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THE GEOMETRICAL NUMBER OF PLATO AND THE COMMENT OF PROCLUS

The famous number in Plato's *Republic* is described as follows (546 B,C):

ἔστι δὲ θείῳ μὲν γεννητῷ περίοδος, ἣν ἀριθμὸς περιλαμβάνει τέλειος, ἀνθρωπείῳ δὲ ἐν ᾧ πρώτῳ αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι, τρεῖς ἀποστάσεις, τέτταρας δὲ ὅρους λαβοῦσαι, ὁμοιούντων τε καὶ ἀνομοιούντων, καὶ αὐξόντων καὶ φθινόντων, πάντα προσήγορα καὶ ῥητὰ πρὸς ἄλληλα ἀπέφηναν· ὧν ἐπίτριτος πυθμὴν πεμπάδι συζυγεὶς δύο ἁρμονίας παρέχεται τρὶς αὐξηθείς, τὴν μὲν ἴσην ἰσάκις ἑκατὸν τοσαυτάκις, τὴν δὲ ἰσομήκη μὲν τῇ, προμήκη δέ, ἑκατὸν μὲν ἀριθμῶν ἀπὸ διαμέτρων ῥητῶν πεμπάδος, δεομένων ἑνὸς ἑκάστων, ἀρρήτων δὲ δυεῖν, ἑκατὸν δὲ κύβων τριάδος. ξύμπας δὲ οὗτος ἀριθμὸς γεωμετρικὸς τοιούτου κύριος, ἀμεινόνων τε καὶ χειρόνων γενέσεων.

I give the comment of Proclus, *In Platonis Rem Publicam* (Kroll, II, pp. 36-7) on the part beginning with ἐν ᾧ πρώτῳ and ending with ἐπίτριτος πυθμήν.

Kroll

[ἐ]ν ᾧ πρώ[τῳ αὐξήσεις] . . . . πρ . . . . . νατους . . [20] . . P. 36
λόγους εἴτε τησυν . . [18] . . δυνάμεναι ποιοῦσαι τετραγώνους, δυνα[στευ]όμεναι δὲ ἀπ' ἐκείνων τῶν δυνάμεων . . . . τῶν τετραγώνων· τὸ γὰρ δυνάμενον πᾶν πρὸς τὸ δυναστευόμενον ἀποδίδοται. καὶ πρὸς τούτοις ὁμοιούντων τε καὶ ἀνομοιούντων ἀριθμῶν· ὁμοιούντων μὲν τῶν τετραγωνικῶν ἢ κυβικῶν, ἀνομοιούντων δὲ τῶν ἀνίσοις χρωμένων πλευραῖς ἢ ἐπιπέδων ἢ στερεῶν. καὶ ἐπὶ τούτοις καθ' ὑποδιαίρεσιν τῶν ἀνομοιούντων ἑξῆς φησιν· αὐξόντων τε καὶ φθινόντων· αὐξόντων μὲν τῶν ἰσάκις ἴσων μειζονάκις, ὧν ἐπὶ τὸ μεῖζον ἡ πρόοδος ἀπὸ τῆς ἰσότητος, φθινόντων δὲ τῶν ἰσάκις ἴσων ἐλασσονάκις· ὧν τοῖς μὲν ὄνομα πλινθίδες φασὶ τοῖς φθίνουσιν, τοῖς δὲ δοκίδες τοῖς αὔξουσιν. αὗται δ' οὖν αἱ αὐξήσεις μέχρι τεττάρων ὅρων προελθοῦσαι τρεῖς ἐχόντων ἀποστάσεις ἀλλήλων (πάντων γὰρ τεττάρων ὅρων συνεχῶν τρεῖς εἰσιν ἀποστάσεις) πάντα ῥητὰ καὶ προσήγορα ποιοῦσιν, καὶ τοὺς δυναμένους καὶ τοὺς δυναστευομένους, καὶ τοὺς ὁμοιοῦντας καὶ τοὺς ἀνομοιοῦντας ἀλλήλοις, καὶ τοὺς αὔξοντας καὶ φθίνοντας. γίνεται γὰρ διάγραμμα, κατὰ μὲν τὰ πλάγια τοὺς ὁμοιοῦντας ἔχον καὶ ἀνομοιοῦντας, αὔξοντάς τε καὶ φθίνοντας, καθ' ἕνα λόγον συνδεομένους τὸν πυθμένα τὸν ἐκτεθησόμενον· κατὰ δὲ τὰ σκέλη τοὺς P. 37
δυναμένους καὶ δυνα[στευομένους. ἐπεὶ] δέ οὗτός ἐστιν ὁ ἀριθμός, ἐν [ᾧ πάντα ἀλλήλοις] συμβαίνει, καλῶς ποιῶν . . . . ὧν

ἐπίτριτος πυθμήν· . . . . . . τῶν ἀριθμῶν ὧν αἱ αὐξήσεις. [ἔστιν οὖν οὗτος] ὁ ἐπίτριτος πυθμὴν γ′ καὶ δ′· καὶ [τούτων ἑκά]τερος ἐφ' ἑαυτὸν καὶ ἐπ' ἀλλήλους [γίγνεται] θ′ ιβ′ ις′ ἐν λόγῳ τῷ αὐτῷ. καὶ αὖθις ὁ μὲν γ′ κυβικῶς τρὶς τρία τρίς, καὶ ὁ δ′ ὡσ[αύ]τως τετράκις τέσσαρα τετράκις· μετ' ἀλλήλων δὲ τρὶς τρία τετράκις, τετράκις τέσσαρα τρίς· γίγνονται οὖν κυβικοὶ μὲν ἄκροι ὁ κζ′ καὶ ξδ′, δοκὶς δὲ ὁ λς′, δύο πλευρὰς ἔχων τριάδος καὶ μίαν τετράδος, πλινθὶς δὲ ὁ μη′, δύο πλευρὰς ἔχων τετράδος καὶ μίαν τριάδος. τούτων δὴ τῶν τεττάρων ὄντων ἐφεξῆς ἐν τῷ ἐπιτρίτῳ λόγῳ ὅρων, κζ′ λς′ μη′ ξδ′, καὶ τρεῖς ἀποστάσεις ἐχόντων, ὁ μὲν κζ′ μετὰ τοῦ μη′ ποιεῖ τὸν οε′, ὁ δὲ λς′ μετὰ τοῦ ξδ′ τὸν ρ′.

On Proclus' explanation of the phrase *αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι* (p. 36, 9-11) Hultsch (Exkurs III in Kroll's edition, II, p. 400) remarks: im Proklostexte (36, 7-11) sind an sieben Stellen zusammen mehr als 120 Buchstaben ausgefallen; doch geht aus den erhaltenen Resten wenigstens soviel hervor, dass *αὐξήσεις δυνάμεναι* die Erhebung ganzer Zahlen ins Quadrat, und *δυναστευόμεναι* die Wurzeln einer Quadratzahl bedeuten. This statement is correct so far as regards *αὐξήσεις δυνάμεναι*. Proclus' definition, *ποιοῦσαι τετραγώνους* (p. 36, 9), leaves us in no doubt that he took this phrase to mean *square numbers*, or, if you will, *multiplications that make squares*. He gives an illustration (p. 37, 5-6), saying "each of these (i.e., 3 and 4) by itself gives 9, 16." But I cannot agree with Hultsch's view that Proclus understood by *αὐξήσεις δυναστευόμεναι* the *roots of square numbers*. Unfortunately part of his definition (p. 36, 10-11) is lacking, and we are forced to derive the meaning from the other instances of the word.

In the first place it should be noted that *δυναστευόμεναι* in l. 10 agrees with *αὐξήσεις*. The word *αὐξήσεις*, to be sure, has to be supplied in Proclus' text, but, as Plato has *αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι*, there can be no doubt concerning the correctness of the restoration. Now *αὐξήσεις* means *multiplications;* cp. *τῶν ἀριθμῶν ὧν αἱ αὐξήσεις* (p. 37, 3) and the examples in the following lines, such as 3 x 3 = 9, 3 x 3 x 4 = 36. Surely it is impossible to believe that Proclus would have described either the root of a square number or the process of extracting the root as an *αὔξησις*. There can be no question of treating *δυναστευόμεναι* as a noun independent of *αὐξήσεις*, for we have *δυναστευομένους* (p. 36, 23; 37, 2) agreeing with *ἀριθμούς*

(36, 12), and the neuter *τὸ δυναστευόμενον* corresponding to *τὸ δυνάμενον* (36, 11-12). Further, the phrase *αὗται αἱ αὐξήσεις* (36, 19) refers to the words defined in the preceding lines, the *φθίνοντες, αὔξοντες, ἀνομοιοῦντες, ὁμοιοῦντες, δυνάμενοι*, and *δυναστευόμενοι*. Every one of the others according to the definitions is the result of a multiplication. How can *δυναστευόμενοι* be included among them, and even be made an attribute of *αὐξήσεις*, and yet not be the result of a multiplication? The sentence beginning with *αὗται αἱ αὐξήσεις* (36, 19) says: "These multiplications *make* everything rational and proportional, the *δυνάμενοι*, the *δυναστευόμενοι*, the *ὁμοιοῦντες*, etc." *Multiplications make*, it seems, according to Hultsch, has its natural meaning with every one of the other terms except the *δυναστευόμενοι*. I dwell at length upon this point because of the persistence of the view that *δυνάμεναί τε καὶ δυναστευόμεναι* in Plato means *squares and roots* or *roots and squares* (Adam), and because this view is supposed to get support from Proclus. I am not now dealing with Plato, but with Proclus, and I insist that if 3 and 4 are examples of *ἀριθμοὶ δυναστευόμενοι*, as Hultsch assumes, Proclus would not apply to them the term *αὐξήσεις*.

What Proclus really meant by *αὐξήσεις δυναστευόμεναι* can be deduced with certainty from the passage beginning with *αὗται δ' οὖν αἱ αὐξήσεις* (36, 19). But first we should note his definitions of the other terms associated with the *δυναστευόμενοι*.

ἀριθμοὶ δυνάμενοι = *squares* (36, 9-10).

ἀριθμοὶ ὁμοιοῦντες = *squares* or *cubes* (36, 13).

