less than a right angle; thus the angle at *K* is an *acute angle*.

19. An obtuse angle is greater than a right angle; thus the angle at *P* is an *obtuse angle*.

20. An oblique angle is formed by one straight line meeting another straight line on one side of it, so as to make the adjacent angles equal to two right angles.

21. A plane angle is the inclination of two lines to one another in a plane which meet together, but are not in the same direction.

22. A plane rectilineal angle is the inclination of two straight lines to one another which meet together, but are not in the same straight line.

23. A Theorem is a truth requiring demonstration.

24. An Axiom is a self-evident truth.

25. A Problem is a question requiring a solution.

26. A Postulate is a self-evident problem.

Theorems, Axioms, Problems, and Postulates, are all called *Propositions*.

27. A LEMA is an auxiliary proposition.

28. A COROLLARY is an obvious consequence of one or more propositions.

29. A SCHOLIUM is a remark made upon one or more propositions, with reference to their connection, their use, their extent, or their limitation.

30. An HYPOTHESIS is a supposition made, either in the statement of a proposition, or in the course of a demonstration.

31. Magnitudes are equal to each other, when each contains the same unit an equal number of times.

32. Magnitudes are equal *in all their parts*, when they may be so placed as to coincide throughout their whole extent.

Of Plane and Rectilineal Surfaces.

1. A superfices or surface has *two dimensions;* length and breadth

REMARK.—The extremities of superfices are lines. A term or boundary is the extremity of anything. A figure is enclosed by one or more boundaries.

NOTE.—*Rectilineal* figures are contained by straight lines. *Trilateral* figures or *triangles*, by three straight lines; *quadrilateral* figures, by four straight lines; *multilateral* figures or *polygons*, by more than four straight lines.

2. An *equilateral* triangle has three equal sides, as *A*.

3. An *isoceles* triangle has two equal sides, as *B*.

4. A right-angled triangle has *one right* angle, as *C*.

REMARK.—The side opposite the right angle is called the *hypothenuse*.

5. A *scalene* triangle has three unequal sides, as *D*.

6. An *obtuse*-angled triangle has *one* obtuse angle as *O*.

7. An *acute-angled* triangle has three acute angles, as *A*.

8. A *square* has all its sides equal, and all its angles right angles, as *E*.

9. A *rhombus* has all its sides equal, but its angles are not all right angles, as *H*.

10. A *rhomboid* has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles, as *FG*.

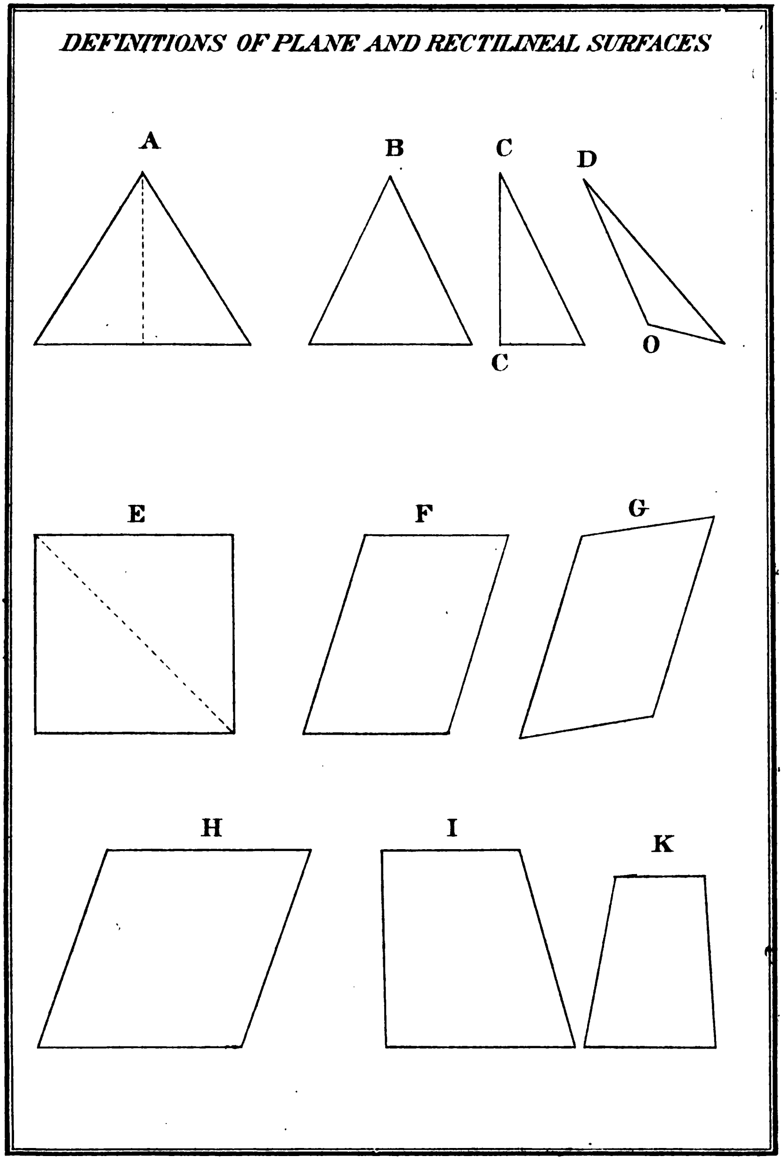
11. A *trapezoid* has only two of its sides parallel, as *IK*.

12. All other four-sided figures beside these are called trapeziums.

NOTE.—The terms *oblong* and *rhomboid* are not often used; practically the following definitions are used: Any four-sided figure is called a *quadrilateral*. A line joining two opposite angles of a quadrilateral is called a *diagonal*. A quadrilateral which has its opposite sides parallel is called a *parallelogram*. The words *square* and *rhombus* are used in the same as defined by Euclid, and the word *rectangle* is used instead of the word oblong.

Some writers propose to restrict the word *trapezium* to a quadrilateral which has two of its sides parallel; and it would certainly be convenient if this restriction were universally adopted.

PLATE II.


DEFINITIONS OF PLANE AND RECTILINEAL SURFACES
A
B
C
D
C
O
E
F
G
H
I
K

Of the Circle, see Plate 3.

1. *A circle* is a plain figure bounded by a curved line called the *circumference*, every point of which is equally distant from a point within.

And this point within is called the *center* *A*, *F*, *C*, *B*, *H*, *D*, *G* is the circumference and *E* the center of the circle. Fig. 2 Plate 3.

2. *A radius* is a straight line drawn from the center to any point of the circumference *AE*, *CE*, *BE*, *DE*. Fig. 2 Plate 3.

3. *A diameter* is a straight line drawn through the center and terminating in the circumference, as *AB*, *CD*. Fig. 2 Plate 3

REMARK.—All radii of the same circle are equal. All diameters are equal, and each is double of the radius.

4. An *arc* is any part of the circumference, as *AF*. Fsg. 1 Plate 3.

5. *A chord* is a straight line joining the extremities of an arc, as *FH*. Fig. 1 Plate 3.

6. *A segment* is that part of a circle between an arc and its chord *AG*, *HB*, and *GHD*; Fig. 2 Plate 3 are segments of the circle *ABCD*.

7. *A sector* is that part of a circle included within an arc and the radii drawn to its extremities *FEC*; Fig. 2 Plate 3 is a sector of the circle *ABCD*, also *AEF*, *CEB*.

REMARK.—*AEF* is also called a *sextant* because it is *one-sixth* part of the circle, and *CEB* is called a *quadrant*, because it is *one-fourth* part of the circle.

TRIGONOMETRY.

Trigonometry *is the science which teaches how to determine the several parts of a triangle from having certain parts given.*

Plane trigonometry treats of plane triangles; spherical trigonometry treats of spherical triangles.

The circumference of a circle is supposed to be divided into 360 equal parts, called *degrees;* each degree into 60 *minutes*, and each minute into 60 *seconds*. Degrees, minutes, and seconds are designated by the characters °, ′, ″. Thus 23° 14′ 35″ is read 23 degrees, 14 minutes, and 35 seconds.

Since an angle at the center of a circle is measured by the arc intercepted by its sides, a right angle is measured by 90°, two right angles by 180°, and four right angles are measured by 360°.

The *complement* of an arc is what remains after subtracting the arc from 90°. Thus the arc *DF*, Fig. 1 Plate 3, is the complement of *AF*. The complement of 25° 15′ is 64° 45′.

In general, if we represent any arc by A, its complement is 90°—A. Hence, if an arc is greater than 90°, its complement must be negative. Thus the complement of 100° 15′ is—10° 15′. Since the two acute angles of a right-angled triangle are together equal to a right angle each of them must be the complement of the other.

The *supplement* of an arc is what remains after subtracting the arc from 180°. Thus the arc BDF is the supplement of the arc AF. The supplement of 25° 15′ is 154° 45′. In general, if we represent any arc by A, its supplement is 180°—A. Hence, if an arc is greater than 180°, its supplement must be negative. Thus the supplement of 200° is—20°. Since in every triangle the sum of the three angles is 180°, either angle is the supplement of the sum of the other two.

The sine of an arc is the perpendicular let fall from one extremity of the arc on the radius passing through the other extremity. Thus FG is the sine of the arc AF, or of the angle ACF.

Every sine is half the chord of double the arc. Thus the sine FG is the half of FH, which is the chord of the arc FAH, double of FA. The chord which subtends the sixth part of the circumference, or the chord of 60°, is equal to the radius (*Loomis' Geom.*, Prop. IV., Book VI.); hence the sine of 30° is equal to half of the radius.

The versed sine of an arc is that part of the diameter intercepted between the sine and the arc. Thus GA is the versed sine of the arc AF.

The tangent of an arc is the line which touches it at one extremity, and is terminated by a line drawn from the center through the other extremity. Thus AI is the tangent of the arc AF, or the angle ACF.

The secant of an arc is the line drawn from the center of the circle through one extremity of the arc, and is limited by the tangent drawn through the other extremity. Thus CI is the secant of the arc AF, or of the angle ACF.

The *cosine* of an arc is the sine of the complement of that arc. Thus the arc DF, being the complement of AF, FK is the sine of the arc DF, or the cosine of the arc AF.

The *cotangent* of an arc is the tangent of the complement of that arc. Thus, DL is the tangent of the arc DF, or the cotangent of the arc AF.

The *cosecant* of an arc is the secant of the complement of that arc. Thus CL is the secant of the arc DF, or the cosecant of the arc AF.

PLATE III.

DEFINITIONS OF THE CIRCLE

Fig. 1

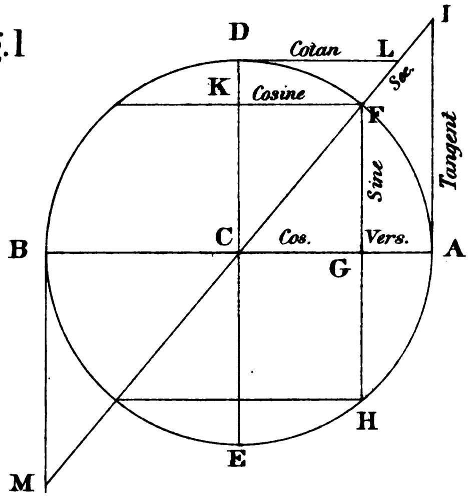
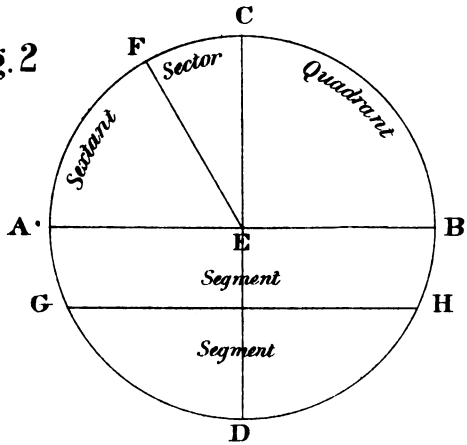


Fig. 2



CF^2; that is, $\sin.^2 A + \cos.^2 A = R^2$; or, ***the square of the sine of an arc, together with the square of its cosine, is equal to the square of the radius***;

$$\text{Hence } \sin. A = \pm \sqrt{R^2 - \cos.^2 A},$$
$$\text{And } \cos. A = \pm \sqrt{R^2 - \sin.^2 A}.$$

Postulates.

Let it be granted:

1. That a straight line may be drawn from any one point to any other point.
2. That a terminated straight line may be produced to any length in a straight line.
3. And that a circle may be described from any center, and with any radius.
4. That no part of the circumference of a circle is straight.
5. That it is possible to find a straight line equal in length to the circumference of a given circle.
6. That it is possible to find a common measure between the side and diagonal of a square in integers to infinity.
7. That it is possible to inscribe in any circle a polygon with any given number of sides expressed by $2^n + 1$, provided, $2^n + 1$ is a prime number.

Axioms.

1. Things which are equal to the same thing are equal to one another.
2. If equals be added to equals the wholes will be equal.
3. If equals be taken from equals the remainders are equal.
4. If equals be added to unequals the wholes will be unequal.
5. If equals be taken from unequals the remainders will be unequal.
6. Things which are double of the same thing are equal to one another.
7. Things which are halves of the same thing are equal to one another.
8. Magnitudes which coincide with one another, that is, which exactly fill the same space are equal to one another.
9. The whole is greater than its part.
10. Two straight lines can not inclose a space.
11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Articles.

1. The double of the cosine or secant is the constantly assumed diameter.

2. The sum of the sines or tangents is the constantly assumed circumference.

3. The sine is a mean proportion between the double of the cosine and the second tangent.

4. The second tangent is a mean proportional between the double of the second secant and the third tangent, etc., etc., etc.

5. The rectangle contained by the sine and cosine is equal to the area of the inscribed double triangle.

6. The rectangle contained by the radius and the tangent is equal to the area of the circumscribed double triangle.

7. The rectangle contained by the radius and the sine is a mean proportional between the inscribed and circumscribed double triangles of *half* the number of sides.

8. The rectangle contained by the inscribed double triangle and the number of sides of the entire polygon is equal to the area of the inscribed polygon.

9. The rectangle contained by the circumscribed double triangle and the number of sides of the entire polygon is equal to the area of the circumscribed polygon.

10. The rectangle contained by the rectangle of the radius and the sine and the number of sides contained in the given polygon, is equal to the area of the entire inscribed polygon of double the number of sides.

Signs.

The following are the principal signs employed:

The *Sign of Addition*, $+$, called *plus*:

Thus, $A + B$, indicates that B is to be added to A.

The *Sign af Subtraction*, —, called *minus*:

Thus, $A - B$, indicates that B is to be substracted from A.

The *Sign of Multiplication*, $\times$:

Thus, $A \times B$, indicates that A is to be multiplied by B.

The *Sign of Division*, $\div$:

Thus, $A \div B$, or, $\frac{A}{B}$, indicates that A is to be divided by B.

The *Exponential Sign*:

Thus, A^3, indicates that A is to be taken three times as a factor, or raised to the third power.

The *Radical Sign*, $\sqrt{\ }$:

Thus, $\sqrt{A}$, $\sqrt[3]{B}$, indicate that the square root of A, and the cube root of B, are to be taken.

When a compound quantity is to be operated upon as a single quantity, its parts are connected by a vinculum or by a parenthesis:

Thus, $\overline{A + B} \times C$, indicates that the sum of A and B is to be multiplied by C; and $(A + B) \div C$, indicates that the sum of A and B is to be divided by C.

A number written before a quantity, shows how many times it is to be taken.

Thus, $3(A + B)$, indicates that the sum of A and B is to be taken three times.

The *Sign of Equality* $=$:

Thus, $A = B + C$, indicates that A is equal to the sum of B and C.

The expression, $A = B + C$, is called an equation. The part on the left of the sign of equality, is called the *first member*; that on the right, the *second member*.

The *Sign of Inequality*, $<$:

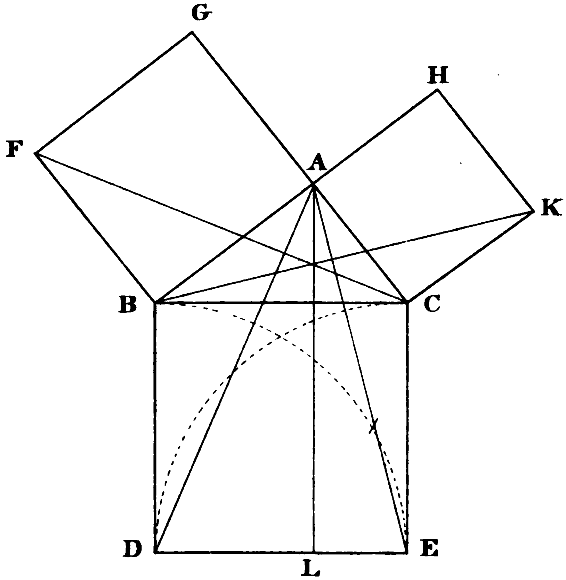
Thus, $\sqrt{A} < \sqrt[3]{B}$, indicates that the square root of A is less than the cube root of B. The opening of the sign is towards the greater quantity.

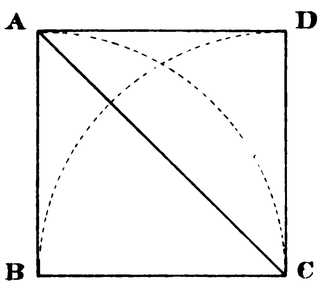
The sign, $\therefore$ is used as an abbreviation of the word *hence*, or *consequently*.

The general truths of Geometry are deduced by a course of logical reasoning, the premises being definitions and principles previously established. The course of reasoning employed in establishing any truth or principle, is called *a demonstration*.

PLATE IV.

PROPOSITION 1 THEOREM





PROPOSITION 1. THEOREM.

In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the sum of the squares described on the side which contains the right angle.

Let *ABC* be a right-angled triangle, having the right angle *BAC*; the square described on the side *BC* shall be equal to the sum of the squares described on *BA*, *AC*. See Plate 4 Fig. 1.

On *BC* describe the square *BDEC*, and on *BA*, *AC* describe the squares *GB*, *HC*; *Todhunter's Euclid*, [1.46,]
through *A* draw *AL* parallel to *BD* or *CE*; [1.31,]
and join *AD*, *EC*.

Then, because the angle *BAC* is a right angle, [*Hypothesis*,]
and that the angle *BAG* is also a right angle, [*Definition* 30,]
the two straight lines *AC*, *AG*, on the opposite sides of *AB*, make with it at the point *A* the adjacent angles equal to two right angles; therefore *CA* is in the same straight line with *AG*. [1.14.]

For the same reason, *AB* and *AH* are in the same straight line.

Now the angle *DBC* is equal to the angle *FBA*, for each of them is a right angle. [*Axiom* 11.]

Add to each the angle *ABC*. Therefore the whole angle *DBA* is equal to the whole angle *FBC*. [*Axiom* 2.]
And because the two sides *AB*, *BD* are equal to the two sides *FB*, *BC*, each to each; [*Definition* 30,]
and the angle *DBA* is equal to the angle *FBC*; therefore the triangle *ABD* is equal to the triangle *FBC*. [1.4.]

Now the parallelogram *BL* is double of the triangle *ABD*, because they are on the same base *BD*, and between the same parallels *BD*, *AL*. [1.41.]
And the square *GB* is double of the triangle *FBC*, because they are on the same base *FB*, and between the same parallels *FB*, *GC*, [1.41.]
But the doubles of equals are equal to one another; [*Axiom* 6.]
Therefore the parallelogram *BL* is equal to the square *GB*.

In the same manner, by joining *AE*, *BK*, it can be shown that the parallelogram *CL* is equal to the square *CH*.

Therefore the whole square *BDEC* is equal to the two squares *GB*, *HC*. [*Axiom* 2.]
And the square *BDEC* is described on *BC*, and the squares *GB*, *HC*

on BA, AC. Therefore the square described on the side BC is equal to the squares described on the sides BA, AC.

CORALLARY 1. Let $ABCD$ be a square, and AC its diagonal; the triangle ABC being right-angled and isoceles, we have $AC^2=AB^2+BC^2=2AB^2$.

Therefore, *the square described on the diagonal of a square, is double of the square described on the side.*

If we extract the square root of each member of this equation, we shall have

$$AB\sqrt{2}=AC; \text{ or } AB : AC :: 1 : \sqrt{2}.$$

See Plate 4 Fig. 2.

PROPOSITION 2. THEOREM.

ARGUMENT 1. If ten (10) right-angled triangles, the base, perpendicular, and hypothenuse of which are respectively three (3), four (4), and five (5), be placed so as to form a polygon of five sides and a little more, that is to say, with all their bases outwards, and the perpendicular of one triangle joined to the perpendicular of another, so that they shall touch one another in those points which contain the right angles, and also in those points which form the verticle angles, so that any two of the bases when taken together shall be equal to six; and the hypothenuse of each triangle joined to the hypothenuse of another triangle, so that the vertical angles, that is the angles opposite the bases shall form a common center, then there will be a lap equal to the *one fiftieth of the circumference of the whole polygon.* See Plate 5 Fig. 1. For, because the straight lines AC, CD, are tangents to the radius BC, therefore they are both in the same straight line, and the radius BC is a perpendicular to them. For the same reason the straight lines DE, EF are in the same straight line, and the radius BE is a perpendicular to them, also FG, GH, and BG. HI, IJ, and BI. JK, KZ, and BK. [*Hypothesis.*]

Again, because the radius BC is perpendicular to the straight line AD, therefore, the angles BCD and BCA are both right angles; for the same reason BED and BEF are right angles, as also BGF and BGH, BIH, and BIJ, BKJ and BKZ. [*Hypothesis.*]

Again, because the angles ADF, DFH, FHJ, HJZ are inscribed in an arc which is less than a semicircle, therefore they are greater than a right angle. [*Hypothesis.*]

PLATE V.

Fig. 1

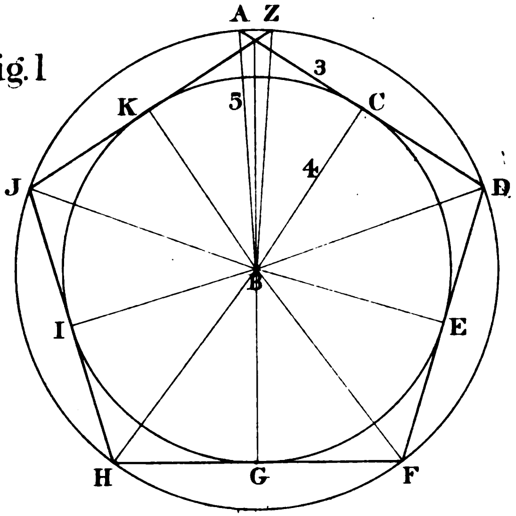
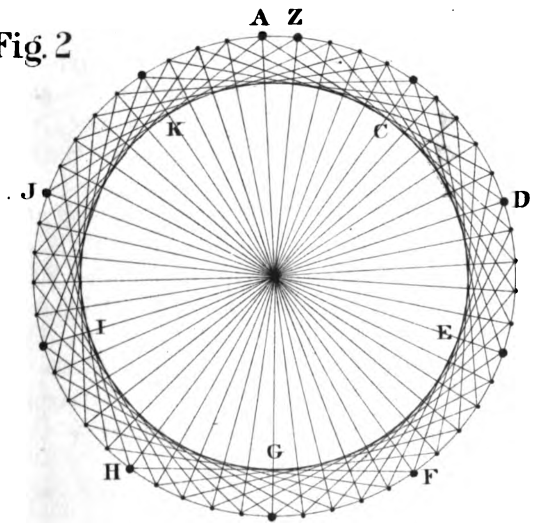


Fig. 2



Now let the right-angled triangles *BCA* and *BCD*, *BED*, and *BEF*, *BGF*, and *BGH*, etc., etc., be the given polygon, and let two circles be drawn around the center *B*; one with a radius of *four* or *BC*, and the other with a radius of *five* or *BD*; so that the first circle shall pass through the points which form the right angles, namely, *CEGIK*, and the second through the points at the other extremity of the bases, namely, *ADFHJZ*; then if a straight line starting at the point *Z* shall be made to pass around the polygon, changing its direction at every point where it intercepts the circle whose radius is *five*, so that it shall not at any time pass beyond it, nor at any time come within the circle whose radius is *four*, this straight line, if continued in this manner till it is applied *exactly forty-nine* times, will return to the point *A*. from whence it first started; thus forming a lap equal to the *one-fiftieth of the whole polygon*, and the circle so formed by the intersection of these lines, which will have a radius of *four*, *will be exactly forty-nine fiftieths of the entire polygon, or exactly one-fiftieth less than the polygon contained within the given circle.* See Plate 5 Fig. 2. Therefore $AZ = \frac{1}{50}$th of entire polygon $= \frac{1}{49}$ of the entire circle. *Consequently,*

If from the number of sides of the above polygon $\frac{1}{50}$th be deducted, the square root of the remaining $\frac{49}{50}$ can be extracted exactly; thus $50 - 1 = 49$, and the $\sqrt{49} = 7$; and if from the sum of the squares of the two sides of any square expressed in integers *the one-fiftieth be deducted, the square root of the remaining forty-nine fiftieths can be extracted exactly.*

Demonstration.

Let 5 be the side of the given square $5 \times 5 = 25$, which multiplied by two gives 50, then $50 - 1 = 49$ and the $\sqrt{49} = 7$;

Again, let the side of the given square be 1, then $1 \times 1 = 1$, which doubled, gives 2; the $\frac{1}{50}$th of $2 = \frac{2}{50}$ or .04 and $2.00 - .04 = 1.96$, and the $\sqrt{1.96} = 1.4$ or $1\frac{2}{5} = \frac{7}{5}$ *QED.*

CORALLARY 1. *If the $\frac{1}{9800}$th part of the sums of the squares of the two sides of any square be added thereto, the square root of the sum can be extracted exactly.*

Demonstration.

Let 1 be the side of the given square, then $1 \times 1 = 1$, which multiplied by $2 = 2$, the $\frac{1}{9800}$ of $2 = \frac{2}{9800}$ or $\frac{1}{4900}$; then $2 + \frac{1}{4900} = \frac{9801}{4900}$ and the $\sqrt{\frac{9801}{4900}} = \frac{99}{70}$ *QED.*

Proposition 3. Theorem.

Argument 1. If a circle be described with the square root of two for a radius, and the *one-fiftieth* of the square described upon the radius be deducted therefrom, the square root of the remaining *forty-nine fiftieths* can be extracted exactly.

2. The square root of the $\frac{1}{50}$ so deducted will be the sine of the given arc.

3. The square root of the remaining $\frac{49}{50}$ will be the cosine of the given arc;

For, let the straight line *fh*, *fm*, Plate 6, Fig. 1 and 2 be the given radius, and *ADBC* the given circle, whose arc is *A*, and let the straight lines *gh*, *ge*, *lm*, *ln*, be the sine of the given arc, then will the straight line *fg*, *fl*, be the cosine of the given arc *A*;

Put $R =$ radius of circle, whose arc is A,
Put sin. $=$ sine A,
Put cos. $=$ cosine A.