ἀριθμοὶ ἀνομοιοῦντες = *rectangles* or *solids with unequal sides* (36, 13-14).

ἀριθμοὶ αὔξοντες or *δοκίδες* (36, 19) = *solids with two equal sides and the third side greater* (36, 16).

ἀριθμοὶ φθίνοντες or *πλινθίδες* (36, 18) = *solids with two equal sides and the third side smaller* (36, 18).

I now translate 36, 19–37, 1. "These multiplications, then, if arranged in sets of four terms with three intervals between them[1] (for there are three intervals between four successive terms always) make everything rational and proportional, the *δυνάμενοι* and the *δυναστευόμενοι*, the *ὁμοιοῦντες* and the *ἀνομοιοῦντες*, both *αὔξοντες* and *φθίνοντες*. For there is produced

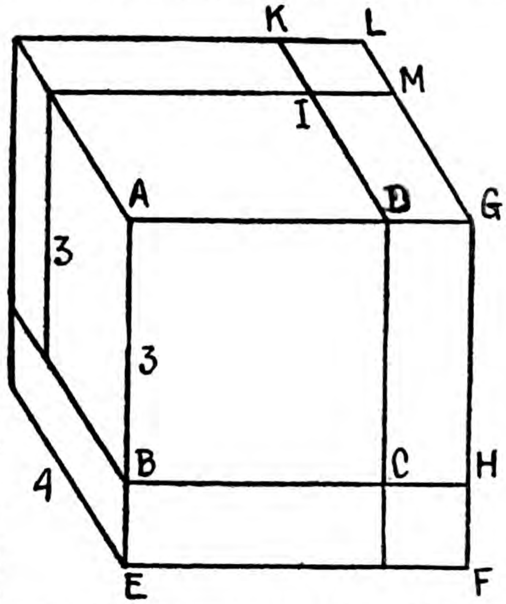
[1] I.e., as in a geometrical proportion.

a diagram, κατὰ μὲν τὰ πλάγια having the ὁμοιοῦντες and ἀνομοιοῦντες, both αὔξοντες and φθίνοντες, bound together in one ratio, the base that shall be assumed; κατὰ τὰ σκέλη having the δυνάμενοι and δυναστευόμενοι." Hultsch has pointed out that we have an example of the proportion called κατὰ τὰ πλάγια in the sentence γίγνονται οὖν κυβικοὶ κτέ. I translate from 37, 6: "And, again, the 3 cubed gives 3 x 3 x 3, and the 4 in like manner 4 x 4 x 4; and with one another they give 3 x 3 x 4, 4 x 4 x 3. There result, then, cubic extreme terms 27 and 64, a δοκίς 36 with two sides of 3 and one of 4, and a πλινθίς 48 with two sides of 4 and one of 3. Of these terms, being in order in the 3 : 4 ratio, 27 36 48 64, and having three intervals, etc." The proportion κατὰ τὰ πλάγια is said (36, 25-6) to have ὁμοιοῦντες and ἀνομοιοῦντες, the latter being both αὔξοντες and φθίνοντες. In 27 36 48 64 the "cubical extreme" terms 27 and 64 are the ὁμοιοῦντες, the δοκίς 36 is the αὔξων, the πλινθίς 48 is the φθίνων. This proportion includes "solid" (στερεοί 36, 14) numbers[2] only. Consequently the words κατὰ τὰ πλάγια refer to solid forms.

What, then, is the proportion κατὰ τὰ σκέλη, and what is meant by the diagram spoken of in 36, 25? Hultsch offers the following explanation (p. 401): Dazu gebe es eine geometrische Figur (διάγραμμα), in welcher auf einer Querlinie zwei Schenkel sich erheben. Auf der horizontalen Geraden seien die ὁμοιοῦντες und von den ἀνομοιοῦντες sowohl die αὔξοντες als die φθίνοντες einzutragen; alle diese Zahlen seien durch den noch darzulegenden ἐπίτριτος πυθμήν verbunden. Auf den Schenkeln des von der Horizontalen aufsteigenden Winkels sollen die Quadratzahlen und ihre Wurzeln eingetragen werden. Die in der Handschrift fehlende Figur hat also drei Gerade etwa in der Zusammenstellung $\underline{\vee}$ gezeigt. Unterhalb der Basis haben, wie es scheint, die Zahlen κζ′, λς′, μη′, ξδ′ gestanden, von denen κζ′ und ξδ′ ὁμοιοῦντες, die beiden anderen aber ἀνομοιοῦντες sind, und zwar λς′ ein αὔξων, μη′ ein φθίνων, wie sofort sich zeigen wird. Zu den Schenkeln mussten zuerst die Zahlen γ′ und δ′, die den ἐπίτριτος πυθμήν bilden, beigeschrieben sein, vermutlich γ′ zu dem einen, δ′ zu dem andern Schenkel. Dann kamen oberhalb von γ′, bez. δ′, die Quadratzahlen θ′ und ις′ und darüber standen, wenn es auch keine δυνάμενοι im eigentlichen

[2] I.e., 3 x 3 x 3, 3 x 3 x 4, etc.

Sinne waren, die Kubikzahlen κζ' und ξδ'. So far as the diagram is concerned I find this anything but enlightening. What possible meaning could Proclus have found in such an arrangement of numbers? And as for the 3 4 9 16, which Hultsch attaches to the σκέλη of his figure, it is to be observed that they are not in proportion. Yet we ought to have a proportion κατὰ δὲ τὰ σκέλη, just as we do κατὰ μὲν τὰ πλάγια; for I suppose it will hardly be claimed that the position of καθ' ἕνα λόγον .... ἐκτεθησόμενον (36, 26-7) restricts its application to the πλάγια part. The whole of what precedes leads us to expect a proportion in both cases alike. Hultsch seems to have felt this himself, for he says (p. 402): "Beide (i.e., 3 and 4) werden ins Quadrat erhoben; da aber diese Quadrate zu einander ein anderes Verhältnis als das von 3 : 4 haben, so ist, um den ἐπίτριτος πυθμήν durchzuführen, zwischen 9 und 16 als mittlere Proportionale die Zahl 3 . 4 = 12 einzuschieben." It is beyond me to fathom how Proclus' text is explained by this remark. It applies, of course, to καὶ τούτων (i.e., 3 and 4) ἑκάτερος ἐφ' ἑαυτὸν καὶ ἐπ' ἀλλήλους γίγνεται θ' ιβ' ις' ἐν λόγῳ τῷ αὐτῷ (37, 4-6). But can anything be clearer than that Proclus in this sentence is giving us an example of the proportion κατὰ τὰ σκέλη, just as the sentence immediately following, καὶ αὖθις ὁ μὲν γ' κυβικῶς κτέ., gives an example of the proportion κατὰ τὰ πλάγια? The numbers κατὰ τὰ σκέλη are δυνάμενοι and δυναστευόμενοι. Of θ' ιβ' and ις' the 9 and 16 are δυνάμενοι. Does it not follow that the 12 in δυναστευόμενος, especially as we are in need of a proportion κατὰ τὰ σκέλη, and 9 : 12 = 12 : 16 is a proportion? The δυναστευόμενος, as we have seen, should be an αὔξησις; and 12, being 3 x 4, is an αὔξησις. It seems to me that only a fixed idea that δυνάμενοι are *squares* and δυναστευόμενοι *roots* can prevent the admission that this is the meaning of Proclus. If conviction has not yet been reached, perhaps the explanation of the *diagram* will attain it. We have seen that the numbers κατὰ τὰ πλάγια are all 'solid'; of those κατὰ τὰ σκέλη the δυνάμενοι, being 9 = 3 x 3 and 16 = 4 x 4, are 'plane' numbers, and the δυναστευόμενοι, if they are 12 = 3 x 4, are also 'plane.' It would seem, then, that the figure should contain planes and solids. It is to be observed that Proclus puts no figure in his text. The diagram just comes about naturally—γίνεται γὰρ διάγραμμα—by the process, described in what precedes, of making plane and solid figures by squaring, cubing, and multiplying equal by equal by less (36, 18), or equal by equal by greater (36, 16). Taking 3 and 4 as the numbers to be multiplied (ὧν αἱ αὐξήσεις 37, 3), the resulting diagram is this.



Explanation is almost unnecessary. AC is $3^2=9$, an ἀριθμὸς δυνάμενος. AH and ED are each $3 \times 4=12$, ἀριθμοὶ δυναστευόμενοι. EG is $4^2=16$, another δυνάμενος. The proportion AC(9) : AH(12) = ED (12) : EG(16) is the proportion κατὰ τὰ σκέλη. Again, BI is $3^3=27$, an ἀριθμὸς κυβικός and ὁμοιῶν. BM and BK are each $3 \times 3 \times 4=36$; they are δοκίδες or ἀριθμοὶ αὔξοντες. EM and EK are each $4 \times 4 \times 3=48$; they are πλινθίδες or ἀριθμοὶ φθίνοντες. EL is $4^3=64$, another ἀριθμὸς κυβικός and ὁμοιῶν. The proportion BI(27) : BM(36) = EK (48) : EL(64) is that κατὰ τὰ πλάγια. The appropriateness of the word πλάγια to describe solid as compared with plane figures will be at once admitted. Why κατὰ τὰ σκέλη should be limited to planes is not so clear. However, the same limitation is to be seen in the application of ἰσο-σκελές to triangles, the only established geometrical use of σκέλος; and, whatever the origin of the phrase, there can be no doubt of what Proclus meant by it.[3]

The next point to be considered is whether my interpretation of the meaning of ἀριθμὸς δυναστευόμενος in Proclus, viz., *the mean proportional between two squares*, or that of Hultsch, viz., the *roots of a square number*, is more consistent with Proclus' use of the word in other connections. His definition, δυναστευόμεναι δὲ ἀπ' ἐκείνων τῶν δυνάμεων . . . . τῶν τετραγώνων

[3] Aristotle (H. A. 2.1.15; 9.44.3) uses κατὰ σκέλος βαδίζειν to describe the gait of the lion and camel, which, he says, walk with the hind foot following the fore on the same side, like a pacing horse. παρὰ σκέλος ἀπαντᾶν is cited in L. & S. in the sense of *cross one's path, thwart*.

(36, 10) lacks an important word and will, consequently, appear indecisive. Still, as I have already pointed out, the fact that δυναστευόμεναι qualifies αὐξήσεις is a strong point against the meaning *roots*. If the ἐλάσσους (?), supplied in the critical note to fill the lacuna, expresses Hultsch's idea, he must, I suppose, translate *roots from those squares, less than the squares.*[4] In this the occurrence of two words for *squares* is peculiar, and there is little point in calling roots *less* than their squares. I take δυνάμεων to be *roots*, not *squares*—either interpretation can find support in Plato, *Theaetetus* 147-8—and, since the mean proportional between two squares is the product of their roots, I translate, δυναστευόμεναι, (*a combination*) *from those roots of the squares*. The words that follow the definition and give a reason (γάρ) for it—τὸ γὰρ δυνάμενον πᾶν πρὸς τὸ δυαστευόμενον ἀποδίδοται—are difficult to interpret from either point of view on account of the uncertainty of the meaning of ἀποδίδοται. I am inclined to believe that Proclus uses δυνάμενον and δυναστευόμενον here less in their technical mathematical meaning than in the general sense of *controlling* and *being controlled*, as in ὅσα κατὰ τὰς δυνάμεις ἀναφαίνεται πᾶσιν ὁμοίως προσήκει τοῖς μαθήμασι τῶν μὲν δυναμένων, τῶν δὲ δυναστευομένων (*In Euclidem*, p. 8), and in a couple of passages to be cited presently. In that case the sentence would mean something like *for everything that controls has its complement* (cf. ἀπόδοσις) *in what is controlled;*[5] cf. πάντῃ οὖν αὐταῖς ἀνισουμέναις ἀποδέδοται τὸ πάντῃ ἄνισον (p. 48, 26) *for what is in every way unequal* (*sc.* the scalene triangle) *is represented by* (or *has its counterpart in*) *them being unequal in every way.*

In the passage now to be cited Proclus applies numbers to the soul in a manner quite meaningless to us. However, the distinction made between δυνάμενος and δυναστευόμενος throws some light upon the meaning of the latter.

ἐκ μὲν οὖν τούτων ἡ ψυχὴ φαίνεται μία δυοειδὴς κατά τε τὸ εἶναι καὶ τὸ ζῆν· ἐκ δὲ τῶν ἀριθμῶν τῶν ἐκ τούτων ἀναφανέντων ἀριθμὸς δυαδικὸς ἀμείνους καὶ χείρους ἔχων δυνάμεις, τὰς μὲν δυναμένας τὰς δὲ δυναστευομένας, ἁπλουστέρας καὶ συνθετωτέρας (δύνανται μὲν γὰρ οἱ πλευρικοί, δυναστεύονται δὲ οἱ ἐκ τούτων), καὶ τὰς μὲν ὁμοιούσας

[4] τετραγώνων might be taken as an adjective qualifying δυνάμεων; but why supply a comparative before a genitive if it is not to govern it?

[5] For πρὸς with ἀποδίδωμι cf. Schol. in Aristoph. *Plut.* 538.

τὰς δὲ ἀνομοιούσας, . . . καὶ τὰς μὲν αὐξούσας τὰς δὲ φθειρούσας τὸ πτέρωμα αὐτῆς (II, p. 51, 9ff).

The closing words show that *δυνάμεις* here has its general sense *powers*. *δυναμένας* and *δυναστευομένας*, as qualifying *δυνάμεις*, must mean *controlling* and *being controlled*. The former are the *ἀμείνους*, the latter the *χείρους δυνάμεις*. The phrase in brackets returns to mathematics. *δύνανται οἱ πλευρικοί* must refer to *squares*. In *δυναστεύονται οἱ ἐκ τούτων* Hultsch doubtless would find support for his view that *δυναστευόμεναι* are *roots*. *οἱ ἐκ τούτων* are *the numbers from those πλευρικοί*. However, *οἱ πλευρικοὶ ἀριθμοί*, with *δύνανται* as a predicate, are not themselves square numbers, but the numbers representing the *sides* of the squares. There are two of these numbers (cp. *ἀριθμὸς δυαδικός*), like our 3 and 4 above—the parallelism of the two passages is clear from the recurrence of the terms applied to the diagram— and *οἱ ἐκ τούτων* means *the numbers from* (the combination of) *these two*. I believe that is made certain by *ἁπλουστέρας* and *συνθετωτέρας*. *ἁπλουστέρας* applies to *τὰς μὲν δυναμένας*, for *squares* are *like* x *like*; *συνθετωτέρας* applies to *τὰς δυναστευομένας*, for *rectangles* are *like* x *unlike*. Squares, then, are *more simple*, rectangles *more composite*.

A passage closely following (p. 51, 26 ff.) still further confirms my view that for Proclus the *ἀριθμὸς δυναστευόμενος* is a rectangle.

εἰσιοῦσα δὲ (sc. *ἡ ψυχή*) *εἰς ἑαυτὴν ἐπιπεδοῦται, καὶ μέχρι μὲν διανοίας ἱσταμένη τετραγωνίζει ἑαυτήν, τὸ ταὐτὸν καὶ ὅμοιον ἐν τῇ διανοητικῇ κινήσει τῷ διάνοια εἶναι πρὸς διάνοιαν κινουμένη ἔτι σώζουσα, δόξαν δὲ μετὰ διανοίας συμμίξασα κινεῖται κίνησιν ἐπίπεδον μὲν ὡς ἐκ δυεῖν γενομένην δυνάμεων ἀλλήλαις συμμιγνυμένων, ἀνίσων δὲ οὐσῶν ἐκείνων προμηκίζει αὐτὴν ἀφ' ἑαυτῆς· εἰς δὲ τὰ μετ' αὐτὴν ῥέπουσα βαθύνει τὰ ἐπίπεδα, τὴν μὲν τετραγωνικὴν ζωὴν κυβίζουσα καὶ φαντασίαν γεννῶσα, ... τὴν δὲ προμήκη κατὰ τὴν ἀνάλογον πρόοδον εἰς τοὺς ἀνομοίους ὑφιζάνουσα στερεοὺς καὶ τὴν αἴσθησιν γεννῶσα.*

This is a curious bit of nonsense, amusing in small doses. What interests me in it in this connection is that the process the soul is said to go through corresponds exactly to what took place in the growth of our diagram above. The *τετραγωνίζει ἑαυτήν* corresponds to the formation of the *ἀριθμοὶ δυνάμενοι*; in *τὴν τετραγωνικὴν ζωὴν κυβίζουσα* we have our cubes; *τοὺς ἀνομοίους στερεούς* are the *δοκίδες* and *πλινθίδες*. What remains must correspond to the δυναστευόμενοι. *When the soul mingles opinion with thought it moves in a plane since the movement arises from two factors* (δυνάμεων) *mingled with one another, and as these factors are unequal the soul makes an oblong rectangle of itself.* Thus the δυναστευόμενος arises from the combination of two unequal factors. It is προμήκης *oblong*, like 12 = 3 x 4. I do not think that anyone will deny that the προμήκης figure formed here ἐκ δυεῖν δυνάμεων ἀνίσων οὐσῶν is the same as the αὔξησις δυναστευομένη formed ἀπ' ἐκείνων τῶν δυνάμεων (p. 36, 10); which δυνάμεων in the ratio selected are 3 and 4, ἄνισοι οὖσαι. And now let us look at a few lines in the definitions on page 36, which were passed over before, namely: καὶ πρὸς τούτοις ὁμοιούντων τε καὶ ἀνομοιούντων ἀριθμῶν· ὁμοιούντων μὲν τῶν τετραγωνικῶν ἢ κυβικῶν, ἀνομοιούντων δὲ τῶν ἀνίσοις χρωμένων πλευραῖς ἢ ἐπιπέδων ἢ στερεῶν (ll. 12-14). Here, though squares have already been mentioned under δυνάμεναι ποιοῦσαι τετραγώνους, the ὁμοιοῦντες are defined as either squares or cubes. The ἀνομοιοῦντες are not only irregular solids, the αὔξοντες and φθίνοντες we hear so much about, but also *irregular rectangles.* They are, of course, the δυναστευόμεναι; but what part have they in Hultsch's interpretation?

On the basis of the word συνθετωτέρας, applied to δυνάμεις δυναστευομένας in the passage cited on p. 7, I suggest that the lacuna in 36, 10 be filled by σύνθεται, a conjecture supported by the τησυν of l. 9. Thus δυναστευόμεναι would be defined as *compounded from those roots of the squares.* To fill the lacunae in ll. 8-9, after Kroll's ἐν ᾧ πρώτῳ αὐξήσεις I suggest [β' ἀριθμῶν] πρ[ώτων ἀ]νὰ τοὺς [ἀεί ποτ' ἐκτεθησομένους][6] λόγους εἴτε τῇ συν[θέσει εἴτε ἐφ' ἑαυτούς][7] meaning *products of two prime numbers in the ratios to be assumed, either by composition with one another or by themselves,* δυνάμεναι *making squares,* δυναστευόμεναι *composite products from those roots of the squares.*

A different explanation of the terms δυνάμενος and δυναστευόμενος is given by Alexander Aphrodisiensis (*In Arist. Met.* A 8, 990ᵃ 23): ἀνικίαν δέ φασιν ὑπὸ τῶν Πυθαγορείων λέγεσθαι τὴν πεντάδα, τοῦτο δὲ ὅτι τῶν ὀρθογωνίων τριγώνων τῶν ἐχόντων ῥητὰς τὰς πλευρὰς πρῶτόν ἐστι τῶν περιεχουσῶν ὀρθὴν γωνίαν πλευρῶν ἡ μὲν τριῶν ἡ δὲ τεττάρων, ἡ δὲ ὑποτείνουσα πέντε. ἐπεὶ τοίνυν ἡ

[6] Cp. καθ' ἕνα λόγον . . . τὸν πυθμένα τὸν ἐκτεθησόμενον 36.26.

[7] Cp. ἐφ' ἑαυτὸν καὶ ἐπ' ἀλλήλους 37.5.