Then by *trigonometry:*

$$R^2 - \text{sin. } A^2 = \text{cos. } A^2 \text{ and } R^2 - \text{cos. } A^2 = \text{sin. } A^2;$$

Consequently, $\text{cos. } A^2 + \text{sin. } A^2 = R^2$.

Substituting the numbers, we have $(\sqrt{2})^2 = 2$, and $\frac{1}{50}$ of $2 = \frac{2}{50} = \frac{1}{25} = .04$. Then $2.00 - .04 = 1.96$, and the $\sqrt{1.96} = 1.4 = 1\frac{2}{5}$ or $\frac{7}{5}$, which equals the cosine of the given arc, and $2.00 - 1.96 = .04$, and the $\sqrt{.04} = .2$ or $\frac{1}{5}$, which equals the sine of the given arc.

Argument 4. Now, if the cosine of the given arc $\frac{7}{5}$ be multiplied by two, it will be $\frac{14}{5}$, which is very near though not quite the true diameter; and if the sine of the given arc $\frac{1}{5}$ be multiplied by two it will be $\frac{2}{5}$, which is very near though not quite the $\frac{1}{22}$ part of true circumference being the chord of twice the arc, whose sine is very near though not quite the $\frac{1}{44}$ part of the true circumference. If then we assume $\frac{14}{5}$ to be the true diameter and $\frac{44}{5}$ to be the true circumference of the given circle, and they are very near it, then dividing the circumference $\frac{44}{5}$ by the diameter $\frac{14}{5}$, we have, by cancellation, $\frac{44}{5} \div \frac{14}{5}$

PLATE VI.

Fig 2

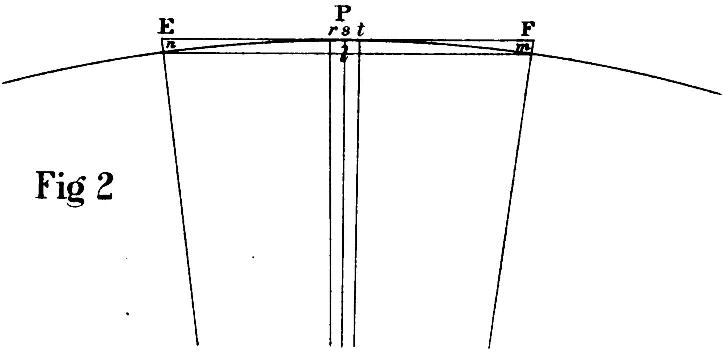
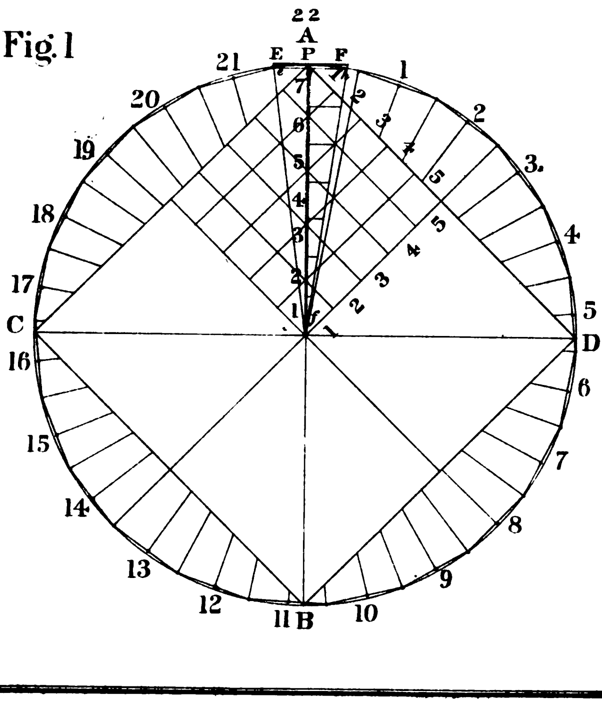


Fig. 1



$= \frac{44}{5} \times \frac{5}{14} = \frac{22}{7}$, which is equal to $3.14285\dot{7}$ or $3\frac{1}{7}$; which is the true ratio of the circumference to the diameter of any given circle.

Consequently,

If the double of the cosine of an arc of any given circle, the sine of which is obtained by extracting the square root of the *one-fiftieth* of the square described upon the radius of the same circle, and the cosine by extracting the square root of the remaining *forty nine fiftieths*, be divided into the sum of the sines, in which case the cos. : sin. : : 7 : 1, then the ratio will be the same as the ratio between the true circumference and the true diameter of the given circle, viz.: $3.\dot{1}4285\dot{7}$ or $3\frac{1}{7}$.

Demonstration.

Let fh, fm, be the radius of the given circle, and gh, ge, or lm, ln, the sine of the given arc A, then will fg, fl, be the cosine of the same arc. See Fig. 1 and 2 Plate 6.

Put $R =$ radius of the given circle
and sin. $=$ sine of the given arc,
and cos. $=$ cosine of the given arc,

then $R^2 - \text{sin.}^2 = \text{cos.}^2$. Consequently the angles fgh, flm, fge, fln, are both right angles, therefore the straight lines hg, ge; ml, ln, are both in the same straight line.

Again, the line gh, lm, is equal to the line ge, ln, each being the sine of the given arc A, and one-half the chord of double the arc A, therefore the straight line he is double the sine of the arc A, for the same reason the chord, which is the base of the double triangle $f1$, is double the sine of the arc A, so also are the chords which are the bases of the double triangles $f2$, $f3$, etc., etc., $f22$ double the sine of the arc A, and each of the chords is assumed to be the $\frac{1}{22}$ part of the whole circumference; also the cosine fg is equal to the cosine $f1$, $f2$, etc., etc., $f22$ and each of them is equal to the one-half of the assumed diameter, therefore the chord $\frac{2}{5} \times 22 = \frac{44}{5} =$ the assumed circumference and the cosine $\frac{7}{5} \times 2 = \frac{14}{5} =$ the assumed diameter. Then, dividing the assumed circumference $\frac{44}{5}$ by the assumed diameter $\frac{14}{5}$, we have by cancellation $\frac{44}{5} \div \frac{14}{5} = \frac{44}{5} \times \frac{5}{14} = \frac{22}{7} = 3.\dot{1}4285\dot{7}$ or $3\frac{1}{7}$ the true ratio of the circumference to the diameter of any given circle.

Consequently, Cor. 1, if the double of the secant be assumed for the true diameter, instead of the double of the cosine, and the sum of the tangents be assumed for the true circumference, instead of the sum of the sines, the ratio will be the same as the ratio between the double of the cosine and the sum of the sines, that is it will be the same as the ratio between the true circumference and true diameter, that is $3.\dot{1}4285\dot{7}$ or $3\frac{1}{7}$.

Proposition 4. Theorem.

To Find a Mean Proportional Between Two Given Straight Lines.

Let *AB*, *BC* be two given straight lines; it is required to find a mean proportional between them. Place *AB*, *BC* in a straight line, and on *AC* describe the semicircle *ADC*; from the point *B* draw *BD* at right angles to *AC*. See Plate 7 Fig. 1. [1.11.]
BD, shall be a mean proportional between *AB* and *BC*.

Join AD, *DC*. Then the angle *ADC* being in a semicircle is a right angle; [*III.* 31,]
and because in the right-angled triangle *ADC*, *DB* is drawn from the right angle perpendicular to the base, therefore *DB* is a mean proportional between *AB*, *BC*, and the segments of the base. [*VI.* 8, *Corollary.*]
Wherefore between the two given straight lines *AB*, *BC*, a mean proportional *DB* is found.

Proposition 5. Theorem.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle and to each other. See Plate 7 *Fig.* 1.

Let *DAC* be a right-angled triangle, having the right angle *ADC*, and from the point *D* let *DB* be drawn perpendicular to the base *AC*, the triangles *BAD*, *DBC*, shall be similar to the whole triangle *DAC* and to one another.

For, the angle *ADC* is equal to the angle *ABD*, each of them being a right angle, [*Axiom* 11.]
and the angle at *A* is common to the two triangles *DAC*, *BAD*, therefore the remaining angle *DCA* is equal to the remaining angle *BDA*; therefore the triangle *DAC* is equiangular to the triangle *BAD*, and the sides about their equal angles proportionals; therefore the triangles are similar. VI. 4. [*Definition* 1.]

PLATE VII.

Fig. 1

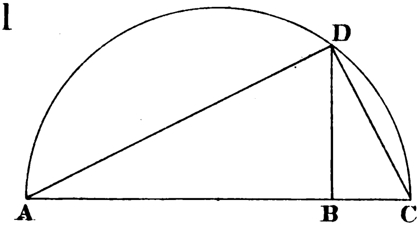


Fig. 2

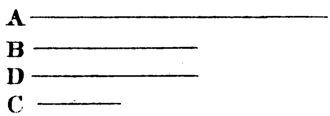


Fig. 4

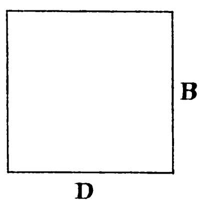
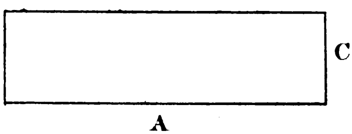


Fig. 3



In the same manner it may be shown that the triangle *BAD* to the triangle *DAC* are similar to each other.

Wherefore in a right-angled triangle, etc., *QED.*

Corallary from this it is manifest that the perpendicular drawn from the right-angle of a right-angled triangle to the base, is a mean proportional between the segments of the base, and also that each of the sides is a mean proportional between the base and the segment of the base adjacent to that side.

For in the triangles *BAD*, *BDC*, *AB* is to *BD* as *BD* is to *BC*; [*VI.* 4.]

And in the triangles *DAC*, *BAD*, *AC* is to *AD* as to *AD* is to *AB*; [*VI.* 4.]

and in the triangles *DAC*, *BDC*, *AC* is to *CD* as *CD* is to *CB*. [*VI.* 4.]

PROPOSITION 6. THEOREM.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square on the mean; and if the rectangle contained by the extremes is equal to the square on the mean, the three straight lines are proportionals.

See Plate 7 *Fig.* 2.

Let the three straight lines *ABC* be proportionals, namely: let *A* be to *B* as *B* is to *C*; the rectangle contained by *A* and *C* shall be equal to the square on *B*. Take *D* equal to *B*. Then, because *A* is to *B* as *B* is to *C*, [*Hypothesis.*]
and that *B* is equal to *D*; therefore *A* is to *B* as *D* is to *C*. [*V.* 7.]
But if four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; [*VI.* 16.]
therefore the rectangle contained by *A* and *C* is equal to the rectangle contained by *B* and *D*. See Plate 7 Fig. 3 and 4. But the rectangle by *B* and *D* is the square on *B*, because *B* is equal to D; [*Construction.*]
therefore the rectangle contained by *A* and *C* is equal to the square on *B*. Next, let the rectangle contained by *A* and *C* be equal to the square on *B*; *A* shall be to *B* as *B* is to *C*. For, let the same construction be made, then, because the rectangle contained by *A* and *C* is equal to the square on *B*, [*Hypothesis.*]
and that the square on *B* is equal to the rectangle contained by *B* and *D*, because *B* is equal to *D*;

therefore the rectangle contained by A and C is equal to the rectangle contained by B and D; but if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals; [*VI.* 16.]
therefore A is to B as D is to C.
But B is equal to D; [*Construction.*]
Therefore A is to B as B is to C. [*V.* 7.]
Wherefore if three straight lines, etc., *QED.*

QUADRATURE OF THE CIRCLE.

Case 1.

In which the Inscribed and Circumscribed Polygons are Carried to 22 Sides.

It is required to find the quadrature of the circle, that is when the radius is known it is required to find a straight line (in terms of the given radius) which shall be equal in length to the circumference of the given circle. [*Postulate* 5.]
For this purpose let the square root of two be taken for the given radius fh, fm, and with this radius describe the given circle $ADBC$. [*Postulate* 3.]
See Plate 6 Fig. 1 and 2. Then let the square $ABCD$ be inscribed in the given circle, the side of the inscribed square will be *two* and the area *four*. [*Hypothesis.*]
For if each of the sides of the inscribed square $ABCD$ be bisected the whole square will be divided into four smaller squares, each of the sides of which as well as their respective areas will be equal to one, consequently they will be equal to one another. [*Axiom* 1.]

Then, by (Proposition 1, Cor.), the diagonals of each of the smaller squares will be equal to the square root of two. But the radius of the given circle is equal to the square root of two, consequently the diagonals of each of the smaller inscribed squares is equal to the radius of the given circle. [*Axiom* 1.]
Again, all the radii are equal and each is half the diameter; [*Definition* 3.]
therefore the diameter of the given circle is twice the square root of two, or the square root of eight, and it is equal to the diagonal of the square whose side is equal to the side of the inscribed square. For the half of the diameter of the given circle is the $\sqrt{2}$, and the half of

the side of the inscribed square is 1, consequently, by (Theorem 1, Cor.), the side of any square is to its diagonal as $1 : \sqrt{2}$; [*Axiom* 7.] therefore the one half of the side of the inscribed square is the side of a square whose diagonal is the square root of two, which is equal to the radius of the circle or half the diameter.

Now, by (PROPOSITION 2), the sine of the given arc is the square root of the $\frac{1}{50}$th of the square described upon the radius, and the cosine of the given arc is the square root of the remaining $\frac{49}{50}$. For, by *trigonometry*, $R^2 - \sin.^2 = \cos.^2$, and $R^2 - \cos.^2 = \sin.^2 =$ consequently $\cos.^2 + \sin.^2 = R^2$. Substituting the numbers we have $(\sqrt{2})^2 = 2.00$ and $\frac{1}{50}$ of $2 = \frac{2}{50} = \frac{1}{25} = .04 =$ and the $\sqrt{.04} = 2$ or $\frac{1}{5} =$ sine of the given arc; and $2.00 - .04 = 1.96$ and the $\sqrt{1.96} = 1.4 = 1\frac{2}{5}$ or $\frac{7}{5} =$ cosine of the given arc.

Again, if the side of the small inscribed square, which is equal to 1, be divided into five equal parts each part is equal to $\frac{1}{5}$, which is equal to the sine of the given arc; and seven of the same parts is equal to $\frac{7}{5}$, which is equal to the cosine of the given arc; consequently $(\frac{1}{5})^2 = \frac{1}{25}$ and $(\frac{7}{5})^2 = \frac{49}{25}$, then $\frac{1}{25} + \frac{49}{25} = \frac{50}{25}$ which is equal to the square described upon the radius; and the $\sqrt{\frac{50}{25}} = \sqrt{2}$ which is equal to the radius. Now, by Theorem 3, the sine $\frac{1}{5}$ is not quite though very near the $\frac{1}{44}$th part of the entire circumference; and the cosine $\frac{7}{5}$ is not quite though very near $\frac{1}{2}$ the entire diameter; therefore $\frac{1}{5} \times 44 = \frac{44}{5} =$ the assumed circumference; and $\frac{7}{5} \times 2 = \frac{14}{5} =$ the assumed diameter; then, dividing the assumed circumference by the assumed diameter, we have by cancellation $\frac{44}{5} \div \frac{14}{5} = \frac{44}{5} \times \frac{5}{14} = \frac{22}{7} = 3.\dot{1}4285\dot{7}$ or $3\frac{1}{7} =$ the true ratio between the assumed circumference and the assumed diameter of the given circle.

Again, the cosine of the given arc is $\frac{7}{5}$, and the sine of the given arc is $\frac{1}{5}$, then, *by trigonometry*, cos. : sine : : R : tang. Substituting the numbers we have cos. $\frac{7}{5}$: sin. $\frac{1}{5}$: : radius $\sqrt{2}$: : tan. $\frac{1}{7}\sqrt{2}$, for $\frac{1}{5} \times \sqrt{2} = \frac{1}{5}\sqrt{2}$; then, by division and cancellation, $\frac{1}{5}\sqrt{2} \div \frac{7}{5} = \frac{1}{5}\sqrt{2} \times \frac{5}{7} = \frac{1}{7}\sqrt{2} =$ tangent of the given arc.

Again, by *trigonometry*, $R^2 + \tan.^2 = \sec.^2$. Substituting the numbers we have rad. $(\sqrt{2})^2 = 2$ and tang. $(\frac{1}{7}\sqrt{2})^2 = \frac{2}{49}$, then $2 + \frac{2}{49} = \frac{100}{49}$.

and the $\sqrt{\frac{100}{49}} = \frac{10}{7} = 1.4142857$, or $= 1.4\frac{1}{7}$ = secant of the given arc. $\frac{7}{5} = 1.4$ = cosine of the given arc. $\frac{1}{5} = .2$ = sine of the given arc. $\frac{1}{7}\sqrt{2} = .2020305089$ = tangent of the given arc.

Again, the side of the inscribed square being divided into five parts, by applying the sine to it as a common measure each part is expressed by $\frac{1}{5}$; then $\frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$ = the size of the square which is to be the first unit of superficial measure for the circle.

For, by ARTICLE 6, the area of the circumscribed double triangle *fPE*, *f PE* is equal to the product of the radius and the tangent; thus, radius $\sqrt{2} \times$ tangent $\frac{1}{7}\sqrt{2} = \frac{2}{7}$; and, by ARTICLE 9, the area of the entire circumscribed polygon is equal to the product of the circumscribed double triangle, and the number of sides contained in the given polygon; thus $\frac{2}{7} \times 22 = \frac{44}{7} = 6\frac{2}{7}$ the area required.

Now, $\frac{44}{7} \div \frac{1}{25} = \frac{44}{7} \times \frac{25}{1} = \frac{1100}{7} = 157\frac{1}{7}$; therefore the area of the entire circumscribed polygon is equal to $\frac{157\frac{1}{7}}{25} = \frac{1100}{175} = \frac{44}{7} = 6\frac{1}{7}$; that is, there is $157\frac{1}{7}$ blocks each of which is equal in area to $\frac{1}{25}$, *QED.*

CASE 2.

In which the Inscribed and Circumscribed Polygons are Carried to $311\frac{1}{7}$ *Sides.*

By *Case* 1, the sine is $\frac{1}{5}$ and the double of the cosine is $\frac{14}{5}$, and, by ARTICLE 3, the sine is a mean proportional between the double of the cosine and the tangent No. 2. For, by (PROPOSITIONS 4 and 5), sin. $(\frac{1}{5})^2 = \frac{1}{25}$, and cos. $\frac{7}{5} \times 2 = \frac{14}{5}$, then, by division and cancellation, $\frac{1}{25} \div \frac{14}{5} = \frac{1}{25} \times \frac{5}{14} = \frac{1}{70}$ = tangent No. 2. (By PROPOSITION 6), the rectangle contained by the double of the cosine and the tangent No. 2 is equal to the rectangle contained by the square described upon the sine of the given arc.

For put double cosine $= A$
and put sine $= B$
and put tangent No. 2 $= C$

Then, by proportion, we have $A : B :: B : C$; then $A \times C = B \times B = (A \times C = B^2)$, consequently, $B^2 \div A = C$. Substituting the numbers we have by division and cancellation $\frac{14}{5} \times \frac{1}{70} = \frac{1}{25}$ and $(\frac{1}{5})^2 = \frac{1}{25}$; consequently $\frac{1}{25} \div \frac{14}{5} = \frac{1}{25} \times \frac{5}{14} = \frac{1}{70}$ = tangent No. 2.

By *trigonometry*, $R^2 +$ tang.$^2 =$ sec.^{2}. Substituting the numbers, we have $R\,(\sqrt{2})^2 = 2$; $T(\frac{1}{70})^2 = \frac{1}{4900}$. Then, $2 + \frac{1}{4900} = \frac{9801}{4900}$ = secant sequare; extracting the $\sqrt{\frac{9801}{4900}}$ we have $\frac{99}{70}$ = secant No. 2.

By *Case* 1, the cosine $\frac{7}{5} = \frac{98}{70}$; and, by *Case* 2, the secant No. 2 = $\frac{99}{70}$; then $\frac{99}{70} - \frac{98}{70} = \frac{1}{70}$; therefore $\frac{1}{70}$ has been added to the cosine, the assumed radius; and $\frac{2}{70}$ to the assumed diameter. Now, by hypothesis, the circumference of a circle is to the diameter as $3\frac{1}{7}$ is to 1; therefore $3\frac{1}{7}$ times $\frac{2}{70}$ must be added to the assumed circumference; thus, $\frac{2}{70} \times 3\frac{1}{7} = \frac{6\frac{2}{7}}{70}$.

By *Case* 1, the sine of the given arc is $\frac{1}{5}$; and, by *Case* 2, the tangent of the given arc is $\frac{1}{70}$; then, dividing the sine by the tangent, we have by cancellation $\frac{1}{5} \div \frac{1}{70} = \frac{1}{5} \times \frac{70}{1} = 14$. Therefore the sine has been divided into 14 equal parts, each of which is equal to the tangent of $\frac{1}{70}$. Now, by *Case* 1, there were 22 sides to the given polygon, each of which contained two sines, $\frac{2}{5} \times 22 = \frac{44}{5} = \frac{616}{70}$. Now, by hypothesis, $\frac{6\frac{2}{7}}{70}$ must be added to the circumference to complete the polygon; then $\frac{616}{70} + \frac{6\frac{2}{7}}{70} = \frac{622\frac{2}{7}}{70}$ = the circumference of the polygon No. 2.

By ARTICLE 1, the double of the secant is the constantly assumed diameter, and, by ARTICLE 2, the sum of the sines or tangents is the constantly assumed circumference.

By *Case* 2, the secant is $\frac{99}{70}$, then $\frac{99}{70} \times 2 = \frac{198}{70}$; = the assumed diameter.

Then, dividing the assumed circumference by the assumed diameter, we have by cancellation $\frac{622\frac{1}{7}}{70} \div \frac{198}{70} = \frac{622\frac{1}{7}}{70} \times \frac{70}{198} = \frac{622\frac{2}{7}}{198} = \frac{4356}{1386}$ $= \frac{22}{7} = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, the true ratio between circumference and diameter of the circle.

Then, by *trigonometry*, sec.^{2} : tang.^{2} : : R^2 : sin.^{2}.

By *Case* 2, the secant is $\frac{99}{70}$; the tangent $\frac{1}{70}$; and the radius $\sqrt{2}$.

Then, $(\frac{99}{70})^2 = \frac{9801}{4900}$; and $(\frac{1}{70})^2 = \frac{1}{4900}$; and $(\sqrt{2})^2 = 2$; and by proportion $\frac{9801}{4900}$: $\frac{1}{4900}$: : 2 : : $\frac{2}{9801}$.

For, $\frac{1}{4900} \times 2 = \frac{2}{4900}$; then, by division and cancellation, we have $\frac{2}{4900} \div \frac{9801}{4900} = \frac{2}{4900} \times \frac{4900}{9801} = \frac{2}{9801}$; extracting the square root we have $\sqrt{\frac{2}{9801}} = \frac{1}{99}\sqrt{2}$ = sine No. 2; again, by *trigonometry*, R^2 — sin.^{2} = cos.^{2}.

Let $\frac{19602}{9801} = R^2$; and $\frac{2}{9801}$ = sin.^{2}.

Then, by subtraction, $\frac{19602}{9801} - \frac{2}{9801} = \frac{19600}{9801}$; extracting the square root $\sqrt{\frac{19600}{9801}}$ we have $\frac{140}{99}$ = cosine No, 2.

Now, by **Article** 5, *the rectangle contained by the sine and the cosine, is equal to area of the inscribed double triangle:*

$$\text{Thus } \frac{140}{99} \times \frac{1}{99}\sqrt{2} = \frac{140}{9801}\sqrt{2}.$$

And, by **Article** 6, *the rectangle contained by the radius and the tangent, is equal to the area of the circumscribed double triangle;*

$$\text{Thus } \sqrt{2} \times \frac{1}{70} = \frac{1}{70}\sqrt{2}.$$

Then, by **Article** 7, *the rectangle contained by the radius and the sine, is a mean proportional between the inscribed and circumscribed double triangles of half the number of sides;*

Thus $\frac{140}{9801}\sqrt{2} \times \frac{1}{70}\sqrt{2}$ = by cancellation, $= \frac{4}{9801}$; extracting the square root $\sqrt{\frac{4}{9801}}$ we have $\frac{2}{99}$, the area of the inscribed double triangle for double the number of sides.

And, again, $\sqrt{2} \times \frac{1}{99}\sqrt{2} = \frac{2}{99}$ = area of inscribed double triangle.

By Article 10, *the rectangle contained by the rectangle of the radius and the sine, and the number of sides contained in the given polygon; is equal to the area of the entire inscribed polygon of double the number of sides.*

For, by *Case* 2, the number of sides contained by the given polygon is $311\frac{1}{7}$, and the rectangle of the radius and the sine is $\frac{2}{99}$; then $\frac{2}{99} \times 311\frac{1}{7} = \frac{622\frac{2}{7}}{99} = \frac{4356}{693} = \frac{44}{7}$; and $(\sqrt{2})^2 = 2$; then $\frac{44}{7} \div 2 = \frac{22}{7} =$ $3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, which is the true ratio of the circumference to the diameter of the given circle.

Again, by Article 8, *the rectangle contained by the inscribed double triangle, and the number of sides is equal to the area of the entire inscribed polygon.* For $\frac{140}{9801}\sqrt{2} \times 311\frac{1}{7} = \frac{43560}{9801}$ = area of the inscribed polygon.

And, by Article 9, *the rectangle contained by the double circumscribed triangle and the number of sides, is equal to the area of the entire circumscribed polygon.* For $\frac{1}{70}\sqrt{2} \times 311\frac{1}{7} = \frac{311\frac{1}{7}}{70}\sqrt{2} = \frac{2178}{490}\sqrt{2}$.

Now the area of circumscribed polygon $\frac{2178}{490}\sqrt{2} = \frac{21346578}{4802490}\sqrt{2}$;

Again, the area of inscribed polygon $\frac{43560}{9801}\sqrt{2} = \frac{21344400}{4802490}\sqrt{2}$;

Then $\frac{21346578}{4802490}\sqrt{2}, - \frac{21344400}{4802490}\sqrt{2}, = \frac{2178}{4802490} = \frac{1}{2205}\sqrt{2}$.

Which is the difference between the areas of the circumscribed and inscribed polygons of the given circle whose radius is the square root of 2.

SUMMARY, CASE 2.

Tangent of the given arc $\frac{1}{70}$ $=$ .0142857 +.
Sine of the given arc $\frac{1}{99}\sqrt{2} =$ .0142849 +.
Cosine of the given arc $\frac{140}{99}\sqrt{2} = 1.4141414$ +.
Radius of the given arc $\sqrt{2} = 1.4142135$ +.
Secant of the given arc $\frac{99}{70}$ $= 1.4142857$ +.