ὑποτείνουσα ἴσον δύναται ἀμφοτέραις ἅμα, διὰ τοῦτο ἡ μὲν δυναμένη καλεῖται, αἱ δὲ δυναστευόμεναι, καὶ ἔστι πέντε. τήν τε πεντάδα ἀνικίαν ἔλεγον ὡς μὴ νικωμένην ἀλλ' ἀήττητον καὶ κρατοῦσαν. That is, according to this authority, in the right-angled triangle with sides 3, 4, and 5, known as the Pythagorean, the hypotenuse 5 was called *ἡ δυναμένη*, the sides *αἱ δυναστευόμεναι*. Proclus, while he gives a different definition of the terms himself, shows that he is acquainted with that of Alexander. In the sections preceding his own detailed explanation of the Platonic number he cites the views of "the Pythagoreans" and others upon various points connected with the problem. One of these runs as follows: *ὅτι τὸ μὲν τρίγωνον αὐτό φασιν οἱ περὶ τὸν Δερκυλλίδην ἐοικέναι τοῖς πρώτοις φύλαξιν διὰ τὴν τῶν λόγων κοινωνίαν, τῶν μὲν περιεχουσῶν τὸν πρῶτον ἐν συμφωνίᾳ λόγον ἐχουσῶν, τῆς δ' ὑποτεινούσης †ἣ δυναμένης ἀμφοῖν.* The † is Kroll's. His critical note is *ἣ*] expectas *διπλάσιον*. This is clearly wrong. The preceding section discusses *equilateral* right-angled triangles in which *διπλάσιον ἡ διάμετρος δύναται τῆς πλευρᾶς* *the square of the diameter is double the square of the side* (p. 25, 9). The present section is a citation from a different authority, and the subject has changed from the mathematical to the philosophical properties of the triangle. That it is the Pythagorean 3-4-5 triangle which is now in question is evident from *τὸ τρίγωνον αὐτό*; from *τρὶς αὐξηθείς*, the phrase appearing in Plato in connection with the same triangle (*ἐπίτριτος πυθμὴν πεμπάδι συζυγείς*); from the numbers 75 ,100, 7500, 10000, which are derived by Proclus from the 3-4-5 triangle (II, 37 ff.); and from the reference to its influence on births. Consequently, there can be no mention here of the hypotenuse squaring to *double* the square of the side. *ὑποτεινούσης ἣ δυναμένης* simply gives us the alternative name for the hypotenuse which is mentioned by Alexander; and *τὸν λόγον (ἀμφοῖν) ἐχούσης* must be supplied from the preceding clause.

It would seem, then, that the scholars who preceded Proclus took different sides on the question of the meaning of *δυναμένη* and *δυναστευομένη*. One set connected the terms with the famous 3-4-5 triangle of Pythagoras, and the proof, attributed to him, that the square of the hypotenuse is equal to the sum of the squares of the sides; by others they were applied to geometric proportions with one mean term, another point in which Pythagoras was greatly interested. Modern attempts to explain the number of Plato by using the definitions of Alexander have met with no success. Hitherto the comment of Proclus has brought little light. Geometrical proportions were found in Plato's words before Proclus was discovered. No one, however, has attempted to apply to the phrase αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι the meaning that Proclus really gives it; *squares and roots* has had too firm a hold on modern thought, and it was easy to see the same meaning in his mutilated text. As a matter of fact he illustrates the meaning by the proportion 9 : 12 = 12 : 16. With this it is interesting to compare the solution of J. Adam.[8] The ἁρμονία which Plato describes as a square (ἴσην ἰσάκις) Adam and Hultsch[9] took to be 3600^2. The *rectangular* harmony (προμήκη) plainly has one side of 2700 (ἑκατὸν κύβων τριάδος), and one side of 4800 (ἑκατὸν μὲν . . . δυεῖν)[10]. 3600^2 is equal to 2700 x 4800; and, if a square is equal to a rectangle, it follows that the side of the square is a mean proportional between the two sides of the rectangle. Therefore 2700 : 3600 = 3600 : 4800. This proportion is of the same kind as that of Proclus; but in his, according to the definition, the extreme terms must be squares, as in the example 9 : 12 = 12 : 16. What I shall now attempt to prove is that Proclus retained a partially correct tradition concerning the meaning of αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι; that the proportion 9 : 12 = 12 : 16 is a correct example of what was meant by the phrase; that Proclus was right in his view that αὐξήσεις δυνάμεναι were *squares* and δυναστευόμεναι were *rectangles*; but that in the 9 : 12 = 12 : 16, whereas he took the 9 and 16 to be the αὐξήσεις δυνάμεναι and the 12 to be the δυναστευομένη, the actual δυναμένη of Plato and the Pythagoreans was 12^2, the δυναστευομένη 9 x 16. Thus, while Proclus' definition limits the phrase to geometrical proportions with a mean term *between two squares*, in reality it applied to all geometrical proportions with a mean term. I shall also attempt to prove that

[8] *The Nuptial Number of Plato* (1891), and in his *Republic*, II, 264 ff. Cf. the solution of Hultsch, *Zeit. f. Math. u. Phys.* 27, *Hist. lit. Abth.*, p. 41.

[9] Their methods of reaching the number are wrong; see below.

[10] *One hundred squares of rational diameters of five* (i.e., squares of seven) *each lacking one, of irrational* (i.e., squares of $\sqrt{50}$) *each lacking two.* See Adam or Jowett & Campbell.

Adam's *number*, $3600^2 = 2700 \times 4800$, is the correct one, though his method of reaching the 3600^2 is wrong and his interpretation of the sentence ἐν ᾧ πρώτῳ . . . ἀπέφηναν as far from the truth as it well could be. This sentence, instead of containing a *number*,[11] as has been almost universally assumed, contains a general definition of the geometrical truth of which the second sentence with its $3600^2 = 2700 \times 4800$ gives a particular example. It states that, if a square is equal to a rectangle, then the side of the square is a mean proportional between the sides of the rectangle, i.e., if a^2 is equal to bc, then $b : a$ equals $a : c$.

My interpretation of this sentence was reached without a copy of Proclus, *In Rempublicam*, and the few phrases from it cited by Adam were more confusing than helpful. Consequently, the explanation of his diagram came as a result of the conclusion that Plato's αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι meant *if a square is equal to a rectangle.* The discussion of Proclus has been presented first, because the results obtained might be called mathematically certain, and the material is comparatively fresh. The Plato passage has been written about to such an extent that it is difficult to believe that anything new can be said of it. Without the support of Proclus I could not expect the following argument to carry conviction, but it will be presented in its original form because the material in the commentary *In Rempublicam* had no part in its formulation.

The Meaning of Republic 546 b,c.

Writing in 1903, P. Tannery expressed the opinion that the sentence ὧν ἐπίτριτος πυθμὴν . . . κύβων τριάδος, in Plato's definition of the number, had been almost completely interpreted.[12] One may agree with this without accepting 10,000 as the square and 7500 as the rectangle, as Tannery did, with the strong support of Proclus. Of the preceding sentence, ἐν ᾧ πρώτῳ. . . ἀπέφηναν, he says: "Vient une phrase, qui est restée parfaitement obscure. . . . Tous les efforts (y compris, bien entendu,

[11] Kafka, whose article in *Philologus* 73.109, is the most recent I have seen on the subject, is correct in his theory that the first sentence does not contain a *different* number from the second, wrong in supposing it to contain a number at all. The acceptance of Adam's *raised to the fourth power* for τρὶς αὐξηθεὶς is enough to condemn his conclusions.

[12] *R. E. G.* 1903, p. 171.

mon *juvenile tentamen* de 1876) pour expliquer cette phrase d'un nombre déterminé, n' ont, à mon avis du moins, abouti jusqu' à présent qu' à des interprétations qui ne sont pas réellement plus claires que le texte même." So far as my knowledge of the literature goes, this statement is as true now as it was in 1903. I shall begin my interpretation of the sentence with the all-important words αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι.

To reach the mathematical meaning of αὔξησις we note first that αὐξηθείς may mean *multiplied*. It is so used in ὁ λε′ ἑξάδι αὐξηθεὶς ἑπτάμηνον χρόνον ἀποτελεῖ τὸν τῶν σι′ ἡμερῶν (*Theol. Ar.* p. 39, Ast; cf. Nicom. *Introd. Ar.* p. 105, Hoche), and, presumably, in τρὶς αὐξηθείς in our passage. The idea of *multiplication* is also present in the phrases δευτέρα and τρίτη αὔξη applied to planes and solids (Plato, *Rep.* 528 B). αὔξησις, then, since it is by formation an abstract noun denoting an act, should mean *an increasing by multiplication*. Yet it is not necessary that the meaning be limited to a *process*, as Adam insists (*Rep.* II, p. 268, 270). Our word *growth* means both the act and the result of growing. δόσις is *a gift* as well as *a giving*. Monro remarks (*J. P.* 8, p. 286), "The combination of words in αὐξήσεις δυνάμεναι is open to the criticism that both are words denoting operations, not quantities," but instead of taking this as an indication "of deliberate obscurity of language," why not admit the possibility that αὔξησις in this case denotes the *product* instead of the *process* of multiplication? In Herodotus II, 13— ἢν οὕτω ἡ χώρη αὕτη κατὰ λόγον ἐπιδιδῷ ἐς ὕψος καὶ τὸ ὅμοιον ἀποδιδῷ ἐς αὔξησιν—, the word is contrasted with ὕψος and means *extent* in width and length. I conclude, then, that a possible meaning of αὐξήσεις is *plane surfaces*, or, if we think of numbers merely, *products* of two numbers. I do not deny that it might be applied to solids and products of three numbers, but, as we shall see, the context shows that we are dealing with planes, not solids.[13]

The mathematical meaning of δύναμαι in Plato is fairly clear. κατὰ δύναμιν is contrasted with κατὰ τὸν τοῦ μήκους ἀριθμόν

[13] In his comment Proclus treats αὔξησις as a *process* in ποιοῦσαι τετραγώνους (36.10) and αἱ αὐξήσεις . . . ποιοῦσι τοὺς δυναμένους κτὲ (36.21, 24), but his interpretation pays no attention whatever to the syntax of the sentence. He deals finally only with the *products*, δυναμένους, αὔξοντας, etc.

and κατὰ τρίτην αὔξην (*Rep.* 587 D), and means *in square measure.* In the *Theaetetus* 148 B—ὅσαι μὲν γραμμαὶ τὸν ἰσόπλευρον καὶ ἐπίπεδον ἀριθμὸν τετραγωνίζουσι, μῆκος ὡρισάμεθα, ὅσαι δὲ τὸν ἑτερομήκη, δυνάμεις, ὡς μήκει μὲν οὐ ξυμμέτρους ἐκείναις, τοῖς δ' ἐπιπέδοις ἃ δύνανται— where the subject is γραμμαί *lines*, the meaning of δύνανται is they *produce when squared.* The meaning *to equal* is a common one of δύνασθαι and in the *Theaetetus* passage we may translate by *equal when squared.* Alexander Aphrodisiensis, indeed, inserts ἴσον in ἡ ὑποτείνουσα ἴσον δύναται ἀμφοτέραις (see above, p. 10) *the hypotenuse equals when squared* (*the squares of*) *both* (*the sides*), but, though the ἴσον is necessary here,[14] it would not be necessary in ἓξ ἑξάκις δύναται ἐννέα τετράκις. Taking αὐξήσεις as *plane surfaces* or *products of two numbers*, we may infer that it means *squares* when modified by δυνάμεναι, and that the combination αὐξήσεις δυνάμεναι means *squares that equal.* If this interpretation is in agreement with what follows, no objection can be raised against it. It is certainly not contrary to Plato's usage elsewhere.