By *Case* 2, the tangent is $\frac{1}{70}$; consequently, the square which forms the unit of comparison is $(\frac{1}{70})^2 = \frac{1}{4900}$.

Again, by *Case* 2, the rectangle contained by the radius and the sine is $\frac{2}{99}$; and the number of sides $311\frac{1}{7}$; therefore $\frac{2}{99} \times 311\frac{1}{7} = \frac{622\frac{2}{7}}{99}$ $= \frac{4356}{693} = \frac{44}{7} =$ area of the inscribed polygon. Now, $\frac{44}{7} \div \frac{1}{4900} =$ $\frac{44}{7} \times \frac{4900}{1} = \frac{215600}{7} = 30800$; therefore there are exactly 30800 squares in the given polygon, each of which is expressed by $\frac{1}{4900}$; then $\frac{1}{4900} \times 30800 = \frac{30800}{4900} = \frac{44}{7} = 6\frac{2}{7} =$ area of the inscribed polygon.

CASE 3.

In which the Inscribed and Circumscribed Polygons are Carried to $61606\frac{2}{7} - 3\frac{1}{7} = 61603\frac{1}{7}$ *Sides.*

By *Case* 1, the tangent is $\frac{1}{70}$; and the secant is $\frac{99}{70}$. Then, by ARTICLE 3 and (PROPOSITIONS 4 and 5), tangent $(\frac{1}{70})^2$ $\frac{1}{4900}$ and sec. $(\frac{99}{70})$ $\times 2 = \frac{198}{70}$. Then, by division and cancellation, $\frac{1}{4900} \div \frac{198}{70} = \frac{1}{4900} \times \frac{70}{198} = \frac{1}{13860} =$ tangent No. 3.

By (PROPOSITION 6) and ARTICLE 4, *the rectangle contained by the double of the secant No. 2 and tangent No. 3, is equal to the rectangle contained by the square described on tangent No. 2.*

For, put double of secant $= A$;
and tangent No. 2 $= B$;
and tangent No. 3 $= C$.

Then, by proportion, we have $A : B :: B : C$; then $A \times C = B \times B = (A \times C = B^2)$. Substituting the numbers we have, by division and cancellation, $\frac{198}{70} \times \frac{1}{13860} = \frac{1}{4900}$; and $(\frac{1}{70})^2 \times \frac{1}{4900}$.

By *trigonometry*, $R^2 + \tan.^2 = \sec.^2$, substituting the numbers we have $R(\sqrt{2})^2 = 2$; $T(\frac{1}{13860})^2 \frac{1}{192099600}$; then, $2 + \frac{1}{192099600} = \frac{384199201}{192099600}$, extracting the square root, we have $\sqrt{\frac{384199201}{192099600}} = \frac{19601}{13860}$ = secant No. 3.

By *Case* 2, secant No. $2 = \frac{99}{70} = \frac{19602}{13860}$; and by *Case* 3, secant No. 3, $= \frac{19601}{13860}$; then $\frac{19602}{13860} - \frac{19601}{13860} = \frac{1}{13860}$; therefore $\frac{1}{13860}$ has been subtracted from secant No. 2, the assumed radius; and $\frac{2}{13860}$ from the assumed diameter. Now, by hypothesis, the circumference of a circle is to the diameter as $3\frac{1}{7}$ is to 1; therefore $3\frac{1}{7}$ times $\frac{2}{13860}$ or $\frac{6\frac{2}{7}}{13860}$ must be subtracted from the circumference.

By *Case* 2, the tangent of the given arc is $\frac{1}{70}$, and, by *Case* 3, the tangent of the given arc is $\frac{1}{13860}$; then dividing tangent No. 2 by tangent No. 3 we have, by cancellation, $\frac{1}{70} \div \frac{1}{13860} = \frac{1}{70} \times \frac{13860}{1} =$ 198. Therefore the tangent No. 2 has been divided into 198 equal parts, each of which is equal to the tangent No. 3, or $\frac{1}{13860}$.

By *Case* 2, the given polygon was carried to $311\frac{1}{7}$ sides, each of which contained two tangents each of $\frac{1}{70}$; then $\frac{2}{70} \times 311\frac{1}{7} = \frac{622\frac{2}{7}}{70} = \frac{123212\frac{4}{7}}{13860}$, and, by hypothesis, $\frac{6\frac{2}{7}}{13860}$ must be subtracted from the circumference to complete the given polygon. Then $\frac{123212\frac{4}{7}}{13860} - \frac{6\frac{2}{7}}{13860} = \frac{123206\frac{2}{7}}{13860}$ = the circumference of the polygon No. 3.

By Article 1, *the double of the secant is the constantly assumed diameter; and, by* Article 2, *the sum of the tangents is the constantly assumed circumference.*

By *Case* 3, the secant No. 3 is $\frac{19601}{13860}$; then $\frac{19601}{13860}\times 2=\frac{39202}{13860}=$ the assumed diameter.

Then, dividing the assumed circumference by the assumed diameter, we have by cancellation $\frac{123206\frac{2}{7}}{13860}\div\frac{39202}{13860}=\frac{123206\frac{2}{7}}{13860}\times\frac{13860}{39202}=\frac{123206\frac{2}{7}}{39202}=\frac{862444}{274414}=\frac{22}{7}=3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$ the true ratio between the circumference and diameter of the given circle.

Now, by *trigonometry*, sec.^{2} : tang.^{2} : : R^2 : sin.^{2}, by *Case* 3, the secant is $\frac{19601}{13860}$, the tangent $\frac{1}{13860}$, and the radius $\sqrt{2}$.

For $\left(\frac{19601}{13860}\right)^2=\frac{384199201}{192099600}$ and $\left(\frac{1}{13860}\right)^2=\frac{1}{192099600}$ and $(\sqrt{2})^2=2$.

By proportion we have $\frac{384199201}{192099600}:\frac{1}{192099600}::2:\frac{2}{384199201}$.

For, by division and cancellation, we have $\frac{2}{192099600}\div\frac{384199201}{192099600}=\frac{2}{192099600}\times\frac{192099600}{384199201}=\frac{2}{384199201}$; extracting the square root, we have $\sqrt{\frac{2}{384199201}}=\frac{1}{19601}\sqrt{2}=$ sine No. 3.

Again, by *trigonometry*, $R^2-\text{sin.}^2=\text{cos.}^2$.

Let $\frac{768398402}{384199201}=R^2$ and $\frac{2}{384199201}=\text{sin.}^2$.

Then, by subtraction, $\frac{768398402}{384199201}-\frac{2}{384199201}=\frac{768398400}{384199201}$; extracting the square root, $\sqrt{\frac{768398400}{384199201}}$, we have $\frac{27720}{19601}$=cosine No. 3.

Now, by Article 5, *the rectangle contained by the sine and the cosine, is equal to the area of the inscribed double triangle;* thus $\frac{27720}{19601}\times\frac{1}{19601}\sqrt{2}=\frac{27720}{384199201}\sqrt{2}$.

And, by ARTICLE 6, *the rectangle contained by the radius and the tangent, is equal to the area of the circumscribed double triangle;* thus $\sqrt{2} \times \frac{1}{13860} = \frac{1}{13860}\sqrt{2}$.

Then, by ARTICLE 7, *the rectangle contained by the radius and the sine, is a mean proportional between the inscribed and circumscribed double triangles of half the number of sides;* thus, by cancellation $\frac{27720}{384199201}\sqrt{2} \times \frac{1}{13860}\sqrt{2}$, we have $\frac{4}{384199201}$; extracting the square root, $\sqrt{\frac{4}{384199201}}$, we have $\frac{2}{19601}$.

And, again, $\sqrt{2} \times \frac{1}{19601}\sqrt{2} = \frac{2}{19601}$ = area of inscribed double triangle for half the number of sides.

By ARTICLE 10, *the rectangle contained by the rectangle of the radius and the sine, and the number of sides contained in the given polygon; is equal to the area of the entire inscribed polygon of double the number of sides.*

For, by *Case* 3, the number of sides contained by the given polygon is $61603\frac{1}{7}$, and the rectangle of the radius and sine is $\frac{2}{19601}$; then $\frac{2}{19601} \times 61603\frac{1}{7} = \frac{123206\frac{2}{7}}{19601} = \frac{862444}{137207} = \frac{44}{7}$; and $(\sqrt{2})^2 = 2$; then $\frac{44}{7} \div 2 = \frac{22}{7} = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, which is the true ratio of the circumference to the diameter of the given circle.

Again, by ARTICLE 8, *the rectangle contained by the inscribed double triangle and the number of sides, is equal to the area of the entire inscribed polygon;* then $\frac{27720}{384199201}\sqrt{2} \times 61603\frac{1}{7} = \frac{1707639120}{384199201}\sqrt{2}$.

And, by ARTICLE 9, *the rectangle contained by the circumscribed double triangle and the number of sides, is equal to the area of the entire circumscribed polygon;* thus $\frac{1}{13860}\sqrt{2} \times 61603\frac{1}{7} = \frac{431222}{97020}\sqrt{2}$.

Now, the area of the circumscribed polygon is $\frac{431222}{97020}\sqrt{2} = \frac{165675147853622}{37275006481020}$.

Again, the area of the inscribed polygon is $\frac{1707639120}{384199201}\sqrt{2} = \frac{165675147422400}{372750064810 20}$.

Then, $\frac{165675147853622}{37275006481020}\sqrt{2} - \frac{165675147422400}{37275006481020}\sqrt{2} = \frac{431222}{37275006481020}\sqrt{2} = \frac{1}{86440410}\sqrt{2}$, which is the difference between the areas of the circumscribed and inscribed polygons of the given circle, whose radius is the square root of two.

Summary of Case 3.

Tangent No. 3 $= \frac{1}{13860} =$ .00007215007215.

Sine No. 3 $= \frac{1}{19601}\sqrt{2} =$ .00007215007205.

Cosine No. 3 $= \frac{27720}{19601}$ = 1.41421356237309504.

Radius No. 3 $= \sqrt{2}$ = 1.41421356237309504.

Secant No. 3 $= \frac{19601}{13860}$ = 1.41421356237309504.

By *Case* 3, the tangent is $\frac{1}{13860}$, consequently the square which forms the unit of comparison is $\left(\frac{1}{13860}\right)^2 = \frac{1}{192099600}$

Again, by *Case* 3, the rectangle contained by the radius and the sine is $\frac{2}{19601}$ and the number of sides $61603\frac{1}{7}$; therefore $\frac{2}{19601} \times 61603\frac{1}{7} = \frac{123206\frac{2}{7}}{19601} = \frac{862444}{137207} = \frac{44}{7} =$ area of the inscribed polygon.

Now, $\frac{44}{7} \div \frac{1}{192099600} = \frac{44}{7} \times \frac{192099600}{1} = \frac{8452382400}{7} = 1207483200$; therefore there are exactly 1207483200 squares in the given polygon, each of which is expressed by $\frac{1}{192099600}$; then $\frac{1}{192099600} \times 1207483200 = \frac{1207483200}{192099600} = \frac{44}{7} = 6\frac{2}{7} =$ area of inscribed polygon for double the number of sides.

Case 4.

In which the Inscribed and Circumscribed Polygons are Carried to $24149664 06\frac{4}{7} - 3\frac{1}{7} = 2414966403\frac{1}{7}$ *Sides.*

By *Case* 3, the tangent is $\frac{1}{13860}$ and the secant is $\frac{19601}{13860}$. And by Article 3 and (Propositions 4 and 5), tang. $\left(\frac{1}{13860}\right)^2 = \frac{1}{192099600}$ and sec. $\frac{19601}{13860} \times 2 = \frac{39202}{13860}$; then, by division and cancellation, we have $\frac{1}{192099600} \div \frac{39202}{13860} = \frac{1}{192099600} \times \frac{13860}{39202} = \frac{1}{543339720} =$ tangent No. 4.

By (Proposition 6) *and* Article 4, *the rectangle contained by the double of the secant No.* 3 *and tangent No.* 4 *is equal to the rectangle contained by the square described in tangent No.* 3.

For, put double of secant $= A$;
and tangent No. 3 $= B$;
and tangent No. 4 $= C$;
Then, by proportion, we have $A : B :: B : C$:; then $A \times C = B \times B = (A \times C = B^2)$.

Substituting the numbers, we have, by division and cancellation, $\frac{39202}{13860} \times \frac{1}{543339720} = \frac{1}{192099600}$ and $\left(\frac{1}{13860}\right)^2 = \frac{1}{192099600}$.

By trigonometry, $R^2 + \text{tang.}^2 = \text{sec.}^2$, substituting the numbers, we have rad. $(\sqrt{2})^2 = 2$. tang. $\left(\frac{1}{543339720}\right)^2 = \frac{1}{295218051329678400}$; then $2 + \frac{1}{295218051329678400} = \frac{590436102659356801}{295218051329678400}$; extracting the square root, we have $\sqrt{\frac{590436102659356801}{295218051329678400}} = \frac{768398401}{543339720} =$ secant No. 4.

By *Case* 3, sec. No. $3 = \frac{19601}{13860} = \frac{768398402}{543339720}$; and, by *Case* 4, sec. No. 4, $= \frac{768398401}{543339720}$; then $\frac{768398402}{543339720} - \frac{768398401}{543339720} = \frac{1}{543339720}$;

therefore $\frac{1}{543339720}$ has been subtracted from the assumed radius, and $\frac{2}{543339720}$ from the assumed diameter.

Now, by hypothesis, the circumference of a circle is to the diameter as $3\frac{1}{7}$ is to 1; therefore $3\frac{1}{7}$ times $\frac{2}{543339720}$, or $\frac{6\frac{2}{7}}{543339720}$, must be subtracted from the circumference.

By *Case* 3, the tangent of the given arc is $\frac{1}{13860}$, and, by *Case* 4, the tangent of the given arc is $\frac{1}{543339720}$; then, dividing the tangent No. 3 by the tangent No. 4, we have by cancellation $\frac{1}{13860} \div \frac{1}{543339720}$ $= \frac{1}{\cancel{13860}} \times \frac{\cancel{543339720}}{1} = 39202$; therefore the tangent No. 3 has been divided into 39202 equal parts, each of which is equal to the tangent No. 4, or $\frac{1}{543339720}$.

By *Case* 3, the given polygon was carried to $61603\frac{1}{7}$ sides, each of which contained two tangents, and each tangent was $\frac{1}{13860}$; then $\frac{2}{13860} \times 61603\frac{1}{7} = \frac{123206\frac{2}{7}}{13860} = \frac{48299328124\frac{4}{7}}{543339720}$; and, by hypothesis, $\frac{6\frac{2}{7}}{543339720}$ must be subtracted from the circumference to complete the polygon; then $\frac{48299328124\frac{4}{7}}{543339720} - \frac{6\frac{2}{7}}{543339720} = \frac{48299328806\frac{2}{7}}{543339720} =$ the circumference of the polygon No. 4.

By ARTICLE 1, *the double of the secant is the constantly assumed diameter; and, by* ARTICLE 2, *the sum of the tangents is the constantly assumed circumference.*

By *Case* 4, the secant No. 4 is $\frac{768398401}{543339720}$; then $\frac{768398401}{543339720} \times 2 =$ $\frac{1536796802}{543339720} =$ the assumed diameter.

Then, dividing the assumed circumference by the assumed diameter, we have by cancellation $\frac{48299328806\frac{2}{7}}{543339720} \div \frac{1536796802}{543339720} = \frac{48299328806\frac{2}{7}}{\cancel{543339720}}$

$\times \frac{543339720}{1536796802} = \frac{48299328062\frac{2}{7}}{1536796802} = \frac{33809529644}{10757577614} = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7} =$ the true ratio between the circumference and diameter of the given circle.

Now, by *trigonometry*, sec.^{2} : tang.^{2} : : R^2 : S^2, and, by *Case* 4, the secant is $\frac{768398401}{543339720}$ and the tangent is $\frac{1}{543339720}$ and the radius is $\sqrt{2}$.

For $\left(\frac{768398401}{543339720}\right)^2 = \frac{590436102659356801}{295218051329678400}$ and $\left(\frac{1}{543339720}\right)^2 = \frac{1}{295218051329678400}$ and $(\sqrt{2})^2 = 2$.

Then, by proportion, we have $\frac{590436102659356801}{295218051329678400} : \frac{1}{295218051329678400} : : 2 : \frac{2}{590436102659356801}$.

For, by division and cancellation, we have $\frac{2}{295218051329678400} \div \frac{590436102659356801}{295218051329678400} = \frac{2}{295218051329678400} \times \frac{295218051329678400}{590436102659356801} = \frac{2}{590436102659356801}$; extracting the square root, $\sqrt{\frac{2}{590436102659356801}}$, we have $\frac{1}{768398401}\sqrt{2} =$ sine No. 4.

Again, by *trigonometry*, R^2 — sin.^{2} = cos.^{2}.

Let $\frac{1180872205318713602}{590436102659356801} = R^2$; and $\frac{2}{590436102659356801} =$ sin.^{2}.

Then, by subtraction, $\frac{1180872205318713602}{590436102659356801} - \frac{2}{590436102659356801} = \frac{1180872205318713600}{590436102659356801}$; then, extracting the square root, $\sqrt{\frac{1180872205318713600}{590436102659356801}}$, we have $\frac{1086679440}{768398401} =$ cosine No. 4.

Now, by ARTICLE 5, *the rectangle contained by the sine and the cosine is equal to the area of the inscribed double triangle; thus* $\frac{1086679440}{768398401} \times \frac{1}{768398401}\sqrt{2} = \frac{1086679440}{590436102659356801}\sqrt{2}$.

And, by ARTICLE 6, *the rectangle contained by the radius and the tangent is equal to the area of the circumscribed double triangle; thus* $\sqrt{2} \times \frac{1}{543339720} = \frac{1}{543339720}\sqrt{2}$.

Then, by ARTICLE 7, *the rectangle contained by the radius and the sine, is a mean proportional between the inscribed and circumscribed double triangles of half the number of sides;* thus, by cancellation, we have $\frac{\cancel{1086679440}}{590436102659356801}\sqrt{2} \times \frac{1}{\cancel{543339720}}\sqrt{2} = \frac{4}{590436102659356801}$; extracting the square root, $\sqrt{\frac{4}{590436102659356801}}$ we have $\frac{2}{768398401}$.

And, again, $\sqrt{2} \times \frac{1}{768398401}\sqrt{2} = \frac{2}{768398401}$ = the area of the inscribed double triangle for half the number of sides.

By ARTICLE 10, *the rectangle contained by the rectangle of the radius and the sine, and the number of sides contained in the given polygon; is equal to the area of the inscribed double triangle of double the number of sides.*

For, by *Case* 4, the number of sides contained by the given polygon is $2414966403\frac{1}{7}$; and the rectangle of the radius and the sine is $\frac{2}{768398401}$; then $\frac{2}{768398401} \times 2414966403\frac{1}{7} = \frac{4829932806\frac{2}{7}}{768398401} = \frac{33809529644}{5378788807} = \frac{44}{7}$, and $(\sqrt{2})^2 = 2$; then $\frac{44}{7} \div 2 = \frac{22}{7} = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$; which is the true ratio of the circumference to the diameter of the given circle.

Again, by ARTICLE 8, *the rectangle contained by the inscribed double triangle, and the number of sides is equal to the area of the entire inscribed polygon;* then $\frac{1086679440}{590436102659356801}\sqrt{2} \times 2414966403\frac{1}{7} = \frac{2624294338586094240}{590436102659356801}\sqrt{2}$.

And, by ARTICLE 9, *the rectangle contained by the circumscribed double triangle, and the number of sides is equal to the area of the entire circumscribed polygon;* thus $\frac{1}{543339720}\sqrt{2} \times 2414966403\frac{1}{7} = \frac{2414966403\frac{1}{7}}{543339720}\sqrt{2} = \frac{16904764822}{3803378040}\sqrt{2}$.

Now the area of the circumscribed polygon is $\frac{16904764822}{3803378040}\sqrt{2} = \frac{998118345787467549869125442 2}{224565170687778325744805004 0}\sqrt{2}$.

Again, the area of the inscribed polygon is $\frac{2624294338586094240}{590436102659356801}\sqrt{2} = \frac{9981183457874675481786489600}{2245651706877783257448050040}\sqrt{2}$.

Then, subtracting, we have $\frac{16904764822}{2245651706877783257448050040}\sqrt{2} = \frac{1}{132841345651568820}\sqrt{2}$, which is the difference between the areas of the circumscribed and inscribed polygons of the given circle, whose radius is the square root of two.

SUMMARY, CASE 4.

Tangent No. 4 is $\frac{1}{543339720} = .00000000018404691635649 2398$.

Sine No. 4 is $\frac{1}{768398401}\sqrt{2} = .000000000184046916356492398$.

Cosine No. 4 is $\frac{1086679440}{768398401} = 1.41421356237309504$.

Radius No. 4 is $\sqrt{2} = 1.41421356237309504$.

Secant No. 4 is $\frac{768398401}{543339720} = 1.41421356237309504$.

By *Case* 4, the tangent is $\frac{1}{543339720}$ and the square, which forms the unit of comparison, is $\left(\frac{1}{543339720}\right)^2 = \frac{1}{295218051329678400}$.

Again, by *Case* 4, the rectangle contained by the radius and the sine is $\frac{2}{768398401}$, and the number of sides $2414966403\frac{1}{7}$; therefore $\frac{2}{768398401} \times 2414966403\frac{1}{7} = \frac{4829932806\frac{2}{7}}{768398401} = \frac{33809529644}{5378788807} = \frac{44}{7} =$ area of the inscribed polygon.

Now $\frac{44}{7} \div \frac{1}{295218051329678400} = \frac{44}{7} \times \frac{295218051329678400}{1} = \frac{12989594258505849600}{7} = 1855656322643692800$; therefore there are exactly 1855656322643692800 squares in the given polygon, each of which is expressed by $\frac{1}{2952180513296784000}$; then $\frac{1}{2952180513296784000} \times 18556322643692800 = \frac{18556322643692800}{2952180513296784000} = \frac{44}{7} = 6\frac{2}{7} =$ area of inscribed polygon for double the number of sides.

Case 5.

In which the Inscribed and Circumscribed Polygons are Carried to $371131264528738560 6\frac{2}{7} - 3\frac{1}{7} = 371131264528738560 3\frac{1}{7}$ *Sides.*

By *Case* 4, the tangent is $\frac{1}{543339720}$ and the secant is $\frac{768398401}{543339720}$.

By Article 3 and (Propositions 4 and 5), tangent $\left(\frac{1}{543339720}\right)^2 = \frac{1}{295218051329678400}$, and sec. $\frac{768398401}{543339720} \times 2 = \frac{1536796802}{543339720}$.

Then, by division and cancellation, we have $\frac{1}{295218051329678400} \times \frac{543339720}{1536796802} = \frac{1}{835002744095575440} =$ tangent No. 5.

By (Proposition 6) and Article 4, we have, by division and cancellation, $\frac{1536796802}{543339720} \times \frac{1}{835002744095575440} = \frac{1}{295218051329678400}$, which equals the square on the tangent as above.

By *trigonometry*, $R^2 +$ tang.$^2 =$ sec.^{2}. Rad. $= \sqrt{2}$; and tang. $= \frac{1}{835002744095575440}$; rad. square $= (\sqrt{2})^2 = 2$, and tang. square $= \frac{1}{697229582647141045327149384731193600}$.

Then, adding to the square of the radius and extracting the square root of the same, $\sqrt{\frac{1394459165294282090654298769462387201}{697229582647141045327149384731193600}}$, we have secant No. 5 $= \frac{1180872205318713601}{835002744095575440}$.

By *Case* 4, secant No. 4 is $\frac{768398401}{543339720}$, which equals $\frac{118087220531-}{835002744095-}$

$\frac{8713602}{575440}$; subtracting from this secant No. 5 above, we find that $\frac{1}{83500274409557544 0}$ has been deducted from the assumed radius and $\frac{2}{835002744095575440}$ from the assumed diameter.

Now, by hypothesis, the circumference of a circle is to the diameter as $3\frac{1}{7}$ is to 1; therefore $\frac{6\frac{2}{7}}{835002744095575440}$ must be deducted from the circumference.

By *Case* 4, the tangent of the given arc is $\frac{1}{543339720}$, and, by *Case* 5, the tangent No. 5 is $\frac{1}{835002744095575440}$; then, by division and cancellation, we have $\frac{1}{543339720} \times \frac{835002744095575440}{1} = 1536796802$; therefore the tangent No. 4 has been divided into 1536796802 equal parts, each of which is equal to tangent No. 5, or $\frac{1}{835002744095575440}$.

By *Case* 4, the polygon was carried to $2414966403\frac{1}{7}$ sides, each of which contained two tangents, and each tangent was $\frac{1}{543339720}$; then $\frac{2}{543339720} \times 2414966403\frac{1}{7} = \frac{4829932806\frac{2}{7}}{543339720} = \frac{7422625290574771212\frac{4}{7}}{835002744095575440}$; and, by hypothesis, $\frac{6\frac{2}{7}}{835002744095575440}$ must be subtracted from the circumference to complete the polygon, which gives $\frac{742262529057477 12 06\frac{2}{7}}{835002744095575440}$; which, by ARTICLE 2, is the assumed circumference.

By *Case* 5, the secant No. 5 is $\frac{1180872205318713601}{835002744095575440} \times 2 = \frac{2361744410637427202}{835002744095575440}$; which, by ARTICLE 1, is the assumed diameter. Then, dividing the assumed circumference by the assumed diameter, we have, by cancellation $\frac{7422625290574771206\frac{2}{7}}{\cancel{835002744095575440}} \times \frac{\cancel{835002744095575440}}{2361744410637427202} = \frac{7422625290574771206\frac{2}{7}}{2361744410637427202} = \frac{22}{7} = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$ = the true ratio between the circumference and diameter of the given circle.

Now, by *trigonometry*, sec.^{2} : tang.^{2} : : R^2 : sin.^{2}, and, by *Case* 5, the secant is $= \frac{1180872205318713601}{835002744095575440}$; and the tangent is $\frac{1}{835002744095575440}$; and the radius is $\sqrt{2}$.

Then sec.$^2 = \frac{1394459165294282090654298769462387201}{697229582647141045327149384731193600}$: tang.^{2} $\frac{1}{697229582647141045327149384731193600}$: : $R^2 = 2$: sin.$^2 = \frac{1}{1394459165294282090654298769462387201}$; then extracting the square root, we have, sine No. 5 $= \frac{1}{1180872205318713601}\sqrt{2}$.

Again, by *trigonometry*, $R^2 -$ sin.$^2 =$ cos.^{2}:

Let $\frac{2788918330588564181308597538924774402}{1394459165294282090654298769462387201} = R^2$, and $\frac{2}{1394459165294282090654298769462387201} =$ sin.^{2}; then subtracting the square of the sine from the square of the radius, and extracting the square root, $\sqrt{\frac{2788918330588564181308597538924774400}{1394459165294282090654298769462387201}}$, we have cosine No. 5 $= \frac{1670005488191150880}{1180872205318713601}$.

Now, by Article 5, cos. $\frac{1670005488191150880}{1180872205318713601} \times$ sin. $\frac{1}{1180872205318713601}\sqrt{2}$, we have the area of the inscribed double triangle $= \frac{1670005488191150880}{1394459165294282090654298769462387201}\sqrt{2}$.

And, by Article 6, rad. $\sqrt{2} \times \frac{1}{835002744095575440} = \frac{1}{835002744095575440}\sqrt{2}$, we have the area of the circumscribed double triangle.

Then, by ARTICLE 7, $\frac{1}{835002744095575440}\sqrt{2} \times \frac{167\text{-}0005488191150880}{1394459165294\text{2-}8209065429876946238720 1}\sqrt{2}$; by division and cancellation, and extracting the square root $\sqrt{\frac{4}{1394459165294282090654298769462387201}}$, we have the area of the inscribed double triangle for half the number of sides $= \frac{2}{1180872205318713601}$.

Again, $\sqrt{2} \times \frac{1}{1180872205318713601} = \frac{1}{1180872205318713601}\sqrt{2}$.

By ARTICLE 10, and *Case* 5, the rectangle of the radius and the sine $\frac{2}{1180872205318713601} \times 3711312645287385603\frac{1}{7} =$ the number of sides in the given polygon $= \frac{7422625290574771206\frac{2}{7}}{1180872205318713601} = \frac{44}{7}$; and $(\sqrt{2})^2 = 2$; then $\frac{44}{7} \div 2 = \frac{22}{7} = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, which is the true ratio of the circumference to the diameter of the given circle.

Again, by ARTICLE 8, the inscribed double triangle $\frac{1670\text{-}0054881911 50880}{139445916529\text{-}4282090654298769462387201}\sqrt{2}$, multiplied by the number of the sides in the given polygon, $3711312645287385603\frac{1}{7}$, is equal to the area of the inscribed polygon $= \frac{6197912486023151972522573087 51\text{-}1514480}{13944591652942820906542987694 6\text{-}2387201}\sqrt{2}$.

And, by ARTICLE 9, the circumscribed double triangle $\frac{1}{835002744\text{-}095575440}\sqrt{2}$, multiplied by the number of sides, $37113126452873\text{-}85603\frac{1}{7}$, is equal to the area of the circumscribed polygon $= \frac{2597918851701169922}{5845019208669028080}\sqrt{2}$; consequently the area of the circumscribed polygon $= \frac{3622691753445493230262962410981366860218523820 55\text{-}14457622}{8150640606849658130440009777367248472858140349 30\text{-}1604080}\sqrt{2}$, and the area of the inscribed polygon $= \frac{3622691753445\text{-}4932302629624109813668576206049688502758400}{8150640606849\text{-}6581304400097773672484728581403493016040 80}\sqrt{2}$; then, subtracting, we have $\frac{25979188517011699222}{815064060684965813044900097773672484728581403493016040080}\sqrt{2} = \frac{1}{31373730559414715563389057984304164O}\sqrt{2}$, which is the difference between the areas of the circumscribed and inscribed polygons of the given circle, whose radius is the square root of two.

SUMMARY, CASE 5.

Tangent No. 5 is $\frac{1}{835002744095575440} =$ .0000000000000000001197600854693165860264915019144180288.

Sine No. 5 is $\frac{1}{1180872205318713601}\sqrt{2} =$ .0000000000000000001197600854693165860264915019144180288.

Cosine No. 5 is $\frac{1670005488191150880}{1180872205318713601} =$ 1.41421356237309504880168872420969927 6.

Radius No. 5 is $\sqrt{2}$ = 1.414213562373095048801688724209699276.

Secant No. 5 is $\frac{1180872205318713601}{835002744095575440} =$ 1.414213562373095048801688724209699276.

By *Case* 5, the tangent is $\frac{1}{835002744095575440}$, and the square, which forms the unit of comparison, is $\left(\frac{1}{835002744095575440}\right)^2 = \frac{1}{697229582647141045327149384731193600}$.

Again, by *Case* 5, the rectangle contained by the radius and the sine is $\frac{2}{1180872205318713601}$, and the number of sides $3711312645287385603\frac{1}{7}$; multiplying these together we have $\frac{74226252905747712062\frac{2}{7}}{1180872205318713601}$ $= \frac{519583770340233984444}{8266105437230995207} = \frac{44}{7}$.

Now, $\frac{44}{7} \div \frac{1}{697229582647141045327149384731193600} = \frac{44}{7} \times$

$\frac{697229582647141045327149384731193600}{1} = \frac{3067810163647420599\text{-}4394572928117251840 0}{7} = 438258594806774371348493898973893\text{1-}$

200; therefore there are exactly 43825859480677437134849389897389 31200 squares in the given polygon, each of which is expressed by $\frac{1}{697229582647141045327149384731193600}$; then, multiplying the number of squares by the value of each square, we have $\frac{4382585948\text{-}06774371348938989738931200}{6972295826\text{-}471410453271493847311193600} = \frac{44}{7} = 6\frac{2}{7}$ = area of the inscribed polygon for double the number of sides.

Case 6.

In which the Inscribed and Circumscribed Polygons are Carried to 876517189613548742696987797947786240$3\frac{1}{7}$ *Sides.*

By *Case* 5, the tangent is $\frac{1}{835002744095575440}$ and the secant is $\frac{1180872205318713601}{835002744095575440}$.

Then, by ARTICLE 3 and (PROPOSITIONS 4 and 5), tang.^{2} = $\frac{1}{697229582647141045327149384731193600}$ and the double of the secant is $\frac{2361744410637427202}{835002744095575440}$.

Then, by division and cancellation, we have $\frac{1}{19720630637346392\text{6-}398445507329911 8880}$ = tangent No. 6.

By (PROPOSITION 6 and ARTICLE 4) and by cancellation, $\frac{2361744410637427202}{835002744095575440} \times \frac{1}{19720630637346392639844550732991188 80} = \frac{1}{697229582647141045327149384731193600}$, and the square described on the tangent equals the same. (See above.)

By *Case* 1, the cosine is $\frac{1}{7}$; then $(7)^2 \times 2 - 1 = 99$.

Multiplying this by tangent No. 2, we have $\frac{99}{70}$ = secant No. 2.

By *Case* 2, the secant is $\frac{99}{70}$; then $(99)^2 \times 2 - 1 = 19601$

Multiplying this by tangent No. 3, we have $\frac{19601}{13860}$ = secant No. 3.

By *Case* 3, the secant is $\frac{19601}{13860}$; then $(19601)^2 \times 2 - 1 = 768398\text{-}401$.

Multiplying this by tangent No. 4, we have $\frac{768398401}{543339720}$ = secant No. 4.

By *Case* 4, the secant is $\frac{768398401}{543339720}$; then $(768398401)^2 \times 2 - 1 = 1180872205318713601$.

Multiplying this by tangent No. 5, we have $\frac{1180872205318713601}{835002744095575440}$ = secant No. 5.

By *Case* 5, the secant is $\frac{1180872205318713601}{835002744095575440}$; then $(1180872205\text{-}318713601)^2 \times 2 - 1 = 2788918330588564181308597538924774401$.

Multiplying this by tangent No. 6, we have $\frac{298891833058856418\text{-}1308597538924774401}{197206306373463926\text{-}3984455073299118880}$ = secant No. 6.

By *Case* 2, the numerator of the secant is 99, and the radius is the square root of two.

Then, by division, we have $\sqrt{2} \times \frac{1}{99} = \frac{1}{99}\sqrt{2}$ = sine No. 2.

By *Case* 3, the numerator of the secant is 19601.

Then, by division, we have $\sqrt{2} \times \frac{1}{19601} = \frac{1}{19601}\sqrt{2}$ = sine No. 3.

By *Case* 4, the numerator of the secant is 768398401.

Then, by division, we have $\sqrt{2} \times \frac{1}{768398401} = \frac{1}{768398401}\sqrt{2}$ = sine No. 4.

By *Case* 5, the numerator of the secant is 1180872205318713601.

Then, by division, we have $\sqrt{2} \times \frac{1}{1180872205318713601} = \frac{1}{1180872205318713601}\sqrt{2}$ = sine No. 5.

And, by *Case* 6, the numerator of the secant is 2788918330588564-181308597538924774401.

Then, by division, we have $\frac{1}{2788918330588564181308597538924774401}\sqrt{2}$ = sine No. 6.

By *Case* 2, the sec. is $\frac{99}{70}$, and the square described on the radius is 2.

Then, by division, we have $2 \times \frac{70}{99} = \frac{140}{99}$ = cosine No. 2.

By *Case* 3, the secant is $\frac{19601}{13860}$.

Then, by division, we have $2 \times \frac{13860}{19601} = \frac{27720}{19601}$ = cosine No. 3.

By *Case* 4, the secant is $\frac{768398401}{543339720}$.

Then, by division, we have $2 \times \frac{543339720}{768398401} = \frac{1086679440}{768398401}$ = cosine No. 4.

By *Case* 5, the secant is $\frac{1180872205318713601}{835002744095575440}$.

Then, by division, we have $2 \times \frac{835002744095575440}{1180872205318713601} = \frac{1670005488191150880}{1180872205318713601}$ = cosine No. 5.

And, by *Case* 6, the secant is $\frac{2788918330588564181308597538924774401}{1972063063734639263984455073299118880}$.

Then, by division, we have $\frac{3944126127469278527968910146598237760}{2788918330588564181308597538924774401}$ = cosine No. 6.

By *Case* 5, the secant No. 5 is $\frac{1180872205318713601}{835002744095575440} = \frac{2788918330588564181308597538924774402}{1972063063734639263984455073299118880}$.

And, by *Case* 6, the secant No. 6 is $\frac{2788918330588564181308597538924774401}{1972063063734639263984455073299118880}$.

Therefore $\frac{1}{1972063063734639263984455073299118880}$ has been deducted from the assumed radius, and twice that amount from the assumed diameter.

Therefore, by hypothesis, $\dfrac{6\frac{2}{7}}{1972063063734639263984455073299118880}$ must be deducted from the circumference to complete the polygon.

By *Case* 5, the tangent is $\dfrac{1}{835002744095575440}$.

And, by *Case* 6, the tangent is $\dfrac{1}{197206306373463926398445507329\text{-}9118880}$.

Then, by division and cancellation, we have 236174441063742 7202; therefore the tangent No. 5 has been divided into 2361744410637427202 parts, each of which is equal to tangent No. 6.

By *Case* 5, the polygon was carried to $3711312645287385603\frac{1}{7}$ sides, each of which contained two tangents of $\dfrac{1}{835002744095575440}$.

Then $\dfrac{2}{835002744095575440} \times 3711312645287385603\frac{1}{7} = \dfrac{74226252\text{-}90574771206\frac{2}{7}}{83500274\text{-}4095575440} = \dfrac{175303437922709748539397559589557248124\frac{4}{7}}{1972063063734639263984455073299118880}$; and, by hypothesis, $\dfrac{6\frac{2}{7}}{1972063063734639263984455073299118880}$ must be subtracted from the circumference to complete the polygon.

Then $\dfrac{175303437922709748539397559589557248124\frac{4}{7}}{1972063063734639263984455073299118880} - \dfrac{6\frac{2}{7}}{197206306\text{-}3734639263984455073299118880} = \dfrac{175303437922709748539397559\text{-}58955724806\frac{2}{7}}{197206306373463926398445507\text{-}3299118880}$ = the circumference of the polygon No. 6.

By *Case* 6, secant No. 6, $\dfrac{2788918330588564181308597538924774\text{-}401}{1972063063734639263984455073299118\text{-}880} \times 2 = \dfrac{5577836661177128362617195077849548802}{1972063063734639263984455073299118880}$ = the assumed diameter.

Then, by Articles 1 and 2, dividing the assumed circumference by the assumed diameter, we have by cancellation $\dfrac{175303437922709748\text{-}}{197206306373463926\text{-}}$

$$\frac{53939755958955724806\frac{2}{7}}{63984455073299118880} \times \frac{1972063063734639263984455073299118880}{5577836661177128362617195077849544802} = \frac{17530343792270974853939755958955724806\frac{2}{7}}{5577836661177128362617195077849544802} = \frac{22}{7} = 3.\dot{1}4285\dot{7},$$

or $3\frac{1}{7}$, the true ratio between the circumference and diameter of the given circle.

By *Case* 2, the sine of the given arc is $\frac{1}{99}\sqrt{2}$, and the given radius is $\sqrt{2}$.

Then, by ARTICLE 7, $\sqrt{2} \times \frac{1}{99}\sqrt{2} = \frac{2}{99}$ = area of the inscribed double triangle for half the number of sides.

By *Case* 3, the sine is $\frac{1}{19601}\sqrt{2}$.

Then, by ARTICLE 7, $\sqrt{2} \times \frac{1}{19601}\sqrt{2} = \frac{2}{19601}$ = area as above.

By *Case* 4, the sine is $\frac{1}{768398401}\sqrt{2}$.

Then, by ARTICLE 7, $\sqrt{2} \times \frac{1}{768398401}\sqrt{2} = \frac{2}{768398401}$ = area as above.

By *Case* 5, the sine is $\frac{1}{1180872205318713601}\sqrt{2}$.

Then, by ARTICLE 7, $\sqrt{2} \times \frac{1}{1180872205318713601}\sqrt{2} = \frac{2}{1180872205318713601}$ = area as above.

By *Case* 6, the sine is

$$\frac{1}{2788918330588564181308597538924774401}\sqrt{2}.$$

Then, by ARTICLE 7, multiplying the radius by the sine, we have, $\frac{2}{2788918330588564181308597538924774401}$ = area of the inscribed double triangle for half the number of sides.

By ARTICLE 10, $\frac{2}{2788918330588564181308597538924774401} \times 8765171896135487426969877979477862403\frac{1}{7} = \frac{17530343792270974\text{-}}{2788918330588564\text{1-}}$

$\frac{85393975595895572480 6\frac{2}{7}}{81308597538924774401} = \frac{44}{7}$; and $(\sqrt{2})^2 = 2$; then $\frac{44}{7} \div 2 = \frac{22}{7}$ $3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$; which is the true ratio of the circumference to the diameter of the given circle.

By *Case* 2, the difference between the inscribed and circumscribed polygons is $\frac{1}{2205}\sqrt{2}$; and, by *Case* 3, the difference between the inscribed and circumscribed polygons is $\frac{1}{86440410}\sqrt{2}$.

Then, dividing the difference of *Case* 2 by the difference of *Case* 3, we have $\frac{1}{2205}\sqrt{2} \div \frac{1}{86440410}\sqrt{2} = \frac{1}{2205}\sqrt{2} \times \frac{86440410}{1}\sqrt{2} =$ 39202, which shows that the difference between the polygons has diminished 39202 times.

By *Case* 3, the tangent is $\frac{1}{13860}$; and, by *Case* 4, the tangent is $\frac{1}{543339720}$.

Then, dividing tangent No. 3 by tangent No. 4, we have $\frac{1}{13860} \div \frac{1}{543339720} = \frac{1}{13860} \times \frac{543339720}{1} = 39202$; therefore the tangent No. 3 is 39202 times as large as tangent No. 4. Consequently the difference between the tangents Nos. 3 and 4 is exactly the same as the difference between the differences of the inscribed and circumscribed polygons of *Cases* 2 and 3.

By *Case* 3, the difference between the inscribed and circumscribed polygons is $\frac{1}{86440410}\sqrt{2}$; and, by *Case* 4, the difference between the inscribed and circumscribed polygons is $\frac{1}{132841345651568820}\sqrt{2}$.

Then, dividing the difference of *Case* 3 by the difference of *Case* 4, we have $\frac{1}{86440410}\sqrt{2} \div \frac{1}{132841345651568820}\sqrt{2} = \frac{1}{86440410}\sqrt{2} \times \frac{132841345651568820}{1}\sqrt{2} = 1536796802$; which shows that the difference between the polygons has diminished 1536796802 times.

By *Case* 4, the tangent is $\frac{1}{543339720}$; and, by *Case* 5, the tangent is $\frac{1}{835002744095575440}$.

Then, dividing tangent No. 4 by tangent No. 5, we have $\frac{1}{543339720} \div \frac{1}{835002744095575440} = 1536796802$; therefore the tangent No. 4 is 1536796802 times as large as tangent No. 5; consequently the difference between the tangents Nos. 4 and 5 is exactly the same as the difference between the differences of the inscribed and circumscribed polygons of *Cases* 3 and 4.

By *Case* 4, the difference between the inscribed and circumscribed polygons is $\frac{1}{132841345615688820}\sqrt{2}$; and, by *Case* 5, the difference between the inscribed and circumscribed polygons is $\frac{1}{313737305594147\text{-}1555633890579843041640}\sqrt{2}$.

Then, dividing the difference of *Case* 4 by the difference of *Case* 5, we have by cancellation $\frac{1}{132841345615688820}\sqrt{2} \times \frac{313737305594147\text{-}1555633890579843041640}{1}\sqrt{2} = 2361744410637427202$; therefore the difference between the inscribed and circumscribed polygons of *Case* 4 is 2361744410637427202 times as large as the difference between the inscribed and circumscribed polygons of *Case* 5.

By *Case* 5, the tangent is $\frac{1}{835002744095557440}$; and, by *Case* 6, the tangent is $\frac{1}{1972063063734639263984455073299118880}$.

Then, dividing tangent No. 5 by tangent No. 6, we have by cancellation $\frac{1}{835002744095575440} \times \frac{19720630637346392639844550732991\text{-}18880}{1} = 2361744410637427202$; therefore the tangent No. 5 is 2361744410637427202 times as large as tangent No. 6; consequently the difference between the tangents Nos. 5 and 6 is exactly the same as the difference between the differences of the inscribed and circumscribed polygons of *Cases* 4 and 5; and, consequently, the difference between the differences of the inscribed and circumscribed polygons of *Cases* 5 and 6 will be the same as the differences between the tangents Nos. 6 and 7; thus the tangent No. 6 is $\frac{1}{1972063063734639263984455073299118880}$ and the tangent No. 7 is $\frac{1}{10999845655052358763719716313591640029823644051645720435317842392159581760}$.

Then, dividing tangent No. 6 by tangent No. 7, we have by cancellation 55778366611771283626171950778495488O2; therefore the tangent No. 6 is 5577836661177128362617195077849548802 times as large as tangent No. 7.

Again, the difference between the inscribed and circumscribed polygons of *Case* 5 is $\frac{1}{3137373055941471556338905798430416 40}\sqrt{2}$; then, dividing this by the difference between tangents Nos. 6 and 7, we have $\left(\frac{1}{313737305594147155633890579843041640}\sqrt{2}\right) \times \left(\frac{1}{5577836661177128362617195077849548802}\right)$; which equals the difference between the areas of the inscribed and circumscribed polygons for *Case* 6.

SUMMARY, CASE 6.

Tangent No. 6 is $\frac{1}{1972063063734639263984455073299118880}$.

Sine No. 6 is $\frac{1}{2788918330588564181308597538924774401}\sqrt{2}$.

Cosine No. 6 is $\frac{3944126127469278527968910146598237 60}{2788918330588564181308597538924774401}$.

Radius No. 6 is $\sqrt{2}$.

Secant No. 6 is $\frac{2788918330588564181308597538924774401}{1972063063734639263984455073299118880}$.

NOTE.—It will no doubt be observed by the student that, in the Summary for *Cases* 6 and 7, the sine and tangent are not extended in decimals, for the reason that the space could be better occupied with matter more important to the subject. The student will find it a very agreeable pastime to verify the result in these Cases for himself, and it was partly with that view that they were omitted. The sine and tangent in *Case* 6 should agree together for 72, and in *Case* 7 for 144, decimal places.

The cosine, radius, and secant have also been omitted, partly for the same reason, and partly because the radius with which they should both agree will be found carried out in the Second Part of this book; for *Case* 6 they should agree like the sine and tangent with each other to 72 decimal places, and for *Case* 7 to 144 decimal places.

By *Case* 6, the tangent is $\frac{1}{197206306373463926398445507329911880}$, and the square which forms the unit of comparison is the square of the tangent, which is $\frac{1}{3889032727346451883806194960659921656367162016449544406781628584372454400}$.

Again, by *Case* 6, the rectangle contained by the radius and the sine is $\frac{2}{2788918330588564181308597538924774401}$ and the number of sides $87651718961354874269698779794778624031\frac{1}{7}$.

Multiplying these together, we have $\frac{17530343792270974853939755958955724806\frac{2}{7}}{2788918330588564181308597538924774401} = \frac{44}{7} \div \frac{1}{3889032727346451883806194960659921656367162016449544406781628584372454400} = \frac{44}{7} \times \frac{3889032727346451883806194960659921656367162016449544406781628584372454400}{1} = \frac{171117440003243882887472578269036552880155128723779953898391657712387993600}{7} = 24445348571891983269638939752719507554307875531968564842627379673198284800$; therefore there are exactly 24445348571891983269638939752719507554307875531968564842627379673198284800 squares in the given polygon, each of which is expressed by $\frac{1}{3889032727346451883806194960659921656367162016449544406781628584372454400}$.

Then, multiplying the number of squares by the value of each square,

. . . 244453485718919832696389397527195075543078753196856-
. . . 272734645188380619496065992165636716201644954-
. . . 8284800 / . . . 54400 $= \frac{44}{7} = 6\frac{2}{7}$ = area of the inscribed poly-
. . . mber of sides.

CASE 7.

In which the Inscribed and Circumscribed Polygons are Carried to 488906971437839665392778795054390151086157510639371296852 5-4759346396569603$\frac{1}{7}$ *Sides.*

By *Case* 6, the tangent is $\frac{1}{1972063063734639263984455073299118880}$ and the secant is $\frac{2788918330588564181308597538924774401}{1972063063734639263984455073299118880}$.

Then, by ARTICLE 3 and (PROPOSITIONS 4 and 5), tangent² $\frac{1}{3889032727346451883806194960659921656367162016449544406781628584372454400}$, and the double of the secant is $\frac{5577836661177128362617195077849548802}{1972063063734639263984455073299118880}$.

Then, by division and cancellation $\frac{1}{3889032727346451883806194960659921656367162016449544406781628584372454400} \times \frac{1972063063734639263984455073299118880}{5577836661177128362617195077849548802}$, we have $\frac{1}{10999845655052358763719716313591640029823644051645720435317842392159581760}$ = tangent No. 7.

By (PROPOSITION 6) and ARTICLE 4, and by cancellation, $\frac{5577836661177128362617195077849548802}{1972063063734639263984455073299118880} \times \frac{1}{10999845655052358763719716313591640029823644051645720435317842392159581760} = \frac{1}{3889032727346451883806194960659921656367162016449544406781628584372454400}$ = square described on the tangent No. 6.

By *Case* 6, the secant is $\frac{2788918330588564181308597538924774401}{1972063063734639263984455073299118880}$.

Then $(2788918330588564181308597538924774401)^2 \times 2 - 1 =$ 15556130909385807535224779842639686625468648065798177627126514337489817601.

Multiplying this by tangent No. 7 we have $\frac{15556130909385807535224779842639686625468648065798177627126514337489817601}{10999845655052358763719716313591640029823644051645720435317842392159581760} =$ secant No. 7.

By *Case* 7, the numerator of the secant is 15556130909385807535224779842639686625468648065798177627126514337489817601, and the given radius is the $\sqrt{2}$.

Then, by division, we have $\frac{1}{15556130909385807535224779842639686625468648065798177627126514337489817601}\sqrt{2} =$ sine No. 7.

By *Case* 7, the secant is $\frac{15556130909385807535224779842639686625468648065798177627126514337489817601}{10999845655052358763719716313591640029823644051645720435317842392159581760}$, and the radius is the square root of two.

Then, by division, we have $\frac{2199969131010471752743943262718328005964728810329144087063568478431916 3520}{15556130909385807535224779842639686625468648065798177627126514337489817601} =$ cosine No. 7.

By *Case* 6, the secant No. 6 is $\frac{2788918330588564181308597538924774401}{1972063063734639263984455073299118880} = \frac{155561309385807535224779842639686625468648065798177627126514337489817602}{10999845655052358763719716313591640029823644051645720435317842392159581760}$.

And, by *Case* 7, the secant No. 7 is $\frac{15556130909385807535224779842639686625468648065798177627126514337489817601}{10999845655052358763719716313591640029823644051645720435317842392159581760}$.

Therefore $\frac{1}{10999845655052358763719716313591640029823644051645720435317842392159581760}$ has been deducted from the assumed radius, and twice that amount from the assumed diameter.

Therefore, by hypothesis, $\frac{6\frac{3}{7}}{10999845655052358763719716313591640029823644051645720435317842392159581760}$ must be deducted from the circumference to complete the polygon.

By *Case* 6, the tangent is $\frac{1}{19720630637346392639844550732991188880}$.

And, by *Case* 7, the tangent is $\frac{1}{10999845655052358763719716313591640029823644051645720435317842392159581760}$.

Then, by division and cancellation, we have 5577836661177128362617195077849548802; therefore the tangent No. 6 has been divided into 5577836661177128362617195077849548802 parts, each of which is equal to tangent No. 7.

By *Case* 6, the polygon was carried to 8765171896135487426969877979477862403$\frac{1}{7}$ sides, each of which contained two tangents of $\frac{1}{19720630637346392639844550732991188880}$.

Then $\frac{2}{19720630637346392639844550732991188880} \times 8765171896135487426969877979477862403\frac{1}{7} = \frac{17530343792270974853939755958955724806\frac{2}{7}}{19720630637346392639844550732991188880} = \frac{97781394287567933078555759010878030217231502127874259379509518692793139212\frac{4}{7}}{10999845655052358763719716313591640029823644051645720435317842392159581760}$.

And, by hypothesis, $\frac{6\frac{3}{7}}{10999845655052358763719716313591640029823644051645720435317842392159581760}$ must be subtracted from the circumference to complete the polygon.

Thus $\frac{97781394287567933078555759010878030217231502127874259370509518692793139212\frac{4}{7}}{10999845655052358763719716313591640029823644051645720435317842392159581760} - \frac{6\frac{3}{7}}{10999845655052358763719716313591359\text{-}}$

1640029823644051645720435317842392159581760 = 977813942875-
109998456550-
6793307855575901087803021723150212787425937050951869279313-
5235876371971631359164002982364405164572043531784239215958-
$9206\frac{2}{7}$
1760 = the circumference of the polygon No. 7.