δυναστεύειν means *to be a ruler.* Its use as an attribute of αὐξήσεις in connection with δυνάμεναι *equaling* suggests that it is here a kind of passive of δύνασθαι, and that part of the meaning is *being equaled*; but there is need of an additional idea, something to correspond to the *being a square*, which δυνάμεναι has when applied to αὐξήσεις. Alexander Aphrod. (*l.c.*) tells us that the hypotenuse of a right-angled triangle was called ἡ δυναμένη, the sides αἱ δυναστευόμεναι. On this basis αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι has been rendered by *products of the hypotenuse and sides* (cp. Gow, J. P., 12, 101). Others, inferring from Plato's definition of δυνάμεις as *roots* in Theaet. 148 B (see above), that δυναστευόμεναι may mean *squares* (or vice versa) have translated the phrase *root and square increases* (Adam, meaning *cubings* of numbers to come), or *incrementum per multiplicationem radicis seu lateris et quadrati mutuam factum a product obtained by multiplying together a root or side and its square* (Schneider, meaning *cubes*). These interpretations do such violence to Greek syntactical usage that they are wrong beyond all question. If δυναμένη means *the hypotenuse, multiplication of the hypotenuse* should be expressed in some other way than by

[14] Cf. Proclus (*In Remp.* II, 25.8) ἐπειδὴ διπλάσιον ἡ διάμετρος δύναται τῆς πλευρᾶς *the square of the diameter equals double the square of the side* (so 38.18).

αὐξήσεις δυνάμεναι.[15] If Plato had wished to speak of the *multiplication of a root* he had a noun δύναμις to make use of.

Returning, then, to the idea that αὐξήσεις δυνάμεναι may mean *squares that equal*, can we, from Alexander's definition, get a corresponding meaning for αὐξήσεις δυναστευόμεναι? Taking the former to refer to squares *of the hypotenuse* that equal, we might make the latter squares *of the sides* that are equaled. We are then faced with the difficulty of bringing this into relation with the following words—τρεῖς ἀποστάσεις, τέτταρας δὲ ὅρους λαβοῦσαι. This should mean *putting them in a geometrical progression*.[16] Plato, *Timaeus* 43 D, uses ἀποστάσεις of the *intervals* between the terms of a geometrical progression, and in the *Republic*, 443 D, ὅροι are *notes* between which there are musical intervals. On the basis of the usage in later writers and in Plato himself the only probable interpretation will find here a reference to a progression, and this conclusion is strengthened by the fact that we have a reference to two *harmonies* in the next sentence, and that from the rectangle 2700 x 4800 there mentioned, and the square, if it is 3600^2, we get two harmonies, 2700 : 3600 *equals* 3600 : 4800, and a geometrical progression. Now if αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι applies to the hypotenuse and sides of a right-angled triangle, if it means *when* a^2 *equals* b^2+c^2, it is difficult to see how we are to get a geometrical proportion; but if the phrase means *when* a^2 *equals* bc, the proportion $b : a = a : c$ at once suggests itself.

On the basis of Alexander's statement that the words δυνάμεναι and δυναστευόμεναι were applied to $a^2=b^2+c^2$ let us assume, as a working hypothesis, that they might also be applied to $a^2=bc$. The conditions for their use would seem to be the following: (1) it is necessary that *one* magnitude should be equal to a combination of two or more others, so that the one *equals*, the others *are equaled*; (2) in view of the mathematical meanings of δύναμις and δύναμαι it is evident that the *one* magnitude must be a square. On this assumption αὐξήσεις δυνάμεναι (a^2) would be *squares that equal*, and αὐξήσεις δυναστευόμεναι (bc)

[15] Proclus, citing Dercyllides, gives us τῆς δ' ὑποτεινούσης ἢ δυναμένης where both participles are treated as nouns and put in the genitive absolute like any other noun.

[16] This is the meaning given to the words by Proclus, II, 36.22; cp. the example 37.12-13.

rectangles that are equaled by squares, or *products of two unequal numbers that are equaled by squares.* The proportion, "with four terms and three intervals," would be $b : a = a : c$.

Proceeding upon our hypothesis, we should expect to find in the phrase ὁμοιούντων τε καὶ ἀνομοιούντων καὶ αὐξόντων καὶ φθινόντων a statement concerning the order in which the terms of the proportion should be placed. The rest of the sentence means (hypothetically): *if a square is equal to a rectangle, and if we set down four terms and three intervals . . . everything turns out to be rational and in proportion.* Such a definition is incomplete. The arrangement of the terms should be added. Using symbols we should say; "if $a^2 = bc$, then $b : a = a : c$ or $a : b = c : a$. Without symbols the definition would be *the side of the square is a mean proportional between the sides of the rectangle*, or, *one side of the rectangle is to the side of the square as the side of the square is to the other side of the rectangle.* Part of this Plato tells us in ὁμοιούντων τε καὶ ἀνομοιούντων. Iamblichus (*Ad Nicom.*, p. 115) and Proclus (*In Remp.* II, 36, 14) are authorities for the statement that squares were called ὅμοιοι, rectangles, ἀνόμοιοι. In the proportion we are dealing with, since it is assumed to be derived from $a^2 = bc$, a is ὁμοιῶν *making a square*, and, therefore means the side of the square, b and c are ἀνομοιοῦντες *making a rectangle*, and each is a side of the rectangle. In a proportion we think naturally of two pairs of ratios, $b : a$ and $a : c$, just as Plato in the next sentence speaks of two ἁρμονίαι. Now, if $a^2 = bc$, it follows that $b : a = a : c$ or $a : b = c : a$. Whichever way we put it, each of the 'harmonies' is composed of an ὁμοιῶν and an ἀνομοιῶν; cp. *the side of the square is to the side of the rectangle.*

Weber, who found in the passage the proportion 6400 : 4800 = 3600 : 2700, drew the conclusion that ὁμοιούντων τε καὶ ἀνομοιούντων meant that the first and third terms were squares, the second and fourth rectangles. He failed to interpret καὶ αὐξόντων καὶ φθινόντων, suggesting, indeed, that this was a repetition of ὁμοιούντων and ἀνομοιούντων, which is obviously improbable. Of course, Plato says nothing of first and third, second and fourth terms; but, if we think of the progression as divided into pairs, then each pair in Weber's is composed of a square and a rectangle. This is not true of the alternative form 6400 : 3600 = 4800 : 2700; and in many proportions of the form $a : b = c : d$, for instance $3:6=12:24$, there are no squares at all. Now Plato's statement here is general—at least no acceptable interpretation has found numbers in it—and it should cover all cases. Further, Weber's terms are squares and rectangles, and should be called ὅμοιοι and ἀνόμοιοι, whereas Plato has used ὁμοιοῦντες *square-makers* and ἀνομοιοῦντες *rectangle-makers*, i.e., the terms should be *sides* of squares and rectangles. If these words define the terms of the progression which the phrase τρεῖς ἀποστάσεις τέτταρας δὲ ὅρους almost forces us to assume, it is obvious that they can be applied only to one developed from $a^2=bc$ A definition of the terms is needed, and no other meaning for the words has been offered that is in the slightest degree plausible. ὁμοιοῦντες and ἀνομοιοῦντες fit in so well with the hypothesis that αὐξήσεις δυνάμεναι means *squares equaling* and αὐξήσεις δυναστευόμεναι *rectangles being equaled* by the squares, that they must be considered a strong support for its truth.

Proceeding to καὶ αὐξόντων καὶ φθινόντων, we observe that the order of the terms is not sufficiently defined if we say that each ratio must be composed of the side of the square and a side of the rectangle. The form $a:b=a:c$, for instance, is not correct. We must have either $b:a=a:c$ or $a:b=c:a$; that is, in a particular case, either $27:36=36:48$ or $36:27=48:36$. In the former both ratios are composed of αὔξοντες *increasing* terms; in the latter both have φθίνοντες *decreasing* terms. As to the syntax, ὅρων should be supplied with these participles, and the genitive is absolute. A comma should be placed after ἀνομοιούντων. The translation will be—*if the terms* (of each ratio) *are a side of the square and a side of the rectangle, both if they are increasing and if they are decreasing*. The close connection of ὁμοιούντων and ἀνομοιούντων by τε καί, as contrasted with the καί–καί, that join αὐξόντων and φθινόντων to what precedes and to one another, justifies taking with each ratio the combined ὁμοιούντων τε καὶ ἀνομοιούντων, while applying αὐξόντων and φθινόντων to each ratio separately.

The closing words of the sentence, πάντα προσήγορα καὶ ῥητὰ πρὸς ἄλληλα ἀπέφηναν, look simple enough. Many interpreters, after wresting from αὐξήσεις δυνάμεναι κτέ. something that seemed to them to make sense, have been content to let the rest take care of itself. So broad a meaning has been given to the words that they have in fact meant nothing. With my explanation of αὐξήσεις . . . φθινόντων there is nothing vague about the conclusion. It gives us what is absolutely essential. *A square and a rectangle being equal, if four terms be set down in the proper order, show all things rational and corresponding to one another.*[17]

In αὐξήσεις . . . ἀπέφηναν, then, we have a definition of the geometrical law that the side of a square is a mean proportional between the sides of a rectangle equal to the square. The definition is obscure. In some respects, probably, not so obscure to the Greek mathematician as to us. Such a beginning as *a square and rectangle, being equal, if they take four terms* might lead us to ask what four terms are meant. But this would surely be simple to the Greek who divided all numbers into square and rectangular forms, 4 being 2 x 2, 6 being 2 x 3 (*Theaet.* 147-8). Nevertheless, the definition is obscure and probably intentionally so. As the ἐπίτριτος πυθμὴν πεμπάδι συζυγείς of the next sentence is the Pythagorean triangle with sides of 3, 4, and 5, so here we may have the Pythagorean definition of mean proportionals. At a time when the science still lacked a technical terminology it must have been exceedingly difficult to give a clear definition of a newly discovered truth, even if clearness had been an object. Secrecy, tradition says, was the aim of the Pythagoreans.

The sentence αὐξήσεις . . . ἀπέφηναν is introduced by ἐν ᾧ πρώτῳ. Like προσήγορα ἀπέφηναν, these words have been treated rather lightly. They look simple also, but they create astonishing confusion in some of the interpretations. To lead up to what must be done with ἐν ᾧ πρώτῳ I shall cite Adam's translation as an example of what it can not be. As is usually done, he takes ἀριθμός to be the antecedent of ᾧ, and he finds a *number* in the following words—again the usual thing.[18] His translation is: *But the number of a human creature is the first*

[17] Proclus (II, 36.22) also takes the words to mean *in proportion*, as the examples in his diagram show. L. & S. (*s.* προσήγορος) cite ὁμόφρονα καὶ ποτάγορα ἀλλάλοις Polus ap. Stob.; "so in other late Pythag. writers, σύμφωνα καὶ π., ὁμοῖα καὶ π."

[18] Proclus, as we have seen, does not deal with actual numbers until he comes to ὧν ἐπίτριτος πυθμήν. For him the first sentence is a general statement describing the results of multiplying in various ways any ratio one chooses to take—τὸν πυθμένα τὸν ἐκτεθησόμενον (II, 36.27). Unfortunately his explanation of ἐν ᾧ πρώτῳ, if he gave one, is lost.

number in which root and square increases, having received three distances and four limits, of elements that make both like and unlike and wax and wane, render all things conversable and rational with one another. The words from *root* to *limits* are said to mean *cubings and nothing more* (p. 272); *of elements . . . wane* means of the numbers 3, 4, 5 (p. 273); the cubes of 3, 4, 5 are added together, making 216, and "the justification for adding the cubes together is that the numbers are said to be *contained* in the total (ἐν ᾧ πρώτῳ κτλ.)," (p. 274). "The number 216 is the first number (ἐν ᾧ πρώτῳ κτλ.) in which the cubes of 3, 4, 5 occur," (p. 293). The words πάντα προσήγορα . . . ἀπέφηναν Adam then interprets by comparison with a passage in Censorinus concerning the harmonious development of the embryo. There is nothing in Censorinus about the number 216, but let us grant to Adam that the Pythagoreans connected 216 with the development of the child. His translation of Plato's sentence then comes to this: *But the number of a human creature is 216, the first number in which cubings of 3, 4, 5 make the development of the embryo harmonious.* Surely it is asking a good deal to expect us to believe that Plato or the Pythagoreans meant by this "the number of a human creature is the first number in which the cubes of 3, 4, 5 occur, namely, 216, for this 216 controls the harmonious development of the embryo." The simple fact is that in Adam's interpretation, even if we accept his explanation of every other phrase, the ἐν ᾧ πρώτῳ is quite meaningless; and no other translation that I have seen offers any satisfactory reason for the number selected being *the first number in which* all things are made rational and in proportion.

If my interpretation of αὐξήσεις . . . ἀπέφηναν is correct, we must ask, in taking up ἐν ᾧ πρώτῳ, what that is *in which first* it was proved that $b : a = a : c$, if $a^2 = bc$. In the eighth proposition of the sixth book of Euclid it is proved that if, in a right-angled triangle, a perpendicular be drawn from the right angle to the hypotenuse, the triangles thus formed are similar to one another and to the whole triangle; and (corollary) the perpendicular is a mean proportional between the segments of the base. From the same figure we know, by Euclid 1, 47 (solved by Pythagoras) and 2, 4, that the square of the perpendicular is equal to the rectangle contained by the segments of the base.[19] It would appear, then, that ἐν ᾧ πρώτῳ refers to this figure, in which first a^2 and bc, being equal, showed[20] $b : a$ to be equal to $a : c$.

A strong proof that ἐν ᾧ πρώτῳ refers to a geometrical figure comes from the fact that in this way we get a perfect connection with the following sentence. Perhaps no greater objection can be brought against the various solutions of the problem than their failure to furnish this connection. The relative ὧν joins the two sentences together. What is its antecedent? "If there is anything clear about the number, it surely is that ὧν in Plato has for its antecedent αὐξήσεις." So wrote Adam in 1892 (*C. R.*, p. 241). In 1902 (*Rep.* II, p. 273) he thinks that 'no one will deny that the relative is most obviously and naturally connected with" ὁμοιούντων . . . φθινόντων. In seeking this elusive antecedent we must bear in mind, not only the words in the preceding sentence to which ὧν might refer, but its possible connections in its own clause. The genitive must depend either upon ἐπίτριτος πυθμήν or upon ἁρμονίας. There is a suggestion of *harmonies* in the proportion which τρεῖς ἀποστάσεις τέτταρας δὲ ὅρους implies, so that we might try *of these harmonies the* ἐπίτριτος πυθμὴν κτἑ. *supplies two.* Yet, if Plato had meant this, he would hardly have said *of which* the ἐπίτριτος πυθμήν *supplies two harmonies.* Again, since the two harmonies are combinations of numbers, we might say *of which numbers* (i.e., such numbers as those in the preceding sentence) *the* πυθμήν *supplies two harmonies.* The objection to this is that no number can be made out of what precedes. It remains to try making ὧν dependent on ἐπίτριτος πυθμήν.

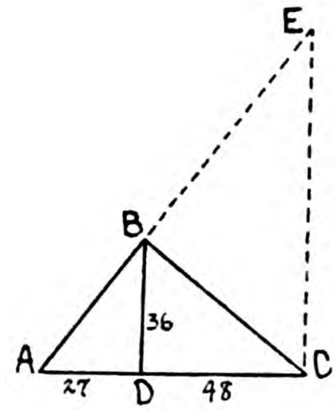
It is admitted by ancient and modern scholars that ἐπίτριτος πυθμὴν πεμπάδι συζυγείς refers to the Pythagorean right-angled triangle with sides in the proportion 3-4-5. The best ancient

[19] Euclid 1.47 proves that the square of the hypotenuse of a right-angled triangle equals the sum of the squares of the sides; 2.4 proves that, if a line be divided into two parts, the square of the whole line equals the squares of the parts and twice the rectangle contained by the parts.

[20] The aorist ἀπέφηναν has its ordinary past sense here. It is commonly, I believe, taken to be a gnomic aorist. In connection with the presents περιλαμβάνει and παρέχεται the past sense of the aorist is more natural, and this allows πρώτῳ to have its temporal force. This is at least a slight argument against the view that the number is the *first* from unity *in which* such and such a thing holds good.

reference for the whole phrase is that of Aristides Quintilianus (*De Musica*, p. 152)—*αἱ δὲ τὴν ὀρθὴν περιέχουσαι δηλοῦσι τὸν ἐπίτριτον. τούτου δὴ καὶ Πλάτων φησὶν ἐπίτριτον πυθμένα πεμπάδι συζυγέντα*; cp. Proclus, *In Euclid.* p. 428 and Plutarch, *De Is. et Os.*, p. 373 F. Yet, while admitting the reference to the triangle, scholars have added or multiplied the 3, 4, 5, thinking *συζυγείς* implies some such process. I do not hesitate to affirm that we must take either one of two alternatives. Either *συζυγείς* means that we are to add or multiply the numbers, and then there is no reference to the Pythagorean triangle; or, if this triangle is meant, it takes the whole phrase, *ἐπίτριτος πυθμὴν πεμπάδι συζυγείς*, to describe it; for *πεμπάδι συζυγείς* means that the 5 joins together the 3 and 4, and we have no right to give to it the additional meaning of *multiplied* or *increased by five*.[21] That Plato was speaking of the triangle is certain from Aristotle's reference to it as a diagram—*ἀρχὴν δ' εἶναι τούτων ὧν ἐπίτριτος πυθμὴν πεμπάδι συζυγεὶς δύο ἁρμονίας παρέχεται, λέγων ὅταν ὁ τοῦ διαγράμματος ἀριθμὸς τούτου γένηται στερεός* (*Pol.* E 12, 1316[a]). If now the relative *ὧν* depends upon *πυθμήν* the words *of which the 3-4-5 right-angled triangle* imply that *ὧν* has as its antecedent geometrical figures, and those, most likely, right-angled triangles. We have already found a right-angled triangle in *ἐν ᾧ πρώτῳ*. This, therefore, gives us the antecedent of *ὧν*. No difficulty need be made of *ᾧ* being singular and *ὧν* plural. The figure to which *ἐν ᾧ πρώτῳ* applies, with the perpendicular drawn to the base, really contains three right-angled triangles; and, apart from that, *ἐν ᾧ πρώτῳ* is used of right-angled triangles of any kind. In the next sentence *ὧν ἐπίτριτος κτέ* calls attention to a particular type of the class, those with sides in the proportion 3-4-5.

[21] In this I am in perfect agreement with Monro, *C. R.* 6.154.