By *Case* 7, the secant No. 7 is 155561309093858075352247798426-
109998456550523587637197163135-
39686625468648065798177627126514337489817601
91640029823644051645720435317842392159581760 × 2 = 311122-
109998-
6181877161507044955968527937325093729613159635525425302867-
4565505235876371971631359164002982364405164572043531784239-
4979635202
2159581760 = the assumed diameter.

Then, by ARTICLES 1 and 2, dividing the assumed circumference by the assumed diameter, we have by cancellation 97781394287567933-
10999845655052358-
$078555759010878030217231502127874259370509518692793139206\frac{2}{7}$
763719716313591640029823644051645720435317842392159581760
× 10999845655052358763719716313591595451753047924035846398-
31112261818716150704495596852793732509372961315963552542-
694780218565561760 = 9778139428756793307855575901087803021-
53028674979635202 = 311122618187161507044955968527937325-
$7231502127874259370509518692793139206\frac{2}{7}$ = 22
0937296131596355254253028674979635202 = 7 = $3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$,
the true ratio between the circumference and diameter of the given circle.

By *Case* 7, the sine of the given arc 1
15556130909385807535224779-
842639686625468648065798177627126514337489817601$\sqrt{2}$.

Then, by ARTICLE 7, multiplying the radius by the sine, we have
2
1555613090938580753552247798426396866254686480657981776271-
26514337489817601 = area of the inscribed double triangle for half the number of sides.

By ARTICLE 10, 2
155561309093858075352247798426396866254686-

4806579817762712651433748981 7601 × 4889069714378396653927 7-
87950543901510861575106393712968525475934639656960 3$\frac{1}{7}$ = $\frac{9778-}{1556-}$

13942875679330785557590108780302172315021278742593705095186-
———
13090938580753522477984263968662546864806579817762712651 43-
$\frac{927931392062\frac{2}{7}}{37489817601} = \frac{44}{7}$, and $(\sqrt{2})^2 = 2$; then $\frac{44}{7} \div 2 = \frac{22}{7} = 3.\dot{1}4285\dot{7}$,
or $3\frac{1}{7}$, which is the true ratio of the circumference to the diameter of the given circle.

Summary, Case 7.

Tangent for *Case* 7 is $\frac{1}{10999845655052358763719716313591640029-}$
$\overline{82364405164572043531784239215958 1760}$.

Sine for *Case* 7 is $\frac{1}{15556130909385807535224779842639686625468-}$
$\overline{6480657981776271265143374898 17601}\sqrt{2}$.

Cosine for *Case* 7 is $\frac{21999691310104717527439432627183280059 6-}{15556130909385807535224779842639686625 4-}$
$\frac{4728810329144087063568478431916 3520}{68648065798177627012651433748981 7601}$.

Radius for *Case* 7 is $\sqrt{2}$.

Secant for *Case* 7 is $\frac{15556130909385807535224779842639686625 4-}{10999845655052358763719716313591640029 8-}$
$\frac{6864806579817762712651433748981 7601}{2364405164572043531784239215958 1760}$.

NOTE.—The difference between the inscribed and circumscribed polygons for *Case* 7 has been omitted, for the purpose of allowing the student to make the calculation for himself; the number of squares contained in the given polygon, together with the unit of comparison, has also been left out, as the calculation is made in the same manner as in the previous cases. The decimals of the Summary have also been omitted, as in *Case* 6, and for the reason mentioned therein, viz.: that the decimals for the cosine, the radius, and the tangent will be found in Part Second of this volume. Indeed, the whole of *Cases* 6 and 7 would have been omitted had it not interfered with the original design of the author, as the other cases are sufficient to enable *any one* to get a thorough and comprehensive knowledge of the subject, though the curious and the adept are at liberty to extend the work *ad infinitum*.

PROOFS.

By *Case* 1, the $\frac{1}{50}$th of the square described upon the radius was deducted for the purpose of finding the value of the given arc, and the square root of the $\frac{1}{50}$th so deducted is the sine of the given arc; and the square root of the remaining $\frac{49}{50}$ths is the cosine of the given arc; for, substituting the numbers, we have $(\sqrt{2})^2 = 2$, and $\frac{1}{50}$ of $2 = \frac{2}{50} = \frac{1}{25}$; and the $\sqrt{\frac{1}{25}} = \frac{1}{5}$ = the sine of the given arc.

Again, $2 = \frac{50}{25}$, and $\frac{50}{25} - \frac{1}{25} = \frac{49}{25}$; and the $\sqrt{\frac{49}{25}} = \frac{7}{5}$ = cosine of the given arc.

Then, by ARTICLE 5, $\frac{7}{5} \times \frac{1}{5} = \frac{7}{25}$ = area of the inscribed double triangle.

Now, by *Case* 1, there are 22 double triangles contained in the given polygon.

Then, $\frac{7}{25} \times 22 = \frac{154}{25}$ = area of the entire inscribed polygon.

By *trigonometry*, we have cos. : sin. : : rad.: tang.

Substituting the numbers, we have $\frac{7}{5} : \frac{1}{5} : : \sqrt{2} : \frac{1}{7}\sqrt{2}$; for $\frac{1}{5} \times \sqrt{2} = \frac{1}{5}\sqrt{2}$.

Then, by division and cancellation, we have $\frac{1}{5}\sqrt{2} \div \frac{7}{5} = \frac{1}{\not 5}\sqrt{2} \times \frac{\not 5}{7} = \frac{1}{7}\sqrt{2}$ = tangent of the given arc.

Then, by ARTICLE 6, $\frac{1}{7}\sqrt{2} \times \sqrt{2} = \frac{2}{7}$ = area of the circumscribed double triangle.

Now, by *Case* 1, there are 22 double triangles contained in the given polygon.

Then $\frac{2}{7} \times 22 = \frac{44}{7}$ = area of the entire circumscribed polygon.

Now $\frac{44}{7} = \frac{1100}{175}$, and $\frac{154}{25} = \frac{1098}{175}$.

Then, by subtraction, $\frac{1100}{175} - \frac{1078}{175} = \frac{22}{175}$ = the difference between the inscribed and circumscribed polygons.

Again, $\frac{22}{175} \div \frac{1100}{175} = \frac{22}{\cancel{175}} \times \frac{\cancel{175}}{1100} = \frac{22}{1100} = \frac{1}{50}$; therefore the difference between the inscribed and circumscribed polygons is $\frac{1}{50}$th of the circumscribed polygon, and is equivalent to the $\frac{1}{50}$th of the square described upon the radius; consequently, deducting $\frac{1}{50}$th from the square described upon the radius is equivalent to deducting $\frac{1}{50}$th from the area of the circumscribed polygon.

By *trigonometry*, $R^2 + \tan.^2 = \sec.^2$.

Substituting the numbers, we have $(\sqrt{2})^2; = 2$ and $(\frac{1}{7}\sqrt{2})^2 = \frac{2}{49}$; then $2 + \frac{2}{49} = \frac{100}{49}$, and $\sqrt{\frac{100}{49}} = \frac{10}{7}$ = secant of the given arc.

Deducting the cosine from the secant of the given arc, we have $\frac{10}{7} = \frac{50}{35}$; and $\frac{7}{5} = \frac{49}{35}$; then $\frac{50}{35} - \frac{49}{35} = \frac{1}{35}$ = the difference between the cosine and the secant of the given arc; then, dividing this difference by the secant, we have by cancellation $\frac{1}{35} \div \frac{10}{7} = \frac{1}{\cancel{35}} \times \frac{\cancel{7}}{10} = \frac{1}{50}$; therefore the difference between the cosine and secant is equivalent to the $\frac{1}{50}$th of the secant.

Consequently, deducting $\frac{1}{50}$th from the square described upon the radius is equivalent to deducting the $\frac{1}{50}$th from the secant of the given arc.

Again, by *Case* 1, the circumscribed polygon is $\frac{1100}{175}$; and the inscribed polygon is $\frac{1078}{175}$; then, by division and cancellation, $\frac{1078}{175} \div$

$\frac{1100}{175} = \frac{1078}{\cancel{175}} \times \frac{\cancel{175}}{1100} = \frac{1078}{1100} = \frac{49}{50}$; therefore the inscribed polygon is $\frac{49}{50}$ of the circumscribed polygon.

By *Case* 2, the sine of the given arc is $\frac{1}{99}\sqrt{2}$; and the radius is $\sqrt{2}$; then $\frac{1}{99}\sqrt{2} \times \sqrt{2} = \frac{2}{99}$ = area of the inscribed double triangle; and there are $311\frac{1}{7}$ sides; then $\frac{2}{99} \times 311\frac{1}{7} = \frac{622\frac{2}{7}}{99} = \frac{4356}{693} = \frac{44}{7}$ = area of the inscribed polygon.

But, by *Case* 1, the area of the inscribed polygon is $\frac{154}{25}$.

Now $\frac{44}{7} = \frac{1100}{175}$; and $\frac{154}{25} = \frac{1078}{175}$.

Then, by division and cancellation, we have $\frac{1078}{175} \div \frac{1100}{175} = \frac{1078}{\cancel{175}} \times \frac{\cancel{175}}{1100} = \frac{1078}{1100} = \frac{49}{50}$; therefore the area of the inscribed polygon for *Case* 1 is $\frac{49}{50}$ of the area of the inscribed polygon of *Case* 2; but the area of the inscribed polygon for *Case* 1 is $\frac{49}{50}$ of the area of the circumscribed polygon for *Case* 1; therefore, by (*Axiom* 1), the area of the inscribed polygon of *Case* 2 is the same as the area of the circumscribed polygon of *Case* 1.

But, by *Case* 1, the $\frac{1}{50}$th of the area of the circumscribed polygon was deducted; therefore, by *Case* 2, the same $\frac{1}{50}$th has been added on again.

Again, by *Case* 1, the sine is $\frac{1}{5}$, and there are 44 sines contained in the given polygon; and, by *Case* 2, each of these sines is divided into 14 parts, each equal to tangent No. 2; therefore there are $44 \times 14 = 616$ tangents contained in the polygon for *Case* 1, each equal to $\frac{1}{70}$.

But, by hypothesis, $\frac{6\frac{2}{7}}{70}$ must be added to complete the polygon; then $\frac{616}{70} + \frac{6\frac{2}{7}}{70} = \frac{622\frac{2}{7}}{70} = \frac{4356}{490}$ = sum of the tangents.

Again, by *Case* 2, the tangent is $\frac{1}{70}$ and the sine is $\frac{1}{99}\sqrt{2}$; therefore the tangent is to the sine as $\frac{1}{70}$ is to the $\frac{1}{99}\sqrt{2}$.

Now, by *Case* 2, the sum of the tangents is $\frac{4356}{490}$, and the sum of the sines $\frac{4356}{693}\sqrt{2}$.

Then, by proportion, $\frac{4356}{490} : \frac{1}{70} :: \frac{4356}{693}\sqrt{2} : \frac{1}{99}\sqrt{2}$.

For $\frac{4356}{490} \times \frac{1}{99}\sqrt{2} = \frac{44}{490}\sqrt{2}$, and $\frac{4356}{693}\sqrt{2} \times \frac{1}{70} = \frac{44}{490}\sqrt{2}$.

Again, by *Case* 1, the tangent is $\frac{1}{7}\sqrt{2}$; and there are 22 sides, each of two tangents; then $\frac{1}{7}\sqrt{2} \times 44 = \frac{44}{7}\sqrt{2} =$ the sum of the tangents for *Case* 1.

And, by *Case* 2, the sine is $\frac{1}{99}\sqrt{2}$, and there are $311\frac{1}{7}$ sides, each of two sines; then $\frac{1}{99}\sqrt{2} \times 622\frac{2}{7} = \frac{622\frac{2}{7}}{99}\sqrt{2} =$ the sum of the sines for *Case* 2.

Now $\frac{44}{7}\sqrt{2} = \frac{4356}{693}\sqrt{2}$, and $\frac{622\frac{2}{7}}{99}\sqrt{2} = \frac{4356}{693}\sqrt{2}$; therefore the sum of the tangents for *Case* 1 has become the sum of the sines for *Case* 2.

Again, by *Case* 1, the sine is $\frac{1}{5}$, and there are 22 sides, each of two sines; then $\frac{1}{5} \times 44 = \frac{44}{5} =$ sum of the sines for *Case* 1.

And, by *Case* 2, the tangent is $\frac{1}{70}$, and there are $311\frac{1}{7}$ sides, each of two tangents; then $\frac{1}{70} \times 622\frac{2}{7} = \frac{622\frac{2}{7}}{70}$.

But $\frac{44}{5} = \frac{616}{70}$; therefore $\frac{6\frac{2}{7}}{70}$ *must* be added to the circumference to complete the polygon; thus $\frac{616}{70} + \frac{6\frac{2}{7}}{70} - \frac{622\frac{2}{7}}{70}$; therefore the sum of the sines for *Case* 1 has become the sum of the tangents for *Case* 2.

Again, by *Case* 1, the cosine is $\frac{7}{5}$; and, by *Case* 2, the secant is $\frac{99}{70}$.

Now $\frac{7}{5}=\frac{98}{70}$; then $\frac{99}{70}-\frac{98}{70}=\frac{1}{70}$; therefore $\frac{1}{70}$ has been added to the cosine; but $\frac{1}{70}$ is $\frac{1}{98}$ of $\frac{98}{70}$; therefore the diameter of the circle has been increased its $\frac{1}{98}$th part.

Again, by *Case* 1, the sum of the tangents is $\frac{616}{70}$; and, by *Case* 2, the sum of the tangents is $\frac{622\frac{2}{7}}{70}$; therefore $\frac{6\frac{2}{7}}{70}$ has been added to the sum of the tangents; but $\frac{6\frac{2}{7}}{70}\div\frac{616}{70}=\frac{6\frac{2}{7}}{\not{7}\not{0}}\times\frac{\not{7}\not{0}}{616}=\frac{6\frac{2}{7}}{616}=\frac{44}{4312}=\frac{1}{98}$.

Consequently the circumference of the circle has been increased its $\frac{1}{98}$th part.

Again, by *Case* 1, there is $\frac{1}{50}$ deducted from the square described upon the radius, and there was also $\frac{1}{50}$th deducted from the secant.

And, by *Case* 2, there was $\frac{1}{98}$ added to the diameter and the circumference of the given circle.

Consequently there was $\frac{1}{98}$th added to the area of the given circle.

By *Case* 2, the sine of the given arc is $\frac{1}{99}\sqrt{2}$, and there are $311\frac{1}{7}$ sides, each of which has two sines.

Then $\frac{1}{99}\sqrt{2}\times 622\frac{2}{7}=\frac{622\frac{2}{7}}{99}\sqrt{2}=\frac{4356}{693}\sqrt{2}$.

By *Case* 3, the sine of the given arc is $\frac{1}{1960}\sqrt{2}$, and there are $61603\frac{1}{7}$ sides, each of which has two sines.

Then $\frac{1}{19601}\sqrt{2}\times 123206\frac{2}{7}=\frac{123206\frac{2}{7}}{19601}\sqrt{2}=\frac{862444}{137207}\sqrt{2}$.

Then $\frac{4356}{693}\sqrt{2}=\frac{597673692}{98084451}\sqrt{2}$.

And $\frac{123206\frac{2}{7}}{19601}\sqrt{2}=\frac{597673692}{98084451}\sqrt{2}$.

Consequently the sum of the sines for *Case* 3 is the same as the sum of the sines for *Case* 2.

By *Case* 3, the sine is $\frac{1}{19601}\sqrt{2}$, and there are $61603\frac{1}{7}$ sides contained in the given polygon, each of which has two sines.

Then $\frac{1}{19601}\sqrt{2} \times 123206\frac{2}{7} = \frac{123206\frac{2}{7}}{19601}\sqrt{2}$ = sum of the sines for *Case* 3.

By *Case* 4, the sine is $\frac{1}{768398401}\sqrt{2}$, and there are $2414966403\frac{1}{7}$ sides contained in the given polygon, each of which has two sines.

Then $\frac{1}{768398401}\sqrt{2} \times 4829932806\frac{2}{7} = \frac{4829932806\frac{2}{7}}{768398401}\sqrt{2}$ = sum of the sines for *Case* 4,

Then, by *Case* 3, $\frac{123206\frac{2}{7}}{19601}\sqrt{2} = \frac{946715129360062\frac{2}{7}}{15061377058001}\sqrt{2}$.

And, by *Case* 4, $\frac{4829932806\frac{2}{7}}{768398401}\sqrt{2} = \frac{94671512936006\frac{2}{7}}{1506137705801}\sqrt{2}$.

Therefore the sum of the sines for *Case* 3 is exactly the same as the sum of the sines for *Case* 4, and the length of the perimeter of the inscribed polygon is not changed.

By *Case* 5, the sine is $\frac{1}{118087220531871 3601}\sqrt{2}$, and the number of sides contained in the given polygon is $37113126452873856031\frac{1}{7}$, each of which has two sines.

Then, multiplying the sine by double the number of sides, we have $\frac{742262529057477120 6\frac{2}{7}}{1180872205318713601}$ = the sum of the sines for polygon No. 5.

By *Case* 4, the sum of the sines is $\frac{94671512936006\frac{2}{7}}{15061377058001}\sqrt{2} = \frac{111794958261600865305241217307206\frac{2}{7}}{17785561541618319480379284571601}\sqrt{2}$.

And, by *Case* 5, the sum of the sines is $\frac{7422625290574771206\frac{2}{7}}{1180872205318713601}\sqrt{2}$ $= \frac{111794958261600865305241217307206\frac{2}{7}}{17785561541618319480379284571601}\sqrt{2}$.

Thus we perceive that the sum of the sines, that is the perimeter of the inscribed polygon, from *Case* 1 to *Case* 5, inclusive, is constantly changing the number of its sides at a rate much more rapid than by doubling, while the length of the perimeter, as well as the radius, and the area of the inscribed polygon constantly remains the same; and, if the number of sides were increased *ad infinitum*, they would still continue to be the same, although at every successive step the usual number of sides, according to hypothesis, has been deducted.

For, by *Case* 2, the sine is $\frac{1}{99}\sqrt{2} = \frac{19602}{384218802}\sqrt{2}$.

And, by *Case* 3, the sine is $\frac{1}{19601}\sqrt{2} = \frac{19602}{384218802}\sqrt{2}$.

Then, by subtraction, we have $\frac{1}{384218802}\sqrt{2}$.

Now, by *Case* 3, there are $61603\frac{1}{7}$ sides, each of two sines.

Then $\frac{1}{384218802}\sqrt{2} \times 123206\frac{2}{7} = \frac{123206\frac{2}{7}}{384218802}\sqrt{2} = \frac{862444}{2689531614}$ $= \frac{6\frac{2}{7}}{19602}$; therefore $\frac{6\frac{2}{7}}{19602}\sqrt{2}$ *has* been deducted from the circumference of the given polygon, and *still* the perimeter of the inscribed polygon remains the same.

By *Case* 2, the secant is $\frac{99}{70} = \frac{19602}{13860}$.

And, by *Case* 3, the secant is $\frac{19601}{13860}$.

Therefore $\frac{1}{13860}$ *has* been deducted from the secant, and $\frac{2}{13860}$ from the assumed diameter; consequently, by hypothesis, $\frac{6\frac{2}{7}}{13860}$ *must* be deducted from the circumference.

By *Case* 2, the tangent is $\frac{1}{70}$; and, by *Case* 3, the tangent is $\frac{1}{13860}$.

Then, dividing tangent No. 2 by tangent No. 3, we have by cancellation $\frac{1}{70} \div \frac{1}{13860} = \frac{1}{\cancel{70}} \times \frac{\cancel{13860}}{1} = 198$; therefore each of the tangents for *Case* 2 has been divided into 198 equal parts, each of which is equal to tangent No. 3, or $\frac{1}{13860}$.

Now, by *Case* 2, there are $311\frac{1}{7}$ sides contained in the given polygon, each of which has two tangents; then $622\frac{2}{7} \times 198 = 123212\frac{4}{7} =$ the number of tangents in the polygon for *Case* 3. But, by *Case* 3, the given polygon has only $123206\frac{2}{7}$ tangents; therefore $\frac{6\frac{2}{7}}{13860}$ *has been deducted from the given polygon;* and, in *Case* 4, $\frac{6\frac{2}{7}}{543339720}$ has been deducted from the given polygon, and so in *Cases* 5, 6, 7, etc.; and if it were possible to carry the polygon to infinity it would still be found that in every case $6\frac{2}{7}$ times the tangent of the given arc, or $3\frac{1}{7}$ *sides* of the given polygon, would be deducted for 1 of the values deducted from the diameter.

But in every case the sum of the sines of the given polygon, which is the perimeter of the inscribed polygon, constantly remains the same; that is, the length is not diminished, and the area of the inscribed polygon as well as the radius remains the same, though the number of sides is continually and rapidly approaching infinity; while the $6\frac{2}{7}$ times the tangent is as regularly and constantly deducted in every *case*; *therefore the* $6\frac{2}{7}$ *times the tangent or* $3\frac{1}{7}$ *sides of the given polygon so deducted is exactly equal to the number of sides taken up by the perimeter of the inscribed polygon by the contraction, which necessarily follows the increase of the number of sides, which is one of the strongest proofs that* $3\frac{1}{7}$ *sides in every case is the exact number to be deducted;* consequently, when the *final limit* is reached, the perimeter of the circumscribed polygon, which is the sum of the tangents of the given polygon *will vanish*, and the perimeter of the inscribed polygon, that is the sum of the sines, will become the circumference of the given circle, *which will be infinite! That is no part of the circumference however small is straight!* And, as the radius is the $\sqrt{2}$, and the diameter the $\sqrt{8}$, therefore the circumference of the given circle will be $3\frac{1}{7}$ times the $\sqrt{8}$; consequently the true ratio of the circumference of any circle to its diameter *must be as* $3\frac{1}{7}$ *is to* 1. *QED.*

By *Case* 1, the cosine of the given arc is 1.4, and the secant is $1.4\frac{2}{7}$; then $1.4\frac{2}{7} \times 1.4 = 2$, and $\sqrt{2} =$ the radius of the given circle.

Again, by *Case* 2, the cosine of the given arc is $\frac{140}{99}$, and the secant is $\frac{99}{70}$; then $\frac{140}{99} \times \frac{99}{70} = 2$, and $\sqrt{2} =$ the radius of the given circle.

Again, by *Case* 3, the cosine of the given arc is $\frac{27720}{19601}$, and the secant is $\frac{19601}{13860}$; then $\frac{27720}{19601} \times \frac{19601}{13860} = 2$, and $\sqrt{2} =$ the radius of the given circle; consequently, if the cosine, *which must constantly lie within the circle*, be multiplied by the secant, *which must constantly lie without the circle*, the square root of their product will, in *every case*, be equal to the radius of the given circle. That is, the "*limit of the* *product of two quantities is the product of their limits;*" consequently, when the final limit of the cosine is reached, that limit will be the radius; and, therefore, the radius multiplied by itself, and the square root of the product being extracted, the result will be equal to the radius; thus $\sqrt{2} \times \sqrt{2} = 2$; then $\sqrt{2} =$ the radius of the given circle.

So, also, if the radius of the given circle be $\sqrt{2}$, and a polygon be inscribed within it, and another circumscribed about it, and these two polygons be made to approach the circle and each other, in the manner proposed in the foregoing problem, that is, one from within and the other from without (in such a manner that there is no chance of forcing a limit), and at the same time if the radius of the inscribed polygon be equal to the cosine of the given arc, and the radius of the circumscribed polygon be equal to the secant of the given arc, and these two radii be also made to approach the true radius in the same manner, and at the same time, as the inscribed and circumscribed polygons, to which they belong, approach the circle; that is, one from within and the other from without, so near that the difference between the inscribed and circumscribed polygons, with regard to the circle and to each other, shall be made less than any assignable quantity, then in that case the ratio between the assumed circumference and the assumed diameter will be the same as the ratio between the true circumference and the true diameter.

The following extract from a chapter on the Doctrine of Limits is copied from the May number of the Home and School Journal for 1873, published by Jno. P. Morton & Co., Louisville, Ky. The chapter is the substance of a paper read before the Georgia Teachers' Association by Jno. M. Richardson. It is inserted here, as it is believed that it will establish the theory advanced by the author. For a fuller and more lucid treatise upon the subject, the reader is referred to the Journal itself:

ART. I.—DEFINITION OF A LIMIT.

The constant A is said to be the limit of the variable a when A—a or a—A may be made less than any assignable magnitude, but can never become zero.

ART. II.—DEFINITION OF AN INFINITESIMAL.

The variable difference A—a or a—A, between a variable and its limit, is an infinitesimal. It may be made less than any designated quantity, but can never become zero. Or more generally, *an infinitesimal is a variable whose limit is zero.*

ART. III.—EXAMPLES OF LIMITS AND OF INFINITESIMALS.

1. The common fraction $\frac{1}{7}$ is the limit of the repetend $.\dot{1}4285\dot{7}$; and $\frac{1}{7} - .\dot{1}4285\dot{7}$ is an infinitesimal.

2. The quantity $\sqrt{2}$ is the limit of 1.4142 . . . . .; and $\sqrt{2} - 1.4142$ . . . is an infinitesimal.

3. The sum of the series $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} +$ etc. is $\frac{4}{3}[1 - (\frac{1}{4})^n]$; its limit is $\frac{4}{3}$; and $\frac{4}{3} - \frac{4}{3}[1 - (\frac{1}{4})^n]$ is an infinitesimal.

4. The circle K is the limit of the inscribed and circumscribed polygons I and C; and K — I or C — K is an infinitesimal.

ART. IV.—REMARKS ON THE EXAMPLES.

a. With regard to the first two examples it is evident that, however far the process of division or of extracting the square root be carried, the limits $\frac{1}{7}$ and $\sqrt{2}$ can never be reached.

b. So loosely, though, are the symbols 0 and ∞ used in algebra, that it may be supposed the limit can be reached in the third: "for," say algebras generally, "when $n = \infty$, $(\frac{1}{4})^n = 0$; and then the sum of the series is $\frac{4}{3}$."

But the sum can not be $\frac{4}{3}$ unless the last term is zero; and the last term can not be zero *unless the one-fourth part of something is nothing*, or unless the division of something can be carried so far *that each of its equal parts is nothing*, or *unless something and nothing are equivalent terms*, the very statement of which absurdity should be its own refutation.

c. The distance between two places is *a* miles; a locomotive starting from one travels during the first hour half the distance to the other; during the second hour half the remaining distance; and so on, going each successive hour half the remaining distance. When will it reach its destination? Never; because the half of something can not be zero.

The expression for the sum of the distances is

$$S = \frac{a}{2} + \frac{a}{4} + \frac{a}{8} + \text{etc.} = a(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \text{etc.}) = a[1 - (\tfrac{1}{2})^n].$$

Now, *a* is the limit of the sum of the distances; and, as the locomotive can never reach its destination, the sum of the distances is never equal to *a*.