Let ABC be a right-angled triangle with its sides AB, BC, AC in the proportion 3 : 4 : 5. If BD be drawn perpendicular to AC, the triangles ABD and BDC are similar to ABC, and therefore of the 3-4-5 type; and the perpendicular BD is a mean proportional between AD and DC, or $BD^2 = AD.DC$. Therefore, if AD is 3, DC is $5\frac{1}{3}$; or, if we wish to have whole numbers, AD may be called 9; then BD is 12, and DC 16. Proclus, in his 'geometrical' explanation of the number, starting from the smallest triangle ABD, gets ABC by drawing BC perpendicular to AB and producing AD to meet it; and then goes on to form AEC in the same way. Or, starting from the larger triangle AEC, he drops the perpendiculars CB and BD. He also points out that, if BE and EC are to be whole numbers, we shall be obliged to call AD 27; then BD is 36, DC 48, BC 60, CE 100, and so on (*In Rem.* II, 40).

The figure ABC with BD drawn perpendicular to the base, is the figure which proves that if $BD^2 = AD.DC$, then AD : BD = BD : DC; i.e., that, if a square is equal to a rectangle, then the side of the square is a mean proportional between the sides of the rectangle. This figure, then, is the one to which the sentence ἐν ᾧ πρώτῳ αὐξήσεις . . . ἀπέφηναν refers. If we make this figure of the 3-4-5 type, following the suggestion of ὧν ἐπίτριτος πυθμὴν κτὲ *of which the 3-4-5 type*, and if we give the sides the magnitudes suggested by Proclus, we find that the triangle ABC, with BD perpendicular to the base, *furnishes two harmonies* (δύο ἁρμονίας παρέχεται), namely 27 : 36 = 36 : 48, or $36^2 = 27 \times 48$. One of these 'harmonies' is ἴσην ἰσάκις *equal by equal*, the other is of the same area as the former, but rectangular (προμήκης) and the sides of the rectangle, 27 and 48, are the same as the sides of Plato's rectangular 'harmony,' except that his are multiplied by 100 (2700 and 4800), just as the sides of the square ἴσην ἰσάκις are multiplied by 100—ἑκατὸν τοσαυτάκις.

Plato says that the ἐπίτριτος πυθμὴν πεμπάδι συζυγεὶς furnishes two harmonies when it is *thrice increased* (τρὶς αὐξηθείς), or *multiplied three times*, or *raised to the third power*. What is meant by *thrice increasing* a triangle is no longer clear to us. Aristotle, in his comment on the passage (see p. 21) omits τρὶς αὐξηθείς and substitutes λέγων ὅταν ὁ τοῦ διαγράμματος ἀριθμὸς τούτου γίνεται στερεός *meaning when the number of this diagram is made solid*. A number becomes 'solid' when it is cubed; but what is *the number of this diagram*? The expression is not much clearer than Plato's; but this much seems certain, namely, that Aristotle meant the same thing as Plato, and did not mean to get 216 from $3^3+4^3+5^3$ or from 6^3; for, whether the harmonies are 10000 and 7500, as Proclus and many moderns believe, or 3600^2 and 2700 x 4800, they can not be derived in any simple way from 216. It is possible, as Tannery (*l.c.*) thinks, that τρὶς αὐξηθείς means the kind of *increase* which Proclus applies to the triangle ABD as above described. At any rate this gives us the numbers that Plato's text contains. In line with this we have in the margin of Par. A three right-angled triangles, one with sides 3-4-5, one with 9-12-15 (i.e., with the 3 squared), one with 27-36-45 (i.e., with the 3 cubed). Proclus, indeed, makes no reference to τρὶς αὐξηθείς in his 'geometrical' interpretation, in which he develops the figure above. In his 'arithmetical' section he agrees with Aristotle in taking it to mean *making solids*. Thus he says (p. 37, 20) εἰκότως οὖν εἶπεν τὸν ἐπίτριτον πυθμένα τρὶς αὐξηθέντα τὰς δύο ποιεῖν ἁρμονίας· μέχρι γὰρ τῶν στερεῶν προελθὼν κτέ.; and on p. 39, after getting 100 from 5 x 4 x 5, and 75 from 5 x 3 x 5, he adds (l. 18) καὶ οὕτως ὁ ἐπίτριτος πυθμὴν τρὶς αὐξηθεὶς πεμπάδι συζυγεὶς ἔσται ποιῶν τὰς δύο ἁρμονίας *and thus the 3-4 made solid by multiplying by five, etc.* Compare also the last few lines of p. 39, and p. 25, 18. I believe that Aristotle changed Plato's τρὶς αὐξηθείς, which qualifies the triangle just described, into ὁ ἀριθμὸς γίνεται στερεός because of the uncertainty that may have arisen from the application of the phrase to a geometrical figure. Aristotle means by ἀριθμός *number in the abstract*. τρὶς αὐξηθείς recalls κατὰ τρίτην αὔξην, which applies to solids. Used of a γραμμή *line* it would mean *cubed*; when it qualifies a triangle it means nothing. Aristotle, in saying *when the numbers of the 3-4-5 triangle are made solid*, means when the triangle is called, not 3-4-5, but 27-36-45, 27 being equal to 3 x 3 x 3, 36 to 3 x 3 x 4, 45 to 3 x 3 x 5, and therefore being 'solid' numbers.

I have derived $36^2 = 27 \times 48$ as the meaning of Plato's *δύο ἁρμονίας παρέχεται τρὶς αὐξηθεὶς* from the figures of Euclid and Proclus; but opinions on the meaning of *τρὶς αὐξηθεὶς* vary so much, and so many prefer to accept 10000 (*ἑκατὸν τοσαυτάκις*), and 7500 (from 2700+4800) as the two harmonies—and these numbers have the weight of Proclus' authority— rather than $3600^2 = 2700 \times 4800$, that I shall try to prove that the proper interpretation of *τὴν μὲν ἴσην ἰσάκις . . . κύβων τριάδος*, leaving *τρὶς αὐξηθεὶς* out of the question, is conclusive in favor of the latter numbers.

The word *προμήκης* means *oblong, rectangular.* That Plato means by *προμήκη* a rectangle, produced, as is usual, by the *multiplication* of unequal sides—and not by their *addition* as in 7500 = 2700+4800— is evident from *Theaet.* 148 A—*καὶ πᾶς ὃς ἀδύνατος ἴσος ἰσάκις γενέσθαι, ἀλλ' ἢ πλείων ἐλαττονάκις ἢ ἐλάττων πλεονάκις γίγνεται, μείζων δὲ καὶ ἐλάττων ἀεὶ πλευρὰ αὐτὸν περιλαμβάνει, τῷ προμήκει αὖ σχήματι ἀπεικάσαντες προμήκη ἀριθμὸν ἐκαλέσαμεν.* Everyone now accepts 4800 as the meaning of *ἑκατὸν μὲν . . . δυεῖν*; and *ἑκατὸν δὲ κύβων τριάδος*, of course, means 2700. *ἑκατὸν μὲν* and *ἑκατὸν δὲ* following *προμήκη* give the unequal sides of the rectangle, which is therefore 2700 x 4800. There can be no doubt that this is the natural interpretation. It is based upon ordinary Greek syntactical usage and the undisputed meaning Plato gives to *προμήκης* in another passage. As against these proofs the authority of Proclus is worthless. In the language of Adam "if there is anything clear about the number it surely is" that Plato's rectangular harmony is 2700 x 4800, and not 2700+4800.

There is, I believe, no evidence that the term *ἁρμονία* was applied to a square or a rectangle, at least outside of writers like Proclus, who are trying to explain the number. If we once grant that the rectangle is equal to the square, we understand why the word *ἁρμονίας* is used; for if $a^2 = bc$, we then have a *harmonious relation* between $b : a$ and $a : b$. *ἁρμονία*, mathematically, indicates a relation between quantities, especially a *proportion* (cf. Nicomachus, *Theol. Arith.* p. 47). In the equality of the relation between 2700 : 3600 and 3600 : 4800 we have two harmonies. A correct definition of *ἁρμονίαι* and προμήκης favors 3600^2 and 2700 x 4800, and rejects 10000 and 7500.

A second proof that the square is 3600^2 comes from the correct explanation of ἰσομήκη μὲν τῇ. Hultsch and Adam, who were the first to suggest that the square was 3600^2, finding it difficult to compare the side 3600 with either 2700 or 4800, were driven to the conclusion that ἰσομήκη μὲν τῇ meant *of equal length one way*. This, they said, meant that the opposite sides of a rectangle are equal, though the adjacent are not. Adam admitted (*Rep.* II, p. 284) that the statement was unnecessary. I find it impossible to believe that Plato intended to express any such idea. There are, I think, only two possible explanations of ἰσομήκη μὲν τῇ; or, rather, there is only one correct one, but two may be considered. They are:

(1) If we should say, "The second harmony is of the same length (as the square) in one direction, but of a different length in the other, for it is oblong," the *in one direction* . . . in the *other direction* could be rendered by τῇ μὲν . . . τῇ δέ. The "of a different length in the other" might be omitted, since "but it is oblong" supplies the idea. Thus the τῇ δέ, for which we would look as the contrast to τῇ μέν, is supplied in thought by προμήκη δέ. Now, if we take this view, in spite of the lack of μέν after τῇ, then, since the rectangle is without any doubt 2700 x 4800, the square must be either 2700^2 or 4800^2, and not 100^2. But no one defends either 2700^2 or 4800^2; the argument is all for 100^2 on the basis of ἑκατὸν τοσαυτάκις.