ART. V. THE DUHAMELIAN PROPOSITION.

If two variables, α and β, are constantly equal, and each tends toward a limit, their limits, A and B, are equal. Or, more generally, *if two variables, α and β, have a constant ratio, and each tends toward a limit, their limits, A and B, have the same ratio.*

If $$\alpha : \beta = r, \qquad (1)$$
then $$A : B = r, \qquad (2)$$
and $$\alpha : \beta :: A : B. \qquad (3)$$

Let $A > B$; α and β less than their limits. If (3) be not true, then

$$\alpha : \beta :: A : B' \quad (4)$$

in which B' is either less or greater than B.

1. Suppose $B' < B$. Let α and β increase until β reaches some value intermediate between B' and B. Denoting these new values of α and β by α' and β', (4) becomes

$$\alpha' : \beta' :: A : B' \quad (5)$$

Passing to products,

$$\alpha' B' = A \beta' \quad (6)$$

But $\alpha' < A$, and $B' < \beta'$; hence $\alpha' B'$ can not be equal to $A \beta'$.

2. Suppose $B' > B$. B' is less than A, for β is supposed to be less than α. Some quantity A', less than A, can always be found to satisfy the proportion.

$$A' : B :: A : B' \quad (7)$$

Comparing (4) and (7),

$$\alpha : \beta :: A' : B, \quad (8)$$

Let α and β increase until $\alpha > A'$ but $< A$. Denoting these new values of α and β by α'' and β'' (8) will become

$$\alpha'' : \beta'' :: A' : B, \quad (9)$$

whence $$\alpha'' B = A' \beta'' \quad (10)$$

But $a'' > A'$, and $B > \beta''$; hence $\alpha'' B$ can not be equal to $A' \beta''$.

If, therefore, the fourth term of (3) can be neither less nor greater than B, it must be equal to B, and (3) must be true, or

$$\alpha : \beta :: A : B. \quad (11)$$

In the same way, if $A < B$, α and β increasing toward their limits; or if α and β are decreasing variables, and A either greater or less than B; it can always be proved that

$$\alpha : \beta :: A : B. \quad (12)$$

Hence, *if the variables α and β have a constant ratio, their limits A and B have the same ratio.*

COROLLARY.

If $\alpha : \beta = 1$, then $A : B = 1$. That is, if $\alpha = \beta$, then $A = B$.

Hence, *if two variables are constantly equal, their limits are equal.*

ART. VI.—ADDITIONAL PROPOSITIONS.

1. *The limit of the sum of several variables, each tending toward a limit, is the sum of their limits.*
2. *The limit of the product of several variables is the product of their limits.*
3. *The limit of the quotient of two variables is the quotient of their limits.*
4. *The limit of any power of a variable is the same power of the limit.*

It is, perhaps, hardly necessary to demonstrate the four preceding propositions; the intelligent student can do so for himself readily enough; but the following one, being rather more abstruse, will be established.

5. *When the limit of the ratio of two quantities is unity their difference divided by*

either one has zero for its limit or is an infinitesimal. Reciprocally, *if the difference of two quantities divided by either one is an infinitesimal, the ratio of the two quantities has unity for its limit.*

PROOF.

Let $$a' - a = \delta \tag{1}$$

whence $$1 - \frac{a}{a'} = \frac{\delta}{a'} \tag{2}$$

$$\frac{a'}{a} - 1 = \frac{\delta}{a} \tag{3}$$

If, now, the limit of $\frac{a}{a'}$ or of $\frac{a'}{a}$ is unity, the limit of $\frac{\delta}{a'}$ or of $\frac{\delta}{a}$ must be zero.

It follows, therefore, *that the limits of the ratios or of the sums of infinitesimals, are not changed when the given infinitesimals are replaced by others which differ from them by quantities which are infinitesimals with regard to them.*

NOTE.—The proposition of Article V. may be found in the works of both Duhamel and J. A. Serret, though the former appears to make much more use of it than the latter. The demonstration given is due to Dr. A. T. Bledsoe, formerly Professor of Mathematics in the University of Virginia, now editor of the Southern Quarterly Review, Baltimore, Md.

The propositions of Article VI. are from Duhamel.

The Quadrature of the Circle has been defined as a problem which has been reduced to finding a straight line equal in length to the circumference of the given circle in terms of the radius; or, how often the square described upon the radius is contained in the area within the given circle, both of which, upon examination, may be said to be reducible to the same thing.

For, when the circumference and the radius of the given circle are known, the rectangle under the radius, it is said, will give the area; and when the area and the radius are known, dividing the area by the square of the radius, it is said, will give the circumference of the given circle.

For, supposing the radius equal to unity its square will be equal to 1; then, if the area contained within the given circle be 6.2831853070, etc. by division we have 6.2831853070, etc., which is equal to the circumference of the given circle.

Again, if the circumference of the given circle be 6.2831853070, etc., and the radius 1, by division we have 6.2831853070, etc., which equals the area of the given circle, which would give for the ratio 3.1415926535, etc.

But, if the true radius be the $\sqrt{2}$, and the assumed radius be $\frac{7}{5}$, and the assumed circumference be $\frac{44}{5}$, then, according to the "Duhamelian theorem," dividing the circumference by the diameter, we have by cancellation $\frac{44}{5} \div \frac{14}{5} = \frac{44}{\not{5}} \times \frac{\not{5}}{14} = \frac{22}{7} = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, which is the true ratio of the circumference to the diameter of the given circle.

By *Case* 1, $\frac{1}{7}\sqrt{2}$ = the tangent of the given arc, and the radius is the $\sqrt{2}$.

Then, by ARTICLE 6, $\frac{1}{7}\sqrt{2} \times \sqrt{2} = \frac{2}{7}$ = area of the circumscribed double triangle.

But there are 22 double triangles contained in the given polygon.

Then, by ARTICLE 9, $\frac{2}{7} \times 22 = \frac{44}{7}$ = area of the circumscribed polygon.

Squaring the radius and dividing, we have $(\sqrt{2})^2 = 2$; then $\frac{44}{7} \div 2 = \frac{22}{7} = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, which is the true ratio of the circumference to the diameter of the given circle.

Again, $\frac{1}{7}\sqrt{2}$ = the tangent of the given arc; then $\frac{1}{7}\sqrt{2} \times 44 = \frac{44}{7}\sqrt{2}$ = the circumference of the given polygon.

But the radius is the $\sqrt{2}$, and the diameter is twice the square root of two; thus $\sqrt{2} \times 2 = 2\sqrt{2}$.

Then, by division, we have $\frac{44}{7}\sqrt{2} \div 2\sqrt{2} = \frac{22}{7} = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, the true ratio of the circumference to the diameter of the given circle.

Again, by *Case* 2, the sine of the given arc is $\frac{1}{99}\sqrt{2}$, and the radius is the $\sqrt{2}$.

Then, by ARTICLE 7, $\sqrt{2} \times \frac{1}{99}\sqrt{2} = \frac{2}{99}$ = area of the inscribed double triangle for double the number of sides.

But there are $311\frac{1}{7}$ sides to the given polygon.

Then, by ARTICLE 10, $\frac{2}{99} \times 311\frac{1}{7} = \frac{622\frac{2}{7}}{99} = \frac{44}{7}$ = area of the inscribed polygon for double the number of sides.

Now the tangent of the given arc is $\frac{1}{70}$, and the unit of comparison is $\left(\frac{1}{70}\right)^2 = \frac{1}{4900}$.

Then, by division and cancellation, we have $\frac{44}{7} \div \frac{1}{4900} = \frac{44}{7} \times \frac{4900}{1} = 30800$.

Therefore there are 30800 squares contained in the given polygon, each of which is represented by $\frac{1}{4900}$; and the square described upon the radius is 2.

Then $2 \div \frac{1}{4900} = 2 \times \frac{4900}{1} = 9800$.

Therefore there are 9800 squares contained in the square described upon the radius, each of which is represented by $\frac{1}{4900}$.

Then $30800 \div 9800 = 3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, the true ratio of the circumference to the diameter of the given circle.

But the area of the circle is said to be the rectangle under the radius; that is, the rectangle contained by the circumference and the radius.

If it is meant for the circumference to be a straight line without regard to the circular figure, *then the definition is true;* but it is admitted by all former mathematicians that this straight line has never yet been found. Moreover, it is asserted that it is not likely that it ever will be found, because "innumerable attempts have been made to find a solution of this problem, but these attempts have been made in vain."

If it is meant for the circumference of a circle to be the boundary of the circular figure known as the circle, then, in that case, it is not true "*that the rectangle contained by the radius and the circumference is equal to the area of the given circle,*" as can very readily be shown. For, witness the following demonstration:

Let the straight line fP, Figures 1 and 2, Plate 6, be the radius of the given circle $= \sqrt{2}$, and let the straight lines PF, PE be the tangents of the given arc, each of which $= \frac{1}{7}\sqrt{2}$; then will the triangles *fPF*, *fPE* be the given circumscribed triangle.

For, by ARTICLE 6, $\sqrt{2} \times \frac{1}{7}\sqrt{2} = \frac{2}{7} =$ area of the circumscribed triangle.

Then, dividing this area by the given radius, we have $\frac{2}{7} \div \sqrt{2} = \frac{1}{7}\sqrt{2}$, which is said to be the arc of the given cirele, but which is, in fact, only the tangent of the given arc, a result too large, because the tangent is greater than its arc.

Again, let the straight line *fh*, *fm*, be the radius of the given circle $= \sqrt{2}$, and the straight lines *gh ge*, *lm ln*, be the sines of the given arc, each of which $= \frac{1}{5}$; then will the triangles *fsm*, *fsn*, be the inscribed double triangles for double the number of sides.

For $\frac{1}{5} \times \sqrt{2} = \frac{1}{5}\sqrt{2} =$ area of the double inscribed triangle for double the number of sides.

Dividing this area by the given radius, we have $\frac{1}{5}\sqrt{2} \div \sqrt{2} = \frac{1}{5}$, which is said to be the arc of the given circle, but which, in fact, is only the sine of the given arc—a result too small, because the sine is less than its arc.

Again, let the straight line *fs* be the given radius $= \sqrt{2}$, and let the straight lines *sr*, *st*, be the tangents of the given arc, each of which $= \frac{1}{70}$.

Then, by ARTICLE 6, $\sqrt{2} \times \frac{1}{70} = \frac{1}{70}\sqrt{2} =$ area of the circumscribed double triangle.

Dividing this area by the given radius, we have $\frac{1}{70}\sqrt{2} \div \sqrt{2} = \frac{1}{70}$, which is said to be the arc of the given circle, but which is, in fact, only the tangent of the given arc—a result too large, because the tangent is greater than its arc.

Again, let the straight line *fs* be the given radius $= \sqrt{2}$, and let the straight lines *sr*, *st*, be the sines of the given arc, each of which is equal to $\frac{1}{99}\sqrt{2}$; then will the triangles *fsr*, *fst*, be the inscribed double triangle for double the number of sides; for $\frac{1}{99} \times \sqrt{2} = \frac{2}{99} =$ the area of the inscribed double triangle for double the number of sides.

Then, dividing the area by the given radius, we have $\frac{2}{99} \div \sqrt{2} = \frac{1}{99}\sqrt{2}$, which, it is said, is the arc of the given circle, but which is, in fact, only the sine of the given arc—a result too small, because the sine is less than its arc.

For the next step let us take the sine, radius, and tangent of *Cases* 3, 4, 5, 6, or 7, and if they do not furnish an example sufficiently small, let the result be carried as far as it is possible for human power and endurance to carry it; and if this again is too small, let us picture to our imagination an example small enough to satisfy the most mathematical exactitude; the relative ratio will still be the same, namely: dividing the area of the circumscribed triangle by the radius gives for the result the tangent, which is too large; and, dividing the area of the inscribed double triangle for double the number of sides gives for the result the sine, which is too small; but the inscribed and circumscribed double triangles, by all former methods, are either inscribed or circumscribed polygons, and whatever is true of one member of a class is true of the whole class; therefore the result obtained by any former method is either too great or too small; and if, as there is nothing to prevent the inscribed polygon from extending beyond the circle, nor to prevent the circumscribed polygon from coming within it, which is very possible, for the limit of the two polygons is the limit obtained; then the result obtained by all former methods must be too small; and, consequently, the rectangle under the radius will not give the *true area* of the given circle.

Again, let the radius of the given circle be $\sqrt{2}$, and the tangent of the given arc be $\frac{1}{70}$; then, by ARTICLE 6, $\sqrt{2} \times \frac{1}{70} = \frac{1}{70}\sqrt{2}$. And the secant of the given arc is $\frac{99}{70}$; then, dividing the area of the circumscribed double triangle by the secant of the given arc, we have $\frac{1}{70}\sqrt{2} \div \frac{99}{70} = \frac{1}{70}\sqrt{2} \times \frac{70}{99} = \frac{1}{99}\sqrt{2} =$ the sine of the given arc.

But the area of the inscribed double triangle for double the number of sides divided by the radius gives the same result, namely: $\frac{1}{99}\sqrt{2} =$ the sine of the given arc.

Therefore the area of the circumscribed double triangle divided by the secant is equal to the area of the inscribed double triangle divided by the radius, namely: $\frac{1}{99}\sqrt{2}$, which is a result too small.

But by the present method the sum of the tangents, which is the perimeter of the circumscribed polygon, vanishes together with that polygon.

And the sum of the sines, which is the perimeter of the inscribed polygon, becomes the circumference of the circle at the same time that the inscribed polygon becomes the circle; therefore the secant vanishes and the cosine becomes the radius, which is the $\sqrt{2}$.

Consequently (PROPOSITION 7, THEOREM), *The true ratio of the circumference to the diameter of the circle is as* $3.\dot{1}4285\dot{7}$, or $3\frac{1}{7}$, is to 1; and, therefore, the last term of the ratio vanishes also.

The following demonstration is taken from "Elements of Euclid," by James Thompson, LLD., pp. 124–5. Belfast: Symmes & McIntyre. London: Longman & Co., and Simpson & Co.:

PROP. XIX. THEOR.—In numbers which are continual proportionals, the difference of the first and second is to the first, as the difference of the first and last is to the sum of all the terms except the last.

If A, B, C, D, E be continual proportionals, $A - B : A :: A - E : A + B + C + D$.

For, since (hyp.) $A : B :: B : C :: C : D :: D : E$, we have (Sup. 8) $A : B :: A + B + C + D : B + C + D + E$. Hence (conv.) $A : A - B :: A + B + C + D : A - E$; and (inver.) $A - B : A :: A - E : A + B + C + D$.

It is evident that if A were the least term, and E the greatest, we should get in a similar manner, $B - A : A :: E - A : A + B + C + D$. Therefore, in numbers, etc., $\frac{1}{7} - \frac{1}{49} : \frac{1}{7} :: \frac{1}{7} : \frac{1}{6}$. For $\frac{1}{7} = \frac{7}{49} - \frac{1}{49} = \frac{6}{49}$, and $\frac{1}{7} \times \frac{1}{7} = \frac{1}{49}$.

Then $\frac{1}{49} \div \frac{6}{49} = \frac{1}{49} \times \frac{49}{6} = \frac{1}{6}$. Therefore the sum of the series $\frac{1}{7} + \frac{1}{49} + \frac{1}{343} + \frac{1}{2401} = \frac{1}{6}$.

Cor. If the series be an infinite decreasing one, the last term will vanish, and, if S be put to denote the sum of the series, the analogy will become $A - B : A :: A : S$; and this, if rA be put instead of B, and the first and second terms be divided by A, will be changed into $1 - r : 1 :: A : S$. The number r is called the *common ratio*, or *common multiplier*, of the series, as by multiplying any term by it the succeeding one is obtained.

PART SECOND.

THE SQUARE ROOT OF TWO,

OR

THE COMMON MEASURE OF THE SIDE AND DIAGONAL OF THE SQUARE;

BEING A

SHORT, EASY, AND CONVENIENT METHOD OF FINDING EITHER THE SIDE OR DIAGONAL OF THE SQUARE, WHEN THE OTHER IS KNOWN, BY COMMON MULTIPLICATION AND DIVISION;

ALSO,

THE SQUARE ROOT OF TWO, BY DIVISION ALONE, TO ONE HUNDRED AND FORTY-FOUR DECIMAL PLACES.

THE SQUARE ROOT OF TWO.

THE following pages are intended to explain the nature, uses, and advantages of the *Common Measure*, as applied to Civil Engineering, Architecture, Draughting, Machinery, Building, Painting, and Landscape Gardening.

Numerous instances could be given where it has been already tested and found to be everything that could be desired.

According to PROPOSITION 1, COR. PART I, the side of any square is to the dioganal as "*one is to the square root of two;* that is, if the side of any square be ONE (1), the diagonal *must* be the square root of two ($\sqrt{2}$); therefore, by the ordinary method, to find the diagonal of a square, when the side is known, use the following

RULE.

Take as many figures of the square root of two as will make the result sufficiently exact, and multiply these figures by the number of units in the side of the given square, the product will be the true diagonal in terms of the side of the square.

NOTE.—If the side of the square be expressed in inches, the diagonal will also be expressed in inches; but if the side be expressed in feet or rods, etc., the diagonal will be expressed in feet or rods, etc.; and if the side contain a decimal or a fraction, the result of that decimal or fraction will have its corresponding result in the diagonal. *In all cases the requisite number of figures of the square root of two must first be obtained.*

When the diagonal is given, to find the side, it is not so easy a task, and requires more time and labor than to find the diagonal when the side is known; but, for the benefit of those who still prefer the former method, I shall here insert the rule which embraces two cases.

CASE 1.

If the diagonal is expressed in integers, such as 1, 2, 3, etc., the side of the square is found by dividing these numbers by the square root of two; the result will be the side in the same denomination as the diagonal.

Case 2.

If the diagonal contains either a fraction or a decimal, this fraction must first be reduced to a decimal and the value pointed off as in reduction of decimals. If a decimal only, it must first be pointed off, then divide this decimal by the square root of two, according to the rule for division of decimals, the quotient will be the side of the square in terms of the diagonal; *but, in all cases, the requisite number of figures of the square root of two must first be found.*

To find an *exact* common measure of the *side* and *diagonal* of the square would be equivalent to finding the exact value of the square root of two; but the square root of two is an *irrational quantity*, therefore it has no end. It is believed that the following common measure comes nearer to an *exact* common measure than any that has been heretofore found, as it can be extended *ad infinitum, and when either the side or the diagonal of any square is known the other can be found by common multiplication and division.*

Owing to the difficulty of working out the necessary figures of the square root of two, and the tax upon the mind necessary to remember the same, civil engineers, architects, draughtsmen, builders, etc., have long since felt the want of a COMMON MEASURE expressed in integers, or a series of numbers, *which would express the relation between the side and diagonal of the square;* one that could be easily remembered, plainly understood, and readily applied. Such, for example, as the ratio between the *circumference and diameter of the circle*, found by Mœtius in 1640, namely: $\frac{355}{113}$, or 113|355, which it is said will give the ratio to *six* decimal places correct, viz.: 3.141592.

The author is happy to state that such a common measure of the side and diagonal of the square has been found which may be expressed in *integers*, and he ventures to hope that when it is fully tested the old method of squaring the side, doubling and extracting the square root, will be gladly cast aside as a noble relict of "*Auld Lang Syne*," when simpler and easier methods were unknown.

The *common measure* here introduced combines all the advantages of conciseness, simplicity, and perfection, for by its aid the desired results are reached much more rapidly than by any former method and with equal exactness.

FIRST EXAMPLE.—Suppose a builder is laying the foundation for a house and wants to make each of the corners a right angle—that is, a perfectly square corner; how will he do it?

ANSWER.—First measure 10.5 feet on each side of the right angle in the direction of the sides, commencing at the angular point; then the distance across from the two extreme points will be 14.85 feet, or 14 feet, 10.2 inches. This result is found by multiplying the side of the given square by 99 and dividing the product by 70, which gives the desired result. See Plate 8, Figure 1.

SECOND EXAMPLE.—Suppose a civil engineer, while surveying, wants to lay out a right angle without the use of his instruments; how will he do it?

ANSWER.—Measure with a tape line any distance, on either side of the right angle, say 105 feet, in the direction of the sides, commencing at the angular point. Then will the distance across from the extreme points be 148.5 feet, or 148 feet 6 inches. This result is found precisely like the former, namely, by multiplying the side of the given square by 99, and dividing the product by 70, which gives the desired result. See Plate 8, Figure 2.

THIRD EXAMPLE.—Suppose a mechanic has a given circle, the diameter of which is 9.9 feet, or 9 feet 10.8 inches, and he wants to find the side of the largest square which it is possible to *inscribe* in the given circle; how will he find it?

ANSWER.—Multiply the given diameter by 70, and divide the product by 99, which will give the desired result, which will be 7 feet. See Plate 8, Figure 3.

If any person has a desire to test the correctness of these results by the former method, he is at liberty to do so, and will find them correct to the fourth decimal place.

NOTE.—In the third example above, the diameter of the circle may be regarded as the diagonal of the given square, the side of which was required to be found; therefore the square which is so formed within the given circle is said to be inscribed within it.

From these three examples, which embrace two cases, we deduce the following rules:

CASE 1.

When the side of any square is known, to find the diagonal correct to *four places* of figures.

PLATE VIII.

Fig. 1

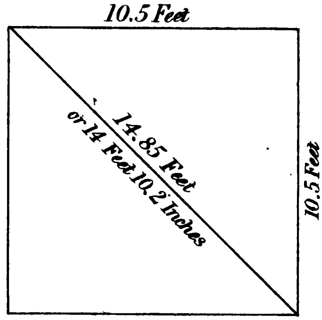


Fig. 2

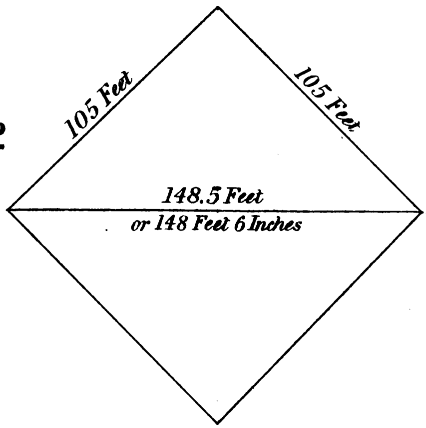
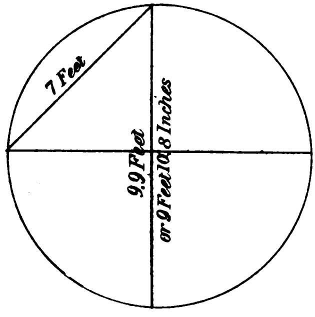


Fig. 3



Rule.

Multiply the side of the given square by 99, and divide the product by 70; the quotient will be the diagonal of the square in terms of the side.

Case 2.

When the diagonal of any square is known, to find the side.

Rule.

Multiply the diagonal of the given square by 70, and divide the product by 99; the quotient will be the side of the given square in terms of the diagonal.

NOTE.—It is believed by the author that the above numbers will answer every purpose for all ordinary measurements; besides this, they are simple and easily remembered by the most ordinary mind. If, however, still greater accuracy is required, attention is invited to the following, which the author ventures to hope will be found to be sufficiently correct to satisfy the most careful.

By *Case* 2, PART 1, it is shown that where the radius of the given circle is the *square root of two*, the side of the inscribed square will be *two*, and the side of the inscribed square, the diagonal of which corresponds with the radius, is *one*. Therefore the secant No. 2 is $\frac{99}{70}$, which, by division, gives the square root of two to five places of figures correct, thus: 1.4142; that is, if the side of the inscribed square, the diagonal of which corresponds with the radius, be divided into 70 equal parts, the diagonal of the same square will be 99 of the same parts nearly.

By *Case* 3, PART 1, it is shown that the secant No. 3 is $\frac{19601}{13860}$, which, by division, gives the square root of two to nine places of figures correct, thus: 1.41421356; that is, if the side of the inscribed square, whose diagonal corresponds with the radius, be divided into 13860 parts, the diagonal of the same square will be 19601 of the same parts nearly.

By *Case* 4, PART 1, it is shown that the secant No. 4 is $\frac{768398401}{543339720}$, which, by division, gives the square root of two to 18 places of figures correct, thus: 1.41421356237309504; that is, if the side of the inscribed square, the diagonal of which corresponds with the radius, be divided into 543339720 parts, the diagonal of the same square will be 768398401 of the same parts nearly.

By *Case* 5, Part 1, it is shown that the secant No. 5 is $\frac{1180872205318713601}{835002744095575440}$, which, by division, gives the square root of two to 36 places of figures correct, thus: 14142135623730950488016887-2420969807; that is, if the side of the inscribed square, the diagonal of which corresponds with the radius, be divided into 835002744095-575440 parts, the diagonal of the same square will be 1180872205318-713601 of the same parts nearly.

By *Case* 6, Part 1, it is shown that the secant No. 6 is $\frac{2788918330588564181308597538924774401}{1972063063734639263984455073299118880}$, which, by division, gives the square root of two to 72 places of figures correct, thus: 14142135-623730950488016887242096980785696718753769480731766797379-90732478; that is, if the side of the inscribed square, the diagonal of which corresponds with the radius, be divided into 1972063063734639-263984455073299118880 sides, the diagonal of the same square will be 2788918330588564181308597538924774401 of the same parts nearly.

By *Case* 7, Part 1, it is shown that the secant No. 7 is $\frac{155561309093858075352247798426396866254686480657981776271265143374898176 01}{10999845055052358763719716313591640029823644051645720435317842392159581760}$, which, by division, gives the square root of two to 144 places of figures correct, thus: 1.41421356237309504880168872420-96980785696718753769480731766797379907324784621070388503-87534327641563643977195724018929160771077122365330384600627; that is, if the side of the inscribed square, the diagonal of which corresponds with the radius, be divided into 10999845655052358763719716313591 sides, the diagonal of the same square will be 1555613-090938580753522477984263968662546864806579817762712651433-7489817601 of the same parts nearly.

By *Case* 8, the square root of two can be found, by division, to 288 places of figures, by *Case* 9 to 576 places, by *Case* 10 to 1152 places, and so on *ad infinitum*; and, in every case, if the side of the inscribed square, the diagonal of which corresponds with the radius, be divided into the requisite number of parts, the diagonal will still be expressed by a certain number of the same parts till it reaches the vanishing point, *when the square root of the sum of the squares of the two sides of any square can be extracted exactly*,

But if by the word "infinite," *indefinite extension* only is meant, then the operation may be continued without end, and in THAT CASE there will forever remain an indivisible unit, the square root of which never can be extracted.

If, then, the true ratio of the circumference of the circle to its diameter be as $3\frac{1}{7}$ is to 1, and the radius of the given circle be the $\sqrt{2}$, then will the area contained within the given circle be $6\frac{2}{7}$, and the side of the inscribed square will be two, while the area will be four. But the area of the circumscribed square is *double* the area of the inscribed square, therefore the area of the circumscribed square is 8.

Again, the square described upon the diameter of any circle is equal to the circumscribed square.

But the diameter of the given circle is $\sqrt{8}$, therefore the area of the circumscribed square is equal to $\sqrt{8} \times \sqrt{8} = 8$.

Again, the area of the inscribed square is *one-half* the area of the circumscribed square; therefore the area of the inscribed square is 4.

Now the area of the given circle is $6\frac{2}{7}$, and the area of the inscribed square is 4, and the area of the circumscribed square is 8; *therefore the area of the circle is to the area of the inscribed square as* 11 *is to* 7; and the area of the circle is to the area of the circumscribed square as 11 is to 14.

Therefore to find the area of any circle, when the diameter is known, use the following

RULE.

Multiply the square of the diameter by 11, *and divide the product by* 14, *the quotient will be the area of the circle;* or, *square the radius, and multiply it by* $3\frac{1}{7}$, *the product will be the area of the circle.*

PART THIRD.

ON THE RIGHT-ANGLED TRIANGLE;

CONTAINING

A VARIETY OF METHODS FOR FINDING AN INFINITE SERIES OF RIGHT-ANGLED TRIANGLES, THE SIDES OF WHICH SHALL BE IN WHOLE NUMBERS, FOR THE USE OF CIVIL ENGINEERS, ARCHITECTS, DRAUGHTSMEN, AND MECHANICS.

ON THE RIGHT-ANGLED TRIANGLE.

OF RIGHT-ANGLED TRIANGLES IN NUMBERS, OR RIGHT-ANGLED TRIANGULAR NUMBERS.

RIGHT-ANGLED triangular numbers are rational numbers so related to each other that the sum of the squares of two of them is equal to the square of the third.

The numbers 3, 4, and 5 have this property—$3^2 + 4^2$ being equal to 5^2.

Right-angled triangular numbers must be severally unequal, for if the two less ones could be each represented by a, and the third or greatest by b, then $2a^2 = b^2$, $b = a\sqrt{2}$, an irrational number, whatever is the value of a.

The area of a right-angled triangle, whose sides are rational, can not be equal to a rational square.

If a, b, and c represent the sides of a triangle, and C be the angle opposite c, then, if $C = 90°$, $a^2 + b^2 = c^2$; if $C = 120°$, $a^2 + ab + b^2 = c^2$; and if $C = 60°$, $a^2 - ab + b^2 = c^2$.

If n represent any number, and m any other number less than n, then $n^2 + m^2$ will represent the hypothenuse of a right-angled plane triangle, of which the other two sides are respectively n^2, m^2, and $2nm$.

For example, if $n=2$ and $m=1$, then $n^2+m^2=5$, $n^2-m^2=3$, and $2nm=4$, which are right-angled triangular numbers.

If $n=7$ and $m=2$, the formulas give 53, 45, and 28 for the numbers, and $53^2=2809=45^2+28^2$.

We shall now propose and solve a few of the most curious problems respecting the right-angled triangles.

PROBLEM 1.

To find as many right-angled triangles in numbers as we please.

This may be effected by the concluding formulas, which we have just given, but we think it right to add the following methods:

Take any two numbers at pleasure, for example 1 and 2, which we shall call generating numbers; multiply them together, having doubled the product, we obtain one of the sides of the triangle, which in this case will be 4. If we then square each of the generating numbers, which in the present example will give 4 and 1, their difference 3 will be the second side of the triangle, having 1 and 2, for their generating numbers are 3, 4, and 5.

If 2 and 3 had been assumed as generating numbers we should have found the sides to be 5, 12, and 13, and the numbers 1 and 3 would have given 6, 8, and 10.

Another Method.—Take a progression of whole or fractional numbers, $1\frac{1}{3}$, $2\frac{2}{5}$, $3\frac{3}{7}$, and $4\frac{4}{9}$, etc., the properties of which are:

1st. The whole numbers are those of the common series and have unity for their common difference.

2nd. The numerators of the fractions annexed to the whole numbers are also the natural numbers.

3rd. The denominators of these fractions are the odd numbers 3, 5, 7, etc.

Take any term of this progression, for example $3\frac{3}{7}$, and reduce it to an improper fraction by multiplying the whole number 3 by 7, and adding to 21 the product, the numerator 3 will give $\frac{24}{7}$. The numbers 7 and 24 will be the sides of a right-angled triangle, the hypothenuse of which may be found by adding together the square of these two numbers, viz.: 49 and 576, and extracting the square root of the sum. The sum in this case being 625, the square root of which is 25, this number will be the hypothenuse required. The sides, therefore, of the triangle produced by the above term of the generating progression are 7, 24, and 25.

In like manner the first term will give the right-angled triangle 3, 4, and 5; the second term, $2\frac{2}{5}$, will give 5, 12, and 13; the fourth, $4\frac{4}{9}$, will give 9, 40, and 41. All these triangles have the ratio of their sides different; and they all possess this property, that the greatest side and the hypothenuse differ only by unity.

The progression, $1\frac{7}{8}$, $2\frac{11}{12}$, $3\frac{15}{16}$, $4\frac{19}{20}$, etc., is of the same kind as the preceding. The first term of it gives the right-angled triangle 8, 15, and 17; the second term gives the triangle 12, 35, and 37; the third

10

triangle 16, 63, 65, etc. All these triangles it is evident are different in regard to the proportion of their sides; and they all have this peculiar property, that the difference between the greater side and the hypothenuse is the number 2.

Problem 2.

To find any number of right-angled triangles in numbers, the sides of which shall differ only by unity.

To resolve this problem we must find out such numbers that the double of their squares, plus or minus unity, shall also be square numbers; of this kind are the numbers 1, 2, 5, 12, 29, 70, etc., for twice the square of 1 is 2, which, diminished by unity, leaves 1, a square number. In like manner twice the square of 2 is 8, to which, if we add 1, the sum 9 will be a square number, and so on.

Having found these numbers, take any two of them which immediately follow each other, as 1 and 2, or 3 and 5, or 12 and 29, for generating numbers; the right-angled triangles arising from them will be of such a nature that their sides will differ from each other only by unity.

The following is a table of these triangles, with their generating numbers:

Generating Numbers.		Sides.		Hypothenuse.
1	2	3	4	5
2	5	20	21	29
5	12	119	120	169
12	29	696	697	985
29	70	4059	4060	5741
70	169	23660	23661	33461

But if the problem were to find a series of triangles of such a nature that the hypothenuse of each should exceed one of the sides only by unity, the solution would be much easier. Nothing, in this case, would be necessary but to assume as the generating numbers of the required triangles any two numbers having unity for their difference.

The following is a table, similar to the preceding of the six first right-angled triangles produced by the first numbers of the natural series:

Generating Numbers.		Sides.		Hypothenuse.
1	2	3	4	5
2	3	5	12	13
3	4	7	24	25
4	5	9	40	41
5	6	11	60	61
6	7	13	84	85

If we assume as generating numbers the respective sides of the preceding series of triangles, we shall have a new series of right-angled triangles, the hypothenuse of which will always be square numbers, as may be seen in the following table:

Generating Numbers.		Sides.		Hypothenuse.	Roots.
3	4	7	24	25	5
5	12	119	120	169	13
7	24	336	527	625	25
9	40	720	1519	1681	41
11	60	1320	3479	3721	61
13	84	2184	6887	7225	85

It may be here observed that the roots of the hypothenuses are always equal to the greater of the generating numbers increased by unity. But if the second side and the hypothenuse of each triangle in the above table, which differ only by unity, were assumed as the generating numbers, we should have a series of right-angled triangles, the least sides of which would always be squares.

A few of these are as follows:

Generating Numbers.		Sides.		Hypothenuse.
4	5	9	40	41
12	13	25	312	313
24	25	49	1200	1201
40	41	81	3280	3281

In the last place, if it were required to find a series of right-angled triangles, the sides of which shall be always a cube, we have nothing to do but to take, as generating numbers, two following terms in the progression of triangular numbers, as 1, 2, 3, 6, 10, 15, 21, etc.

By way of example, we shall here give the first four of these triangles:

Generating Numbers.		Sides.		Hypothenuse.
1	3	6	8	10
3	6	36	27	45
6	10	120	64	136
10	15	300	125	326

The following short and easy method for finding a series of right-angled triangles, together with the rule for their solution, is inserted by the author with the hope that it may be of service to practical men. The triangles which belong to this series must all have these properties in common.

1st. All the sides of the given triangles shall be expressed in integers, and the shortest of the three sides, which we will call the *perpendicular*, shall be expressed by one of the odd numbers, 3, 5, 7, 9, 11, 13, 15, 17, 19, etc.

2nd. The hypothenuse of each triangle shall exceed the longer of the other two sides, which we will call the *base*, by unity.

3rd. The square described upon the *perpendicular* shall be equal to the sum of the other two sides.

4th. If the square of the perpendicular be increased by unity, the sum, divided by *two*, will give the hypothenuse of the given triangle; but if the square of the perpendicular be diminished by unity, the difference, divided by two, will give the base of the given triangle.

5th. If an arc of a circle be described across double the width of the triangle, with a radius equal to the base of the triangle, the double of the area of the given triangle, divided by the hypothenuse, will be the *sine* of the given arc.

6th. If the base of the given triangle be divided by the hypothenuse, it will give the *versed sine* of the given arc.

7th. If the square of the radius be divided by the hypothenuse of the given triangle, it will give the *cosine* of the given arc.

Problem 1.

For example, let 3 be the perpendicular of the given triangle; then $3^2 = 9 =$ sum of the other two sides.

Because $9 + 1 \div 2 = 5 =$ the hypothenuse of the given triangle.

Again. $9 - 1 \div 2 = 4 =$ base of the given triangle.

PLATE IX.

Fig. 1

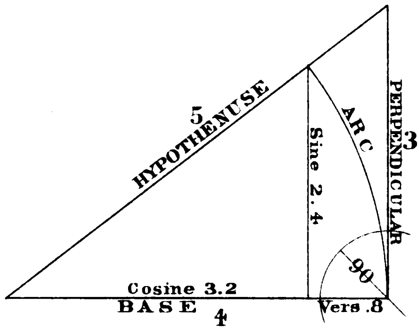


Fig. 2

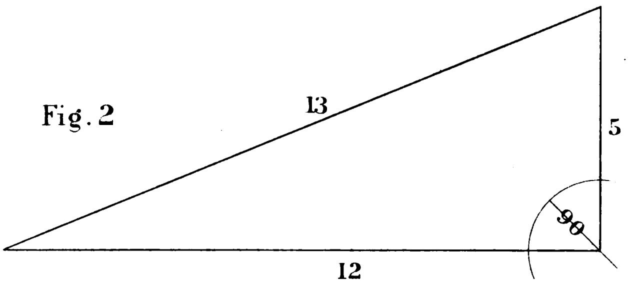


Fig. 3

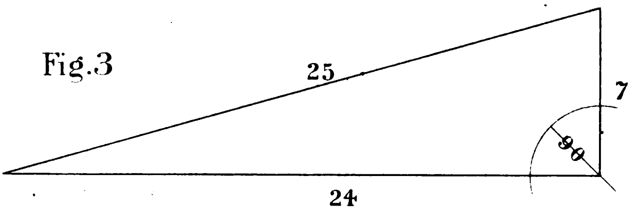
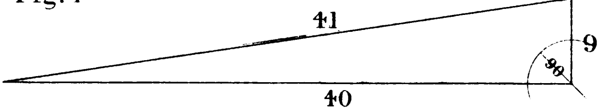


Fig. 4



Therefore the sides of the triangle are expressed in the integers 3, 4, and 5.

Again, $5 - 4 = 1$. Therefore the hypothenuse exceeds the base by unity.

Now $3 \times 4 = 12 =$ double the area of the given triangle.

Then $12 \div 5 = 2.4 =$ the sine of the given arc.

And $4 \times 4 = 16 =$ the square of the radius or base of the triangle.

Then $16 \div 5 = 3.2 =$ cosine of the given arc. See Plate 9, Figure 1.

Subtracting the cosine from the radius will give the versed sine of the arc.

PROBLEM 2.

Let 5 be the perpendicular of the given triangle; then $5^2 = 25 =$ the sum of the other two sides.

For $25 + 1 \div 2 = 13 =$ the hypothenuse of the given triangle.

Again, $25 - 1 \div 2 = 12 =$ the base of the given triangle.

Therefore the sides of the given triangle are expressed by the integers 5, 12, and 13. See Plate 9, Figure 2.

PROBLEM 3.

Let 7 be the perpendicular of the given triangle; then $7^2 = 49 =$ the sum of the other two sides.

For $49 + 1 \div 2 = 25 =$ the *hypothenuse* of the given triangle.

Again, $49 - 1 \div 2 = 24 =$ the *base* of the given triangle.

Therefore the sides of the given triangle are expressed by the integers 7, 24, and 25. See Plate 9, Figure 3.

PROBLEM 4.

Let 9 be the *perpendicular* of the given triangle; then $9^2 = 81 =$ the sum of the other two sides.

For $81 + 1 \div 2 = 41 =$ the *hypothenuse* of the given triangle.

And $81 - 1 \div 2 = 40 =$ the *base* of the given triangle.

Therefore the sides of the given triangle are expressed by the integers 9, 40, and 41, See Plate 9, Figure 4.

The other parts to the triangle are found exactly in the same manner as shown in Problem 1, and the student is recommended to make the calculation for himself. The triangles, as may readily be seen, can be continued *ad infinitum.*

The following table will give the first 14 of the series, which the student can continue as far as he may desire.

Perpendicular.	Base.	Hypothenuse.
3	4	5
5	12	13
7	24	25
9	40	41
11	60	61
13	84	85
15	112	113
17	144	145
19	180	181
21	220	221
23	264	265
25	312	313
27	364	365
29	420	421

APPENDIX.

A CHAPTER ON CONSTRUCTION

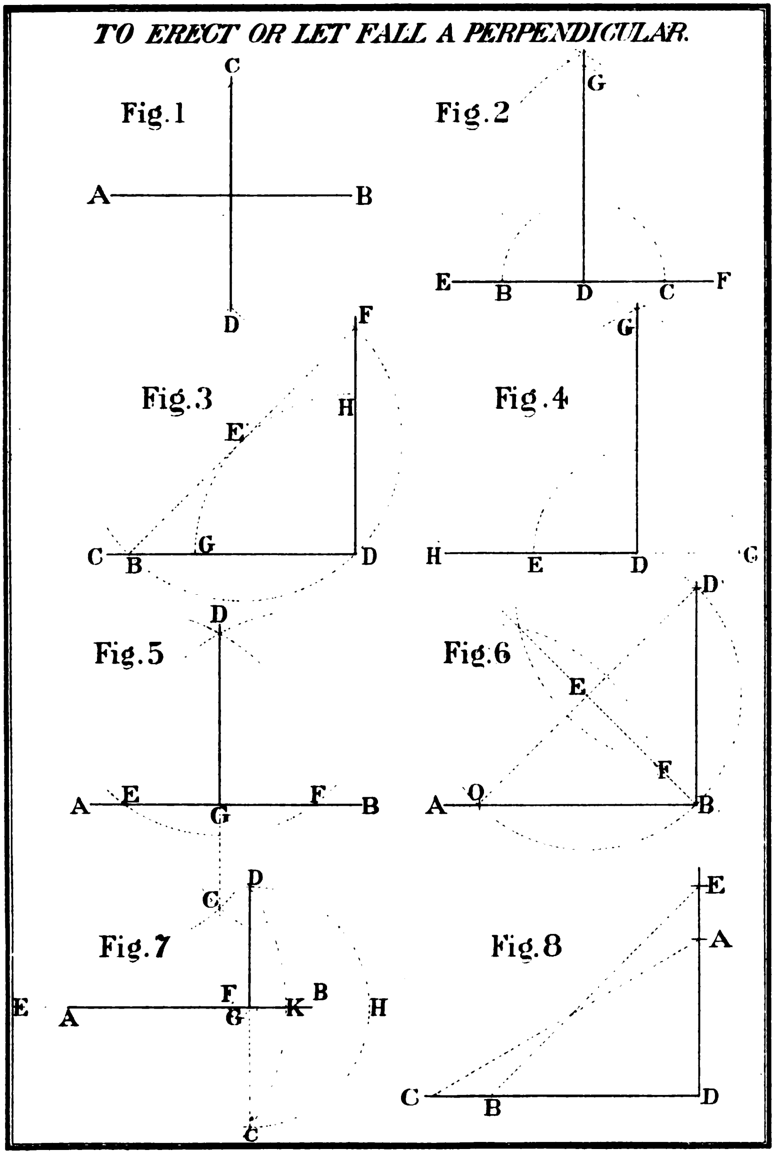
FOR THE USE AND BENEFIT OF

DRAUGHTSMEN, BUILDERS,

AND

MECHANICS.

PLATE X.


TO ERECT OR LET FALL A PERPENDICULAR.
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6
Fig. 7
Fig. 8

CONSTRUCTION.

PLATE 10.

TO ERECT OR LET FALL A PERPENDICULAR.

PROBLEM 1, FIGURE 1.

To Bisect the Right Line AB by a Perpendicular.

1st. With any radius greater than half of the given line, and with one point of the dividers in *A* and *B*, successively draw two arcs intersecting each other in *C* and *D*.

2nd. Through the points of intersection draw *CD*, which is the perpendicular required.

PROBLEM 2, FIGURE 2.

From the Point D in the Line EF to Erect a Perpendicular.

1st. With one foot of the dividers placed in the given point *D*, with any radius less than one-half of the line, describe an arc cutting the given line in *B* and *C*.

2nd. From the points *B* and *C*, with any radius greater than *BD*, describe two arcs cutting each other in *G*.

3rd. From the point of intersection draw *GD*, which is the perpendicular required.

PROBLEM 3, FIGURE 3.

To Erect a Perpendicular when the Point D is At or Near the End of a Line.

1st. With one foot of the dividers in the given point *D*, with any radius, as *DE*, draw an indefinite arc *GH*.

2nd. With same radius and the dividers in any point as *E*, draw the arc *BDF*, cutting the line *CD* in *B*.

3rd. From the point *B*, through *E*, draw a right line cutting the arc in *F*.

4. From F draw FD, which is the perpendicular required.

NOTE.—It will be perceived that the arc BDF is a semicircle, and the right line BF a diameter; if, from the extremities of a semi circle, right lines be drawn to any point in the curve, the angle formed by them will be a right angle; this affords a ready method of forming a "square corner," and will be found useful on many occasions, as its accuracy may be depended on.

Problem 4, Figure 4.

Another Method of Erecting a Perpendicular when At or Near the End of a Line.

Continue the line HD toward C, and proceed as in Problem 2. The letters of reference are the same.

Problem 5, Figure 5.

From the Point D to Let Fall a Perpendicular to the Line AB.

1st. With any radius greater than DG, and one foot of the compasses in D, describe an arc cutting AB in E and F.

2nd. From E and F, with any radius greater than EG, describe two arcs cutting each other as in C.

3rd. From D draw the right line DC, then DG is the perpendicular required.

Problem 6, Figure 6.

When the Point D is Nearly Opposite the End of the Line.

1st. From the given point D, draw a right line to any point of the line AB, as O.

2nd. Bisect OD by Problem 1 in E.

3rd. With one foot of the compasses in E, with a radius equal to ED or EO, describe an arc cutting AB in F.

4th. Draw DF, which is the perpendicular required.

NOTE.—The reader will perceive that we have arrived at the same result as we did by Problem 3, but by a different process, the right angle being formed with a semicircle.

Problem 7, Figure 7.

Another Method of Letting Fall a Perpendicular when the Given Point D is Nearly Opposite the End of the Line.

1st. With any radius, as ED, and one foot of the compasses in the line AB, as at F, draw an arc DHC.

PLATE XI.

CONSTRUCTION AND DIVISION OF ANGLES.

Fig. 1

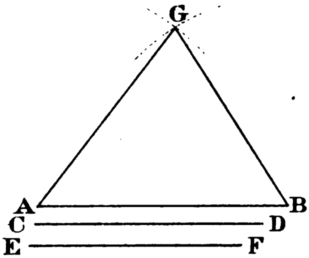


Fig. 2

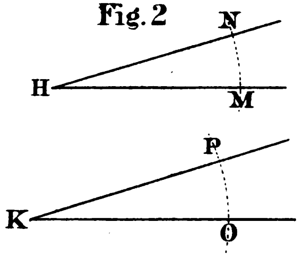


Fig. 3

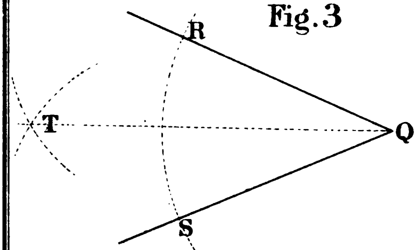


Fig. 4

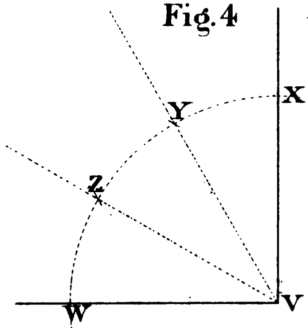


Fig. 5

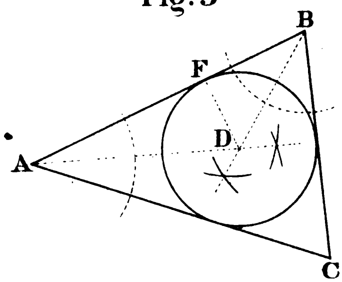
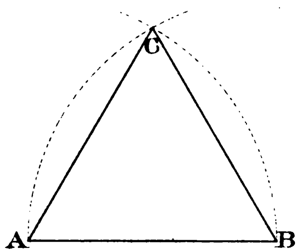


Fig. 6



2nd. With any other radius, as *ED*, draw another arc *DKC*, cutting the first arc in *D* and *C*.

3rd. From *D* draw *DC*, and *DG* is the perpendicular required.

NOTE.—The points *E* and *F*, from which the arcs are drawn, should be as far apart as the line *A B* will admit of, as the exact point of intersection can be more easily found; for it is evident that the nearer two lines cross each other at a right angle the finer will be the point of contact.

PROBLEM 8, FIGURE 8.

To Erect a Perpendicular Line at D, the End of the Line C D, with a Scale of Equal Parts.

1st. From any scale of equal parts take three in your dividers, and, with one foot in *D*, cut the line *CD* in *B*.

2nd. From the same scale take four parts in your dividers, and, with one foot in *D*, draw an indefinite arc toward *E*.

3rd. With a radius equal to five of the same parts, and one foot of the dividers in *B*, cut the other arc in *E*.

4th. From *E* draw *E D*, which is the perpendicular required.

NOTE.—1st. If four parts were first taken in the dividers and laid off on the line *CD*, then three parts should be used for striking the indefinite arc at *A*, and the five parts struck from the point *C*, which would give the intersection *A* and arrive at the same result.

2nd. On referring to the definitions of angles, it will be found that that side of a right-angled triangle opposite the right angle is called the hypothenuse; thus the line *EB* is the hypothenuse of the triangle *EDB*.

3rd. The square of the hypothenuse of a right-angled triangle is equal to the sum of the squares of both the other sides.

4th. The square of a number is the product of that number multiplied by itself. Example: The length of the side *DE* is 4, which, multiplied by 4, will give 16. The length of *DB* is 3, which, multiplied by 3, gives for the square 9. The products of the two sides added together give 25. The length of the hypothenuse is 5, which, multiplied by 5, gives 25.

PLATE 11

CONSTRUCTION AND DIVISION OF ANGLES.

PROBLEM 9, FIGURE 1.

The Length of the Sides of a Triangle AB, CD, and EF being given to Construct the Triangle, the Two Longest Sides to be Joined Together at A.

1st. With the length of the line *CD* for a radius, and one foot in *A*, draw an arc at *G*.

2nd. With the length of the line *EF* for a radius, and one foot in *B*, draw an arc cutting the other arc at *G*.

3rd. From the point of intersection draw *GA* and *GB*, which complete the figure.

Problem 10, Figure 2.

To Construct an Angle at K Equal to an Angle at H.

1st. From *H*, with any radius, draw an arc cutting the sides of the angle as at *MN*.

2nd. From *K*, with the same radius, describe an indefinite arc at *O*.

3rd. Draw *KO* parallel to *HM*.

4th. Take the distance from *M* to *N* and apply it from *OP*.

5th. Through *P* draw *KP*, which completes the figure.

Problem 11, Figure 3.

To Bisect the Given Angle Q by Right Line.

1st. With any radius, and one foot of the dividers in *Q*, draw an arc cutting the sides of an angle as in *R* and *S*.

2nd. With the same or any other radius greater than one-half *RS*, from the points *S* and *R*, describe two arcs cutting each other at *T*.

3rd. Draw *TQ*, which divides the angle equally.

Note.—This problem may be very usefully applied by workmen on many occasions. Suppose, for example, the corner *Q* be the corner of a room, and a washboard or cornice has to be fitted around it. First apply the level to the angle and lay it down on a piece of board; bisect the angle as above, then set the level to the center line, and you have the exact angle for cutting the miter. This rule will apply equally to the internal or external angle. Most good, practical workmen have several means for getting the cut of the miter, and to them this demonstration will appear unnecessary; but I have seen many men make sad blunders for want of knowing this simple rule.

Problem 12, Figure 4.

To Trisect an Angle.

1st. From the angular point *V*, with any radius, describe an arc cutting the sides of the angle, as in *X* and *W*.

2nd. With the same radius, from the points *X* and *W*, cut the arc in *Y* and *Z*.