(2) τῇ may refer to the square τὴν μέν, and the dative be dependent upon ἰσομήκη. Is such a demonstrative use of τῇ possible? Plato uses καὶ τὸν εἰπεῖν (*Symp.* 174 A), but καὶ τόν has special rights which the lonely τῇ has not. A closer parallel is to be found in Plato's *Laws* 701 E:—ἐπὶ δὲ τὸ ἄκρον ἀγαγόντων ἑκατέρων, τῶν μὲν δουλείας, τῶν δὲ τοὐναντίου οὐ συνήνεγκεν οὔτε τοῖς οὔτε τοῖς. Here τοῖς . . . τοῖς refer back to τῶν μὲν . . . τῶν δέ. In our passage there is also the preceding τὴν μὲν . . . τὴν δέ, and the τῇ refers back to the τὴν μέν. Proclus, it may be remarked, turns the phrase by ἰσομήκης μὲν ἐκείνῃ (p. 37, 20). We do not accept all his mathematical calculations, but on a point of this kind his feeling for the Greek may be allowed to have some weight. If need be we may call the

demonstrative article dialectic and ascribe it to Pythagoras.[22]

But, if the rectangle is ἰσομήκη to the square, this would seem to bring us where we were in (1). Not altogether, however. The other explanation, taking προμήκη δέ to mean *of different length in the other direction* forces us to think that ἰσομήκη compares the side of the square with the side of the rectangle. If we reject the contrast of τῇ with προμήκη—and there is no μέν after τῇ—we can take ἰσομήκη μὲν τῇ to mean that the rectangle is *equal in area* to the square. I am aware that this view will be considered improbable, if not impossible; but I believe that it can be successfully defended by the aid of Plato and Euclid. In the first place Plato commonly uses μῆκος not in the sense of *length* as opposed to *width*, but of *lineal* measure as opposed to *square* and *cubic* measure; cf. κατὰ τὸν τοῦ μήκους ἀριθμὸν . . . κατὰ δὲ δύναμιν καὶ τρίτην αὔξην *Rep.* 587. Then we have this passage in the *Theaet.* 147 A: περὶ δυνάμεών τι ἡμῖν Θεόδωρος ὅδε ἔγραφε, τῆς τε τρίποδος πέρι καὶ πεντέποδος ἀποφαίνων ὅτι μήκει οὐ ξύμμετροι τῇ ποδιαίᾳ, which means *Theodorus was writing something for us concerning squares, proving concerning the square containing three feet and the square containing five feet that in linear measure they are not commensurable with the foot unit.* I am aware that others translate otherwise taking δυνάμεων to be *roots* and τῆς τρίποδος to be $\sqrt{3}$, but the translation I have given is the preferable one, for it is absurd to suppose that δυνάμεων in 147 A means the same thing as δυνάμεις in 148 A when the latter is just being explained as a new mathematical term not before in use. This passage shows that the Greeks thought of areas in terms of linear measure if they could reduce them to rational numbers. Euclid, Book X, Def. 2 says: εὐθεῖαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ' αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρῆται *straight lines are commensurable in square when the squares on them are measured by the same area.* I presume that it will not be denied that, if Greek mathematicians

[22] Herodotus has the demonstr. article after prepositions, when there is a direct reference to one demonstrative and a contrast with another; cp. ταῦτα ἀπέπεμπε, καὶ τάδε ἄλλα ἅμα τοῖσι 1.51.2 and ταῦτα ἔλεγε καὶ πρὸς τοῖσι τάδε 5.97.9. Compare also 7.8 B—πρῶτα μὲν . . . ἐνέπρησαν . . τὰ ἰρά· δεύτερα δὲ ἡμέας οἷα ἔρξαν . . . τὰ ἐπίστασθε, where, though there is no preceding demonstrative, there is the contrast πρῶτα μὲν . . δεύτερα δέ, the τά referring to δεύτερα δέ. Cp. Thuc. 3. 61. 1.

could talk of lines *commensurable in square*, they could also talk of areas *commensurable in linear measure*. A square that is 49 and a rectangle that is 48 are commensurable in their surfaces (ἐπίπεδα); but the μῆκος of the one is 7, of the other $\sqrt{48}$, and they are not commensurable in μῆκος. The rectangle 27 x 48, however, has the square surface 1296, the μῆκος of which is 36. Consequently, 27 x 48 is ἰσομήκης to 36^2.

Thus both ἁρμονίας and ἰσομήκη μὲν τῇ go to prove that the square is equal in area to the rectangle. The rectangle is certainly 2700 x 4800, and the square must therefore be 3600^2. Can this 3600^2 be derived in any other way than from the triangle of Proclus 'thrice increased' in the manner above described? The methods of Adam and Hultsch are wrong because they gave to πεμπάδι συζυγεὶς an arithmetical value instead of taking it as part of the definition of the Pythagorean triangle. Further, Hultsch made τρὶς αὐξηθεὶς mean *multiplied by three*, though that meaning is expressed by τριάδι αὐξηθεὶς (Nicom, *Intr. Ar.*, p. 105). I consider Adam's *raised to the fourth power* for τρὶς αὐξηθεὶς so indefensible that it is not worth discussing. Hultsch's $36 = 3 \times (3+4+5)$ cannot be said to *furnish* 3600^2 and 2700 x 4800 in any natural way. Adam's $(3 \times 4 \times 5)^4$ *furnishes* 3600^2, but it is not apparent why the factors 2700 and 4800 should be chosen in preference to others, whereas in the triangle of Proclus 36^2 and 27 x 48 are before our eyes. The triangle *furnishes* them both. Adam also failed to give a good explanation of τοσαυτάκις. Monro (*C. J.*, 6, 153) pointed out that τοσαυτάκις should not refer to a number "discovered by an algebraic process from a subsequent statement;" and when Adam says "$(3 \times 4 \times 5)^4$ furnishes two harmonies, the one equal an equal number of times, so many times 100," it cannot be granted that the *so many times* naturally means 36. I agree with Monro that "the ordinary interpretation of ἑκατὸν τοσαυτάκις—'a hundred taken *that* number of times viz. 100 times'—is unassailable," unless the preceding words clearly supply a 36 to which τοσαυτάκις can refer. Our triangle does supply the 36; and, since the first sentence emphasizes that $b : a = a : c$ if $a^2 = bc$, it is the $36^2 = 27 \times 48$ in this triangle that are especially called to the attention, rather than the proportion $27 : 36 = 48 : 64$, which the Proclus triangle also *furnishes*. When Plato says ἑκατὸν τοσαυτάκις after ἴσην ἰσάκις, inasmuch as the triangle gives 36^2 for the ἴσην ἰσάκις, he probably means 36 x 100, i.e., he gives the μῆκος of the square not its area. I offer no explanation for the numbers 27, 36, and 48 being multiplied by 100, but call attention to the fact that, if the figure which Plato had in mind contained the numbers 2700, 3600, and 4800, the square which it 'furnished' would have been sufficiently described by τὴν μὲν ἴσην ἰσάκις without adding ἑκατὸν τοσαυτάκις.

The full translation of the passage will be:

"There is for a divine creature a period which a perfect number contains; for a human creature (there is a number) in that figure in which first products that are squares and rectangles, equaling and being equaled, if arranged in a proportion with three intervals and four terms, the terms being sides of the squares and sides of the rectangles, both if they are increasing and if they are decreasing, showed all in proportion and rational to one another; of which the 3-4-5- type, if the numbers are made solid, furnishes two harmonies, the one a square with its side multiplied by 100, the other equal in area to the former but oblong, one side of 100 squares of rational diameters of five, each lacking one, or of irrational diameters, each lacking two, the other side of 100 cubes of 3. This total, a geometrical number, is in control of such a creature, of better and of worse births."

Hultsch (*l.c.*, p. 405) accepts the interpretation of Plato given by Proclus as far as p. 37, 12 the part I have cited above; after that, he says, Proclus gives zumeist nur willkürliche Kombinationen. That is, Hultsch believes that Plato is giving us here an example of what he tells us in *Timaeus* 31C–32B, namely that between two squares there is one mean proportional, between two cubes two mean proportionals. As a matter of fact, Proclus, while preserving a partially correct tradition concerning the meaning of αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι, goes utterly wrong in his treatment of ὁμοιούντων . . . . φθινόντων. He pays no attention to the syntax of Plato's sentence. If these words had the meanings he gives them they should stand in some syntactical relation to αὐξήσεις, as do the δυνάμεναι and δυναστευόμεναι, to which Proclus makes ὁμοιοῦντες, etc. parallel. We have no evidence except Proclus' statement that αὔξοντες and φθίνοντες had the same meanings as δοκίδες and πλινθίδες.

I shall add a few remarks on Aristotle's criticism of Plato's passage in *Pol.* E 12, 1316[a]. He says: ἐν δὲ τῇ πολιτείᾳ λέγεται μὲν περὶ τῶν μεταβολῶν ὑπὸ τοῦ Σωκράτους, οὐ μέντοι λέγεται καλῶς· τῆς τε γὰρ ἀρίστης πολιτείας καὶ πρώτης οὔσης οὐ λέγει τὴν μεταβολὴν ἰδίως. φησὶ γὰρ αἴτιον εἶναι τὸ μὴ μένειν μηθὲν ἀλλ' ἔν τινι περιόδῳ μεταβάλλειν, ἀρχὴν δ' εἶναι τούτων ὧν ἐπίτριτος πυθμὴν πεμπάδι συζυγεὶς δύο ἁρμονίας παρέχεται, λέγων ὅταν ὁ τοῦ διαγράμματος ἀριθμὸς τούτου γένηται στερεός, ὡς τῆς φύσεώς ποτε φυούσης φαύλους κτέ. . . . . καὶ διά γε τοῦ χρόνου, δι' ὃν λέγει πάντα μεταβάλλειν, καὶ τὰ μὴ ἅμα ἀρξάμενα γίνεσθαι ἅμα μεταβάλλει, οἷον εἰ τῇ προτέρᾳ ἡμέρᾳ ἐγένετο τῆς τροπῆς, ἅμα ἄρα μεταβάλλει; In this comment ἀρχὴν has been taken (cf. Adam, *Rep.* II, p. 307 ff.) to mean the *beginning* of the change. Now a *number*, whether 216 or another, does not define the beginning of anything, and I refuse to accept Adam's notion that the clause ὅταν . . . στερεός fixes the time of the beginning. These words are Aristotle's definition of Plato's τρὶς αὐξηθείς. Of course, Plato does not say that the number *begins* the change, but he does say that it is *in control* (κύριος) of better and worse births. I take it that Aristotle means "he says that there is a controlling principle in those things of which the 3-4-5- type, etc., on the ground that *nature* (i.e., uncontrolled) sometimes produces inferior men."

In the last sentence some have changed τοῦ χρόνου into τὸν χρόνον or δι' ὃν into δι' οὗ. I think that διὰ τοῦ χρόνου refers to the end of the successive periods, so that the meaning is "and at the end of the period, in the course of which he says all things change, do those things that did not come into being at the same time change at the same time, for example if they were born on the day before the turn, do they change at the same time?"

Aristotle's τούτων refers to the type of figure we have found in Plato's ἐν ᾧ πρώτῳ. The fact that he omits the whole of the first sentence, except as he sums it up in τούτων, is additional evidence of the correctness of the view that the first sentence contains a general statement of the law of which a particular example is given in the second.

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