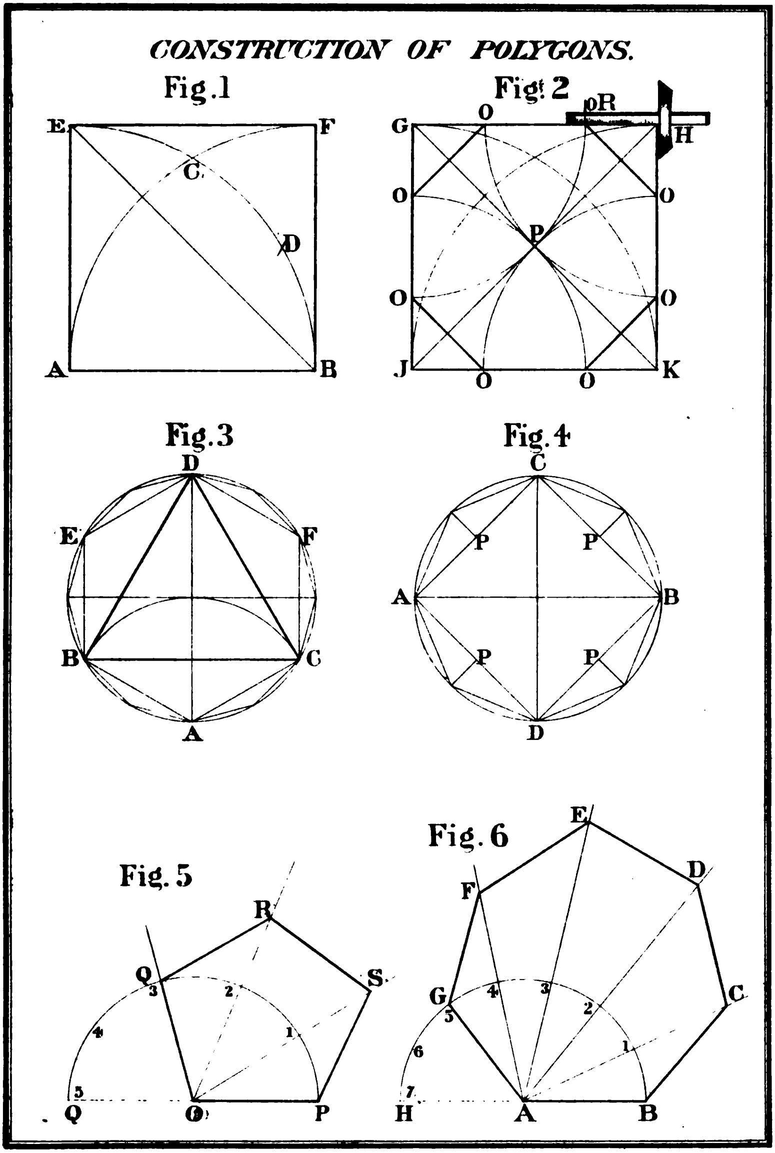
3rd. Draw *YV* and *ZV*, which will divide the angle as required.

Problem 13, Figure 5.

In the Triangle ABC, to Describe a Circle Touching all its Sides.

1st. Bisect two of the angles by Problem 11, as *A* and *B*, the di-

PLATE XII.


CONSTRUCTION OF POLYGONS.
Fig. 1
E F C D A B
Fig. 2
G O R H O P O O J O O K
Fig. 3
D E F B C A
Fig. 4
C P P A B P P D
Fig. 5
R Q S 3 2 1 4 5 Q O P
Fig. 6
E D F G C 4 3 2 1 5 6 7 H A B

viding lines, will cut each other in *D*, then *D* is the center of the circle.

2nd. From *D* let fall a perpendicular to either of the sides, as at *F*, then *DF* is the radius with which to describe the circle from the point *D*.

Problem 14, Figure 6.

On the Given Line AB to Construct an Equilateral Triangle, the Line AB to become One of its Sides.

1st. With a radius equal to the given line, from the points *A* and *B*, draw two arcs intersecting each other in *C*.

2nd. From *C* draw *CA* and *CB* to complete the figure.

Plate 12.

CONSTRUCTION OF POLYGONS.

Problem 15, Figure 1.

On a Line AB to Construct a Square whose Side shall be Equal to the Given Line.

1st. With the length *AB* for a radius, from the points *A* and *B*, describe two arcs cutting each other in *C*.

2nd. Bisect the arc *CA* or *CB* in *D*.

3rd. From *C*, with a radius equal to *CD*, cut the arc *BE* in *E*, and the arc *AF* in *F*.

4th. Draw *AE*, *EF* and *FB*, which complete the square.

Problem 16, Figure 2.

In the Given Square GHKJ to Inscribe an Octagon.

1st. Draw the diagonals *GK* and *HJ* intersecting each other in *P*.

2nd. With a radius equal to half the diagonal from the corners *GHK* and *J*, draw arcs cutting the sides of the square in *O*, *O*, *O*, etc.

3rd. Draw the lines *O*, *O*, *O*, *O*, etc., and they will complete the octagon.

Note.—This mode is used by workmen when they desire to make a piece of wood for a roller, or any other purpose. It is first made square and the diagonals drawn across the end; the distance of one-half the diagonal is then set off, as from *G* to *R* in the diagram, and a gauge set from *H* to *R*, which, run on all the corners, gives the lines for reducing the square to an octagon; the corners are again taken off, and finally finished with a tool appropriate to the purpose. The center of each face of the octagon gives a line in the circumference of the circle, running the whole length of the piece; and, as there are eight of those lines equidistant from each other, the further steps in the process are very simple.

PROBLEM 17, FIGURE 3.

In a Given Circle to Inscribe an Equilateral Triangle, a Hexagon, and Dodecagon.

1st. For the *triangle:* With the radius of the given circle from any point in the circumference, as at *A*, describe an arc cutting the circle in *B* and *C*.

2nd. Draw the right line *BC*, and, with a radius equal to *BC* from the points *B* and *C*, cut the circle in *D*.

3rd. Draw *DB* and *DC*, which complete the triangle.

4th. For the *hexagon:* Take the radius of the given circle and carry it round on the circumference six times; it will give the points *ABEDFC*; through them draw the sides of the hexagon. The radius of a circle is always equal to the side of an hexagon inscribed.

5th. For the *dodecagon:* Bisect the arcs between the points found for the hexagon, which will give the points for inscribing the dodecagon.

PROBLEM 18, FIGURE 4.

In a Given Circle to Inscribe a Square and an Octagon.

1st. Draw a diameter *AB*, and bisect it with a perpendicular by Problem 1, giving the points *CD*.

2nd. From the points *ACBD* draw the right lines forming the sides of the square required.

3rd. For the *octagon:* Bisect the sides of the square, and draw perpendiculars to the circle, or bisect the arcs between the points *ACBD*, which will give the other angular points of the required octagon.

PROBLEM 19, FIGURE 5.

On the Given Line OP to Construct a Pentagon, OP being the Length of the Side.

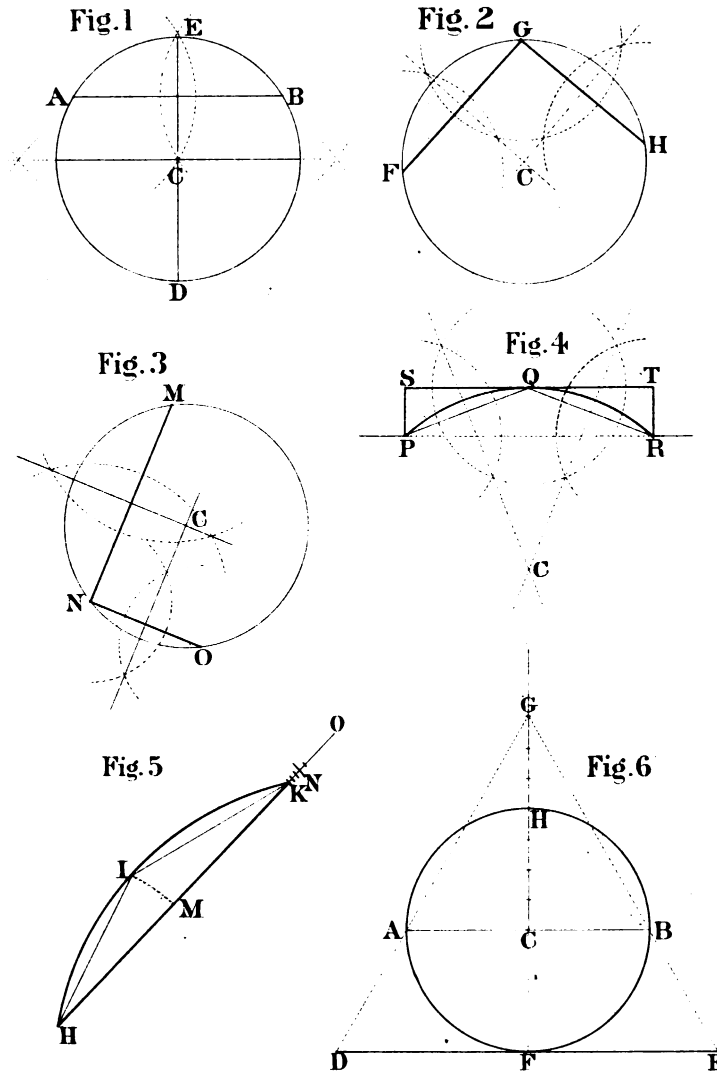
1st. With the length of the line *OP* from *O*, describe the semicircle *PQ*, meeting the line *PO* extended in *Q*.

2nd. Divide the semicircle into five equal parts, and from *O* draw lines through the divisions 1, 2, and 3.

3rd. With the length of the given side from *P* cut *O* 1 in *S*, from *S* cut *O* 2 in *R*, and from *Q* cut *O* 2 in *R*; connect the points *OQRSP* by right lines, and the pentagon will be complete.

PLATE XIII.

PROBLEMS RELATING TO THE CIRCLE.



Problem 20, Figure 6.

On the Given Line AB to Construct a Heptagon, AB being the Length of the Side.

1st. From *A*, with *AB* for a radius, draw the semicircle *BH*.

2nd. Divide the semicircle into seven equal parts, and from *A*, through 1, 2, 3, 4, and 5, draw indefinite lines.

3rd. From *B* cut the line *A* 1 in *C*, from *C* cut *A* 2 in *D*, from *G* cut *A* 4 in *F*, and from F cut *A* 3 in *E*; connect the points by right lines to complete the figure.

Note.—Any polygon may be constructed by this method; the rule is to divide the semicircle into as many equal parts as there are sides in the required polygon, draw the lines through all the divisions except two, and proceed as above. Considerable care is required to draw these figures accurately, on account of the difficulty of finding the exact points of intersection. They should be practiced on a much larger scale

Plate 13.

PROBLEMS RELATING TO THE CIRCLE.

Problem 21, Figure 1.

To Find the Center of a Circle.

1st. Draw any chord, as *AB*, and bisect it by a perpendicular *ED*, which is a diameter of the circle.

2nd. Bisect the perpendicular *ED* by Problem 1; the point of intersection is the center of the circle.

Figure 2.

Another Method of Finding the Center of a Circle.

1st. Join any three points in the circumference as *FGH*.

2nd. Bisect the chords *FG* and *GH* by perpendiculars; their point of intersection at *C* is the center required.

Problem 22, Figure 3

To Draw a Circle Through any Three Points not in a Straight Line, as MNO.

1st. Connect the points by straight lines, which will be chords to the required circle.

2nd. Bisect the chords by perpendiculars; their point of intersection at *C* is the center of the required circle.

3rd. With one foot of the dividers at *C*, and a radius equal to *CMCN* or *CO*, describe the circle.

Problem 23, Figure 4.

To Find the Center for Describing the Segment of a Circle.

1st. Let *PR* be the chord of the segment, and *PS* the rise.

2nd. Draw the chords *PQ* and *QR*, and bisect them by perpendiculars; the point of intersection at *C* is the center for describing the segment.

Problem 24, Figure 5.

To Find a Right Line Nearly Equal to an Arc of a Circle as HIK.

1st. Draw the chord *HK* and extend it indefinitely towards *O*.

2nd. Bisect the segment in *I* and draw chords *HI* and *IK*.

3rd. With one foot of the dividers in *H*, and a radius equal to *HI*, cut *HO* in *M*; then, with the same radius, and one foot in *M*, cut *HO* again, in *N*.

4th. Divide the difference, *KN*, into three equal parts, and extend one of them towards *O*; then will the right line *HO* be nearly equal to the curved line *HIK*.

Problem 25, Figure 6.

To Find a Right Line Equal to the Semi-circumference AFB.

1st. Draw the diameter *AB* and bisect, by the perpendicular *FH*, indefinitely toward *G*.

2nd. Divide the radius *CH* into four equal parts, and extend three of these to *G*.

3rd. At *F* draw an indefinite right line *DE*.

4th. From *G*, through *A*, at the end of the diameter *AB*, draw *GAD*, cutting the line *DE* in *D*; and from *G* through *B*, draw *GBE*, cutting *DE* in *E*; then will the line *DE* be equal to the semi-circumference of the circle, and the triangles *DGE* and *AGB* will be equilateral.

PLATE XIV.

PARABOLA.

Fig. 1

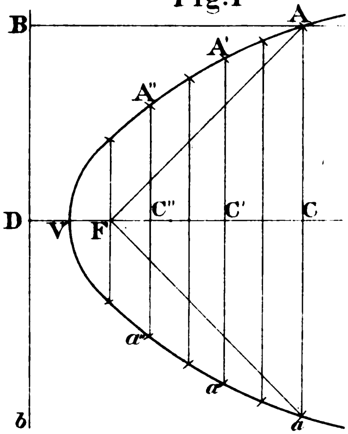
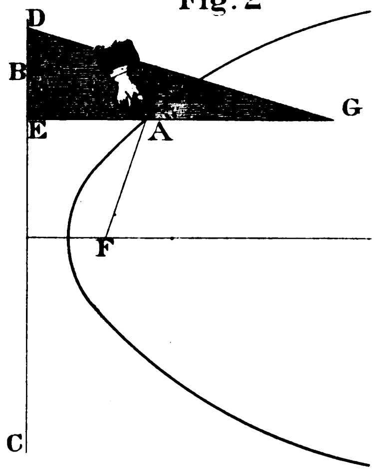


Fig. 2



NOTE.—The right lines found by Problems 24 and 25 are not mathematically equal to the respective curves, but are sufficiently correct for all practical purposes. *Workmen* are in the habit of using the following method for finding the length of a curved line:

They open their compasses to a small distance, and commencing at one end, step off the whole curve, noting the number of steps required and the remainder less than a step, if any; they then step off the same number of times, with the same distance, on the article to be bent around it, and add the remainder, which gives them a length sufficiently true for their purpose. The error in this method amounts to the sum of the differences between the arc cut off by each step and its chord.

PLATE 14, FIGURE 1.

The Focus and Directrix of a Parabola being given, to Describe the Curve.

FIRST METHOD.—*By Points.*

Let F be the focus and Bb the directrix of a parabola. Through F draw DC perpendicular to Bb, and bisect FD in V; then, since $DV = VF$, V is a point on the curve, and CV is the axis of the parabola.

To find other points of the curve, draw any number of lines, Aa, $A'a'$, $A''a''$, etc., perpendicular to CD; then, with the distances DC, DC', DC'', etc., as radii, and the focus F as a center, describe arcs intersecting the perpendiculars in A, A', A'', etc. The points A, A', A'', etc., in which the arcs cut the perpendiculars are points of the curve.

For $FA = DC = AB$ (Def. 1).

We may thus determine as many points on the curve as we please, and the curve line which passes through all the points V, A, A', A'', etc., will be the parabola whose focus is F, and directrix Bb.

Cor. The same radius determines two points of the curve, one above and one below the axis. Since $AF = aF$, FC is common to the two triangles AFC, aFC, and the angles at C are right angles; therefore $AC = aC$; that is, every straight line terminated by the curve and perpendicular to the axis is bisected by it; and, consequently, the parabola consists of two equal branches similarly situated with respect to the axis.

Moreover, since the radius FA is always greater than FC, the arc described with F as a center will always intersect the corresponding perpendicular, and there is, therefore, no limit to the distance to which the curve may extend on both sides of the axis.

PLATE 14, FIGURE 2.

SECOND METHOD.—*By Continuous Motion.*

Let BC be a ruler whose edge coincides with the directrix of the parabola, and let DEG be a square. Take a thread equal in length to EG, and attach one extremity of it at G, and the other at the focus F. Then slide the side of the square DE along the ruler BC, and at the same time keep the thread continually stretched by means of the point of a pencil, A, in contact with the square; the pencil will describe one part of the required parabola. For, in every position of the square,

$$AF + AG = AE + AG,$$

and hence, $AF = AE$; that is, the point A is always equally distant from the focus F and the directrix BC.

If the square be turned over and moved on the other side of the point F the other part of the same parabola may be described.

PLATE 15, FIGURE 1.

The Major Axis and Foci of an Ellipse being given, to Describe the Curve.

FIRST METHOD.—*By Points.*

Let AA' be the major axis and FF' the foci of an ellipse. Take E any point between the foci, and from F and F' as centers, with distances AE, $A'E$ as radii, describe two circles cutting each other in the point D; D will be a point on the ellipse. For, join FD, $F'D$; then $DF + DF' = EA + EA' = AA'$; and, at whatever point between the foci E is taken, the sum of DF and DF' will be equal to AA'. Hence, by Def. 1, D is a point on the curve; and in the same manner any number of points in the ellipse may be determined.

Cor. The same circles determine two points of the curve D and D', one above and one below the major axis. It is also evident that these two points are equally distant from the axis; that is, the ellipse is symmetrical with respect to its major axis, and is bisected by it.

PLATE XV.

Fig. 1

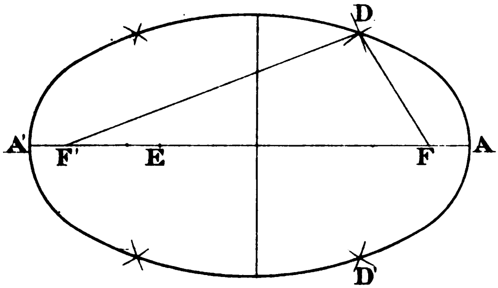


Fig. 2

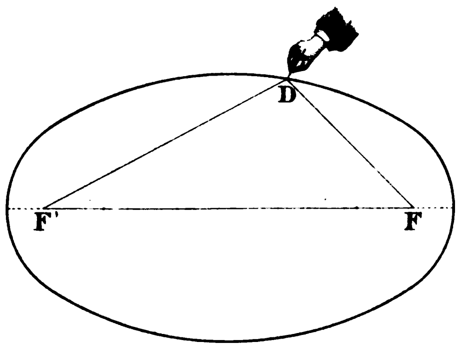


Plate 15, Figure 2.

Second Method.—*By Continuous Motion.*

Take a thread equal in length to the major axis of the ellipse, and fasten one of its extremities at F, the other at F'. Then let a pencil be made to glide along the thread, so as to keep it always stretched; the curve described by the point of the pencil will be an ellipse. For, in every position of the pencil, the sum of the distances DF, DF' will be the same, viz.: equal to the entire length of the string.

Scholium. The ellipse is evidently a continuous and closed curve.

Plate 16, Figure 1.

The Transverse Axis and Foci of an Hyperbola being given, to Describe the Curve.

First Method.—*By Points.*

Let AA' be the transverse axis, and FF' the foci of an hyperbola. In the transverse axis AA' produced, take point E, and from F and F' as centers, with the distances AE, $A'E$ as radii, describe two circles cutting each other in the point D; D will be a point in the hyperbola. For, join FD, $F'D$; then $DF' - DF = EA' - EA = AA'$; and at whatever point of the transverse axis produced E is taken, the difference between DF' and DF will be equal to AA'. Hence, by Def. 1, D is a point on the curve; and, in the same manner, any number of points in the hyperbola may be determined. In a similar manner the opposite branch may be constructed.

Cor. The same circle determine two points of the curve D and D', one above and one below the transverse axis.

It is also evident that these two points are equally distant from the axis; that is, the hyperbola is symmetrical with respect to its transverse axis.

Plate 16, Figure 2.

Second Method.—*By Continuous Motion.*

Take a ruler longer than the distance FF', and fasten one of its extremities at the point F'. Take a thread shorter than the ruler, and fasten one end of it at F and the other to the end H of the ruler. Then move the ruler HDF' about the point F', while the thread is kept constantly stretched by a pencil pressed against the ruler; the curve described by the point of the pencil will be a portion of an hyperbola. For, in every position of the ruler, the difference of the lines DF DF' will be the same, viz.: the difference between the length of the ruler and the length of the string.

If the ruler be turned and move on the other side of the point F, the other part of the same branch may be described.

Also, if one end of the ruler be fixed in F, and that of the thread in F', the opposite branch may be described.

It is evident that each portion of each branch will extend to an indefinitely great distance from the foci and center.

PLATE XVI.

HYPERBOLA.

Fig. 1

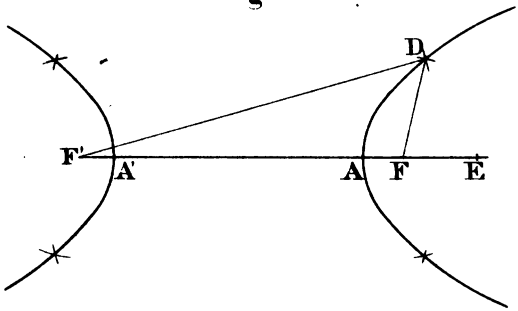
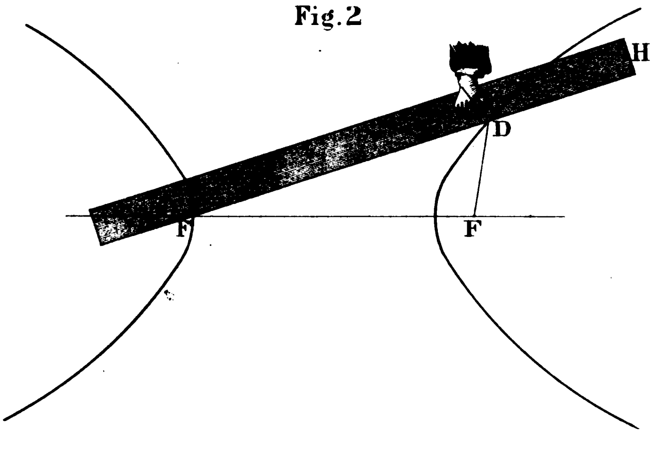


Fig. 2



INDEX.

The extremely subtile theory of imaginary quantities was first placed upon a scientific basis by Gauss. His profound views upon this subject, verbally imparted to us as early as 1830, have since been enunciated in a lecture to the Scientific Society of Gottingen. The report of this lecture is found in the Gott, litrary notices, 1831. But, as these notices are probably available to few, we will give what follows from the aforesaid report.

" Just as integral number is represented by a row of points arranged at equal distances in a straight line, in which the starting-point stands for zero, the next for the number 1, &c.; and just as for the representation of negative numbers only an unlimited prolongation of the series in the opposite direction is requisite, so the representation of imaginary number requires in addition only that the abovementioned row be regarded as situated in an unlimited plane, and that parallel with it, on both sides, we conceive an unlimited number of similar rows at equal distances from each other. Thus, instead of one row of points, we have a system of points before us, which can be arranged in rows of rows in two ways, and which serve to mark out a portion of the plane into absolutely equal squares. The point next to the zero point in one of the parallel rows next to that representing real numbers will then represent the number i, just as the corresponding point in the nearest parallel

row on the other side represents -i, &c. By this means the performance of arithmetical operations on ~~complex~~ imaginary quantities becomes capable of a graphic representation leaving little to be desired.

On the other hand, the true metaphysics of imaginary quantities is thus placed in a new and clear light.

Our universal arithmetic, whose scope so widely transcends ~~so widely~~ the geometry of the ancients, is wholly a creation of modern times. Starting originally from the notion of absolute integral number, it has extended its domain, step by step. To integral number has been added fractional; to rational, irrational; to positive, negative; to real, imaginary. This advance has always been made, at first, with timid, hesitating step. The early algebraists still called the negative roots of equations false roots; and they are such, if the problem to which they refer has been presented in such guise that the nature of the required quantities admits of no opposite. But, just as in universal arithmetic no scruple is felt in regard to the admissibility of fractional number, though there are so many objects capable of being numbered in connection with which a fraction has no meaning, because there innumerable objects having no opposites, ought we to refuse to negative numbers equal rights with positive ones. The reality of negative numbers is sufficiently

justified by the fact that they find an adequate substratum in innumerable objects. As regards them, it is true, an understanding has for some time been arrived at. But the opposite of real, that is imaginary number,—formerly, and even now sometimes, improperly termed impossible—is still rather tolerated than naturalised, is rather regarded as a mere symbolic artifice, to which is utterly denied any objective substratum, though there is no wish to despise the precious contribution yielded by this symbolizm to the treasure-house of real quantitative relations.

The author* has for many years been regarding this highly important part of mathematics from a different point of view, in accordance with which a substratum can be supplied for imaginary, just as well as for negative quantity.

Positive and negative number can only then find an application, when what is numbered has an opposite which, conceived as combined with it, reduces it to nothing. Exactly taken, this assumption is only then an actuality when not substances [objects conceivable per se], but relations between pairs of objects is what is numbered. It is assumed that these objects are arranged in a row in some particular way, as A, B, C, D, ..., and that the relation of A to B can be regarded as equal to that of

*Gauss.

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B to C,&c. Here nothing more is required for the notion of opposition, that the transposition of the terms of the relation, so that, if the relation (or the passage) from A to B amounts to $+1$, the relation of B to A must be denoted by -1. Inasmuch as such a series is without limit in both direction directions, every real integer representing the relation of any member whatever of the row taken as starting-point to some particular member of the row.

But, if the objects are of such a nature that they cannot be arranged in a single, even limitless, row, but only in rows of rows, or, what is the same thing, they form a magnitude of two dimensions, if then the relation of one row to another, or the passage from one to another, is as that above of the transition from one member of a row to another of the same row, there is evidently requisite, in order to measure the transition from one member of the system to another, besides the former units, $+1$ and -1, two others opposites, $+i$ and $-i$. It must evidently be postulated at the same time, that the unit i always denotes the passage from a given member of a row to a particular member of the immediately next row. In this way, accordingly, the system can be arranged in rows of rows in a twofold manner.

The mathematician entirely abstracts the relations of objects

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from their constitution and contents, he has to do only with the measurement and comparison of the relations with each other. Thus, just as he assigns similarity of nature to the relations denoted by +1 and −1 he is entitled to extend that characteristic to the four elements +1, −1, +i, and −i.

These relations can be presented to perception only by a representation in space, and the simplest case is where no reason exists for arranging the symbols of the objects otherwise than in a quadratic form, that is, in an unlimited plane, divided into squares by two systems of parallel lines intersecting at right angles, we select the points of intersection as the symbols. Every such point A has four neighbors, and, if we denote the relation of A to a neighboring point, by +1, then that denoted by −1 is determined of itself, while we [redacted] select for +i whichever we like of the other two, or can take the point that refers to +i either to the right or left. This distinction between right and left is in itself quite determined as soon as we have settled the forward and backward the up and down in relation to the sides of the plane, although we we can ~~establish~~ prove our perception of this distinction only by reference to material objects actually present. But, when we have settled this, we see that it none the less depends on our own

freewill which of the two intersecting rows we would regard as the primary row, and which direction therein should refer to positive numbers. It is further apparent that if we wish to take as +1 the relation previously treated as +i, we must necessarily take for +i that relation before treated as −1. But this, in the language of mathematics, signifies: +i is a mean proportional between +1 and −1, that is, corresponds to the symbol $\sqrt{-1}$. Purposely we do not say <u>the</u> mean proportional, since −i has evidently an equal claim [to that designation]. Hence the possibility of proving an objective signification for the symbol $\sqrt{-1}$ is fully established, and nothing more is needful for the admission of this quantity into the domain of the objects of arithmetical computation